GROUPS OF EMBEDDED MANIFOLDS

BY
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Abstract. This paper defines a group $\theta(M^n, \nu_\phi)$ which generalizes the group of framed homotopy n -spheres in S^{n+k} . Let M^n be an arbitrary 1-connected manifold satisfying a weak condition on its homology in the middle dimension and let ν_ϕ be the normal bundle of some imbedding $\phi: M^n \rightarrow S^{n+k}$, where $2k \geq n+3$. Then $\theta(M^n, \nu_\phi)$ is the set of h -cobordism classes of triples (F, V^n, f) , where $F: S^{n+k} \rightarrow T(\nu_\phi)$ is a map which is transverse regular on M , $V^n = F^{-1}(M^n)$, and $f = F|V^n$ is a homotopy equivalence. ($T(\nu_\phi)$ is the Thom complex of ν_ϕ .) There is a natural group structure on $\theta(M^n, \nu_\phi)$, and $\theta(M^n, \nu_\phi)$ fits into an exact sequence similar to that for the framed homotopy n -spheres.

This paper attempts to generalize in a natural way a well-known exact sequence concerning framed homotopy spheres which is contained in the work of Novikov [11], Kervaire-Milnor [7], and Levine [10]. The author stumbled onto these results partly because of his efforts to prove imbedding theorems for manifolds in the metastable range, and partly because of his recent work on Browder-Novikov theory for maps of degree d , $|d| \neq 0$ (see [2]).

§2 describes the basic constructions used in this paper. The “group of embedded manifolds”, $\theta(M, \nu_\phi)$, is defined in §3. A fairly simple description of that group is given towards the end of that section. §3 also contains the main results about $\theta(M, \nu_\phi)$. In §4 we discuss a few interesting open problems. The author would like to thank the referee for some helpful suggestions.

1. Notation. All manifolds will be C^∞ , compact, and oriented. Maps will be transverse to boundaries.

If M^n is a connected closed manifold, let $[M] \in H_n M$ denote the orientation class. Recall that $f: V^n \rightarrow M^n$ is said to have degree d , i.e., $\deg f = d$, if $f_*([V]) = d[M]$, where $f_*: H_n V \rightarrow H_n M$ is the map induced by f on the integral homology groups.

As usual, D^k denotes the closed unit ball in Euclidean k -space $\mathbf{R}^k$, i.e., $D^k = \{(y_1, \dots, y_k) \in \mathbf{R}^k \mid y_1^2 + \dots + y_k^2 \leq 1\}$. $S^k = \partial D^{k+1} = D_+^k \cup D_-^k$, where $D_+^k = \{(y_1, \dots, y_{k+1}) \in \mathbf{R}^{k+1} \mid y_1^2 + \dots + y_{k+1}^2 = 1, y_1 \geq 0\}$ and $D_-^k = \{(y_1, \dots, y_{k+1}) \in \mathbf{R}^{k+1} \mid y_1^2 + \dots + y_{k+1}^2 = 1, y_1 \leq 0\}$. We have natural inclusions $S^k \subseteq S^{k+1}$ and $D^k \subseteq D^{k+1}$. Let $e = (1, 0) \in S^0 \subseteq S^k$.

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If $f: V^n \rightarrow W^{n+q}$ is an imbedding, we shall consider f as an inclusion map and identify the total space $E = E(\nu_f)$ of the normal disk bundle ν_f with a tubular neighborhood of V in W . $T(\nu_f) = E/\partial E$ is the Thom complex of ν_f , and $T_f: W \rightarrow T(\nu_f)$ is the natural collapsing map. Given $g: U^m \rightarrow T(\nu_f)$ which is transverse regular on V so that $N = g^{-1}(V)$ is an $(m-q)$ -submanifold of U , we shall always assume that a tubular neighborhood T of N has been given a fixed bundle structure which is the pullback of ν_f under g . We also assume that N is given an orientation which is induced from the orientation of V .

If $V^n \subseteq W^{n+q}$, then by a framing of V in W , or by a framing of a tubular neighborhood T of V , we shall mean a diffeomorphism $\mathcal{F}: V \times D^q \rightarrow T$ such that $\mathcal{F}(x, 0) = x$. Two framed submanifolds $(V_1^n, \mathcal{F}_1)$ and $(V_2^n, \mathcal{F}_2)$ in W^{n+q} are framed cobordant if there is a framed submanifold $(N^{n+1}, \mathcal{G})$ in $W \times [1, 2]$ such that $(N, \mathcal{G}) \cap (W \times i) = (V_i, \mathcal{F}_i) \times i$, $i = 1, 2$. They are framed h -cobordant if N is an h -cobordism.

2. Preliminaries. Throughout this paper we shall make the following assumptions: M^n is a 1-connected oriented manifold with $n \geq 5$. Let $t = [n/2]$. Then either $n \equiv 0 \pmod{4}$, or $H_t M = 0$ and $H_{t-1} M$ is torsion-free. φ is an imbedding of M in S^{n+k} , where $2k \geq n+3$.

DEFINITION.

$\mathcal{T}_0^+(M, \nu_\varphi) = \{(F, V^n, f) \mid F: S^{n+k} \rightarrow T(\nu_\varphi) \text{ is a map which is transverse regular on } M \text{ with } V^n = F^{-1}(M) \text{ 1-connected and } f = F|V: V \rightarrow M\}$.

$\mathcal{S}_0^+(M, \nu_\varphi) = \{(F, V, f) \in \mathcal{T}_0^+(M, \nu_\varphi) \mid \deg f > 0 \text{ and } f_*: H_i V \rightarrow H_i M \text{ is an isomorphism for } 0 \leq i \leq [n/2]\}$.

NOTE. $\mathcal{S}_0^+(M, \nu_\varphi) \neq \emptyset$ since it contains $(T_\varphi, M, \text{identity})$.

If $\alpha_i = (F_i, V_i, f_i) \in \mathcal{T}_0^+(M, \nu_\varphi)$, define $\alpha_1 \sim \alpha_2$ if there is a map $H: S^{n+k} \times [1, 2] \rightarrow T(\nu_\varphi)$ which is transverse regular on M such that $H|S^{n+k} \times i = F_i \times i$ and $H^{-1}(M)$ is an h -cobordism between $V_1 \times 1$ and $V_2 \times 2$. Clearly, $\sim$ is an equivalence relation. We set

$$\mathcal{T}^+(M, \nu_\varphi) = \mathcal{T}_0^+(M, \nu_\varphi)/\sim \quad \text{and} \quad \mathcal{S}^+(M, \nu_\varphi) = \mathcal{S}_0^+(M, \nu_\varphi)/\sim \subseteq \mathcal{T}^+(M, \nu_\varphi).$$

If $\alpha \in \mathcal{T}_0^+(M, \nu_\varphi)$, we shall also write α for the equivalence class that α determines in $\mathcal{T}^+(M, \nu_\varphi)$.

Suppose that $\alpha_i = (F_i, V_i, f_i) \in \mathcal{T}^+(M, \nu_\varphi)$. Let ν_i be the normal disk bundle of V_i in S^{n+k} and let $D_+^n \times D^k$ and $D_-^n \times D^k$ be canonical tubular neighborhoods of D_+^n and D_-^n in D_+^{n+k} and D_-^{n+k} , respectively. Without loss of generality we may assume that $D_+^n \subseteq V_1$, $D_-^n \subseteq V_2$, $E(\nu_1|S^{n-1}) = E(\nu_2|S^{n-1}) \subseteq S^{n+k-1}$, $V_1 - D_+^n \subseteq D_-^n \times D^k$, $V_2 - D_-^n \subseteq D_+^n \times D^k$, $f_1(D_+^n) = f_2(D_-^n) = x_0 \in M$, and $f_1|S^{n+k-1} = f_2|S^{n+k-1}$. Now define $F_3: S^{n+k} \rightarrow T(\nu_\varphi)$ by $F_3|D_+^{n+k} = f_2|D_+^{n+k}$ and $F_3|D_-^{n+k} = f_1|D_-^{n+k}$. Let $V_3 = F_3^{-1}(M)$, $f_3 = F_3|V_3$, and $\alpha_1 \# \alpha_2 = (F_3, V_3, f_3)$. Then $\deg f_3 = \deg f_1 + \deg f_2$, $V_3 = V_1 \# V_2$, and $\alpha_1 \# \alpha_2$ is a well defined element of $\mathcal{T}^+(M, \nu_\varphi)$.

Next, define $\theta_f^{n+k,n}$ to be the group of h -cobordism classes of framed homotopy n -spheres in S^{n+k} . If Σ^n is a homotopy n -sphere in S^{n+k} with a framing $\mathcal{F}$ of its normal disk bundle ν_Σ , we let $[\Sigma, \mathcal{F}]$ denote the element it determines in $\theta_f^{n+k,n}$. (Note that in analogy with $\theta_f^{n+k,n}$ we can think of $\mathcal{T}^+(M, \nu_\phi)$ and $\mathcal{S}^+(M, \nu_\phi)$ as h -cobordism classes of certain submanifolds $V^n \subseteq S^{n+k}$ with a given bundle map from the normal disk bundle of V in S^{n+k} to ν_ϕ .)

Keeping the notation of the two previous paragraphs, let $\sigma = [\Sigma, \mathcal{F}] \in \theta_f^{n+k,n}$. Assume that $D_-^n \subseteq \Sigma$, $E(\nu_\Sigma|S^{n-1}) = E(\nu_1|S^{n-1})$, and $X = \Sigma - D_-^n \subseteq D_+^n \times D^k$. Define $F': S^{n+k} \rightarrow T(\nu_\phi)$ by $F'|D_-^{n+k} = f_1|D_-^{n+k}$, $F'(\mathcal{F}(y, u)) = f_1(\mathcal{F}(e, u))$, for $(y, u) \in X \times D^k$, and $F'(D_-^{n+k} - \mathcal{F}(X \times D^k)) = \text{canonical base point of } T(\nu_\phi)$. Set $V' = (F')^{-1}(M)$, $f' = F'|V'$, and $\alpha_1 \# \sigma = (F', V', f')$. Then $\alpha_1 \# \sigma$ is a well defined element of $\mathcal{T}^+(M, \nu_\phi)$, $\deg f' = \deg f$, and $V' = V_1 \# \Sigma$.

One can easily check that if $\alpha_i \in \mathcal{T}^+(M, \nu_\phi)$ and $\sigma_i \in \theta_f^{n+k,n}$, then the two connected sum operations defined above have the following properties:

- (1) $(\alpha_1 \# \alpha_2) \# \alpha_3 = \alpha_1 \# (\alpha_2 \# \alpha_3)$,
- (2) $\alpha_1 \# \alpha_2 = \alpha_2 \# \alpha_1$,
- (3) $(\alpha_1 \# \alpha_2) \# \sigma_1 = \alpha_1 \# (\alpha_2 \# \sigma_1)$,
- (4) $(\alpha_1 \# \sigma_1) \# \alpha_2 = (\alpha_1 \# \alpha_2) \# \sigma_1$,
- (5) $\alpha_1 \# (\sigma_1 + \sigma_2) = (\alpha_1 \# \sigma_1) \# \sigma_2$, and
- (6) if $\alpha_1 \in \mathcal{S}^+(M, \nu_\phi)$, then $\alpha_1 \# \sigma_1 \in \mathcal{S}^+(M, \nu_\phi)$.

LEMMA 1. Let $\alpha_i = (F_i, V_i, f_i) \in \mathcal{S}^+(M, \nu_\phi)$ and suppose $\alpha_1 \# \alpha_2 = (F_3, V_3, f_3)$. Then there exists a triple $\Gamma = (H, W^{n+1}, h)$ such that

- (a) $H: S^{n+k} \times [3, 4] \rightarrow T(\nu_\phi)$ is a map which is transverse regular on M with $W = H^{-1}(M)$ 1-connected and $h = H|W: W \rightarrow M$;
- (b) $H|S^{n+k} \times I = F_i \times I$, $I = 3, 4$;
- (c) $\alpha_4 = (F_4, V_4, f_4) \in \mathcal{S}^+(M, \nu_\phi)$ where $V_4 = F_4^{-1}(M)$ and $f_4 = F_4|V_4$; and
- (d) $H_i(W, V_4) = 0$ for $i \geq t+1$ if $n \not\equiv 0 \pmod{4}$.

Proof. We shall define inductively a sequence $\Gamma_i = (H_i, W_i, h_i)$, for $0 \leq i \leq [n/2] - 1$, such that

- (1) $H_i: S^{n+k} \times [0, i+1] \rightarrow T(\nu_\phi)$ is a map which is transverse regular on M with $W_i = H_i^{-1}(M)$ 1-connected and $h_i = H_i|W_i$;
- (2) $H_i|S^{n+k} \times [0, i] = H_{i-1}$;
- (3) $\partial W_i = V_3 \times 0 \cup N_i$ with N_i 1-connected;
- (4) $(h_i)_*: H_i W_i \rightarrow H_i M$ is an isomorphism for $0 \leq t \leq i$;
- (5) if $j: V_3 \times 0 \rightarrow W_i$ is the natural inclusion, then $j_*: H_t(V_3 \times 0) \rightarrow H_t W_i$ is an isomorphism for $t > i+1$; if $t = i+1$, j_* is one-to-one and $H_{i+1} W_i = j_* H_{i+1}(V_3 \times 0) \oplus G$, where G is a torsion-free group which is zero if $H_i M$ had no torsion and $(h_i)_*(G) = 0$.

Define $H_0: S^{n+k} \times [0, 1] \rightarrow T(\nu_\phi)$ by $H_0(x, t) = F_3(x)$. Then H_0 clearly determines a triple $\Gamma_0 = (H_0, W_0, h_0)$ which satisfies (1)–(5). Suppose $\Gamma_{i-1} = (H_{i-1}, W_{i-1}, h_{i-1})$

has been defined for $1 \leq i \leq [n/2] - 1$ satisfying (1)–(5). Our object will be to add handles to W_{i-1} along N_{i-1} to make h_{i-1} i -connected.

Let $j': N_{i-1} \rightarrow W_{i-1}$ be the natural inclusion and consider the exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{i+1}(W_{i-1}, N_{i-1}) & \longrightarrow & H_i N_{i-1} & \xrightarrow{j'_*} & H_i W_{i-1} = j_* H_i(V_3 \times 0) \oplus G \longrightarrow \cdots \\ & & \cong & & \downarrow & \nearrow (h_{i-1})_* & \\ & & H^{n-i}(W_{i-1}, V_3 \times 0) & & H_i M & & \end{array}$$

Since $i \leq [n/2] - 1$, the universal coefficient theorem for cohomology and (5) imply that j'_* is an isomorphism. Let $K_i = \text{kernel of } (f_3)_*: H_i V_3 \rightarrow H_i M$. K_i is a direct summand of $H_i V_3$ because $V_3 = V_1 \# V_2$ and $(f_l)_*: H_i V_l \rightarrow H_i M$, $l=1, 2$, is an isomorphism. Therefore, if $K = \text{kernel of } (h_{i-1})_*: H_i W_{i-1} \rightarrow H_i M$, then K is a direct summand of $H_i W_{i-1}$ and $K = j_*(K_i) \oplus G$. The diagram

$$\begin{array}{ccc} H_i(V_3 \times 0) & \xrightarrow{j_*} & H_i W_{i-1} \\ (f_3 \times 0)_* \searrow & & \swarrow (h_{i-1})_* \\ & H_i M & \end{array}$$

shows that $(h_{i-1})_*$ is onto $H_i M$. It follows that every element of K can be realized as an imbedded sphere in N_{i-1} with trivial normal bundle. One can now add handles to W_{i-1} along N_{i-1} to kill K as in the case of the usual Browder-Novikov theory. In fact, since $2k \geq n+1$, the handles can be attached in $S^{n+k} \times i$ so that the method of [4] and [9] can be used to obtain $\Gamma_i = (H_i, W_i, h_i)$. Γ_i will satisfy (1)–(4) trivially. The proof of Theorem 2.1 in [1] shows how (5) can be satisfied also. This finishes the inductive definition of Γ_i .

Let $t = [n/2]$. Assume that $n \not\equiv 0 \pmod{4}$. By hypothesis $H_t M = 0$ and $H_{t-1} M$ is torsion-free, and so (5) implies that $(h_{t-1}|N_{t-1})_*: H_t N_{t-1} \rightarrow H_t M$ is an isomorphism for $0 \leq l \leq [n/2]$. Define $H: S^{n+k} \times [3, 4] \rightarrow T(v_\phi)$ by $H(x, u) = H_{t-1}(x, t(u-3))$ and set $W = H^{-1}(M)$, $h = H|W$. Then $\Gamma = (H, W, h)$ satisfies (a)–(d) in Lemma 1 and the lemma is proved in this case. If $n \equiv 0 \pmod{4}$, then there is no obstruction to doing surgery on N_{t-1} in the middle dimension and one can define Γ_t satisfying (1)–(4) very much like the other Γ_i . Therefore we can get a Γ satisfying (a)–(c) in this case also. This completes the proof of Lemma 1.

We can now define an operation $+$ in $\mathcal{S}^+(M, v_\phi)$ as follows: If $\alpha_i \in \mathcal{S}^+(M, v_\phi)$, then we let $\alpha_1 + \alpha_2 = \alpha_4$, where α_4 is defined as in Lemma 1(c).

LEMMA 2. $+$ is a well defined associative and commutative operation.

Proof. Suppose that $\Gamma' = (H', W', h')$ and $\alpha'_4 = (F'_4, V'_4, f'_4)$ are triples which satisfy (a)–(d) in Lemma 1. $+$ will be well defined once we show that $\alpha'_4 = \alpha_4$. Define $H'': S^{n+k} \times [2, 4] \rightarrow T(v_\phi)$ by $H''(x, u) = H'(x, -u+6)$ for $u \in [2, 3]$ and $H''(x, u) = H(x, u)$ for $u \in [3, 4]$. Let $W'' = (H'')^{-1}(M)$ and $h'' = H''|W''$. It suffices to show that we can make W'' into an h -cobordism via framed surgery in $S^{n+k} \times$

(2, 4). To be precise, we are looking for a map $P: S^{n+k} \times [2, 4] \times [0, 1] \rightarrow T(\nu_\phi)$ satisfying

- (1) P is transverse regular on M ;
- (2) $P|_{S^{n+k} \times [2, 4] \times 0} = H'' \times 0$, $P|_{S^{n+k} \times 2 \times u} = F'_4 \times 2 \times u$, $P|_{S^{n+k} \times 4 \times u} = F_4 \times 4 \times u$, for $u \in [0, 1]$; and
- (3) $U = P^{-1}(M) \cap S^{n+k} \times [2, 4] \times 1$ is an h -cobordism.

If $n \equiv 0 \pmod{4}$, then there is no obstruction to doing surgery on W'' , even in the middle dimensions. If $n \not\equiv 0 \pmod{4}$, then it follows from Lemma 1(d) (using our hypothesis on the homology of M in the middle dimensions) that we only have to do surgery on W'' in dimension $\leq [n/2]$. In any case, it is therefore possible to define P inductively, similar to the definition of the Γ_i in the proof of Lemma 1. We shall omit the details and leave it to the reader to translate the construction for the Γ_i so that it is applicable in this situation.

Finally, the fact that $+$ is associative and commutative follows from the fact that $\#$ has these properties. This finishes the proof of Lemma 2.

LEMMA 3. Let $\alpha_i \in \mathcal{S}^+(M, \nu_\phi)$ and $\sigma_i \in \theta_j^{n+k, n}$. Then

$$(\alpha_1 \# (\sigma_1 + \sigma_2)) + \alpha_2 = (\alpha_1 \# \sigma_1) + (\alpha_2 \# \sigma_2).$$

Proof. This lemma is an easy consequence of the observation that

$$(\alpha_1 \# (\sigma_1 + \sigma_2)) \# \alpha_2 = (\alpha_1 \# \sigma_1) \# (\alpha_2 \# \sigma_2).$$

Next, let $\alpha = (F, V, f) \in \mathcal{T}^+(M, \nu_\phi)$. Define

$$\psi'_0: \mathcal{T}^+(M, \nu_\phi) \rightarrow \pi_{n+k}T(\nu_\phi) \quad \text{and} \quad \psi_0: \mathcal{S}^+(M, \nu_\phi) \rightarrow \pi_{n+k}T(\nu_\phi)$$

by $\psi'_0(\alpha) = [F]$ and $\psi_0 = \psi'_0|_{\mathcal{S}^+(M, \nu_\phi)}$.

LEMMA 4. ψ'_0 and ψ_0 are well defined. ψ_0 is additive.

Proof. Since ψ'_0, ψ_0 are clearly well defined, it suffices to show that ψ_0 is additive. Let $\alpha_i \in \mathcal{S}_j^+(M, \nu_\phi)$. It follows from the definitions that $\psi_0(\alpha_1 + \alpha_2) = \psi'_0(\alpha_1 \# \alpha_2) = \psi'_0(\alpha_1) + \psi'_0(\alpha_2) = \psi_0(\alpha_1) + \psi_0(\alpha_2)$. Therefore ψ_0 is additive and the lemma is proved.

We conclude this section with the definition of two more maps. First, note that we can identify $H_{n+k}T(\nu_\phi)$ with the integers $\mathbb{Z}$ in such a way that 1 corresponds to $[M] \in H_n M^n$ via the Thom isomorphism. Let

$$\deg: \pi_{n+k}T(\nu_\phi) \rightarrow H_{n+k}T(\nu_\phi)$$

be the Hurewicz homomorphism and define an additive map $\deg: \mathcal{S}^+(M, \nu_\phi) \rightarrow \mathbb{Z}$ by $\deg(F, V, f) = \deg f$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}^+(M, \nu_\phi) & \xrightarrow{\psi_0} & \pi_{n+k}T(\nu_\phi) \\ \deg \searrow & & \swarrow \deg \\ \mathbb{Z} & = & H_{n+k}T(\nu_\phi). \end{array}$$

3. **The group $\theta(M, \nu_\phi)$.** Let $\alpha, \beta \in \mathcal{S}^+(M, \nu_\phi)$ and set $\varepsilon = (T_\phi, M, \text{identity})$. Define $\alpha \sim_e \beta$ if $\alpha + r\varepsilon = \beta + s\varepsilon$ for some nonnegative integers r and s . Obviously, $\sim_e$ is an equivalence relation and we can define $\theta(M, \nu_\phi) = \mathcal{S}^+(M, \nu_\phi) / \sim_e$. We shall write $[\alpha]$ for the equivalence class in $\theta(M, \nu_\phi)$ determined by $\alpha \in \mathcal{S}^+(M, \nu_\phi)$.

Define an operation $+$ in $\theta(M, \nu_\phi)$ by $[\alpha] + [\beta] = [\alpha + \beta]$. It is easy to check that $+$ is a well defined associative and commutative operation. $[\varepsilon]$ acts as a zero element. The projection $\mathcal{S}^+(M, \nu_\phi) \rightarrow \theta(M, \nu_\phi)$ is additive.

At this point the only thing which keeps $\theta(M, \nu_\phi)$ from being a group is that we do not know whether every element has an inverse. We shall return to this question shortly.

Define P_n as usual by

$$\begin{aligned} P_n &= 0 & \text{if } n \text{ is odd,} \\ &= \mathbf{Z}_2 & \text{if } n \equiv 2 \pmod{4}, \\ &= \mathbf{Z} & \text{if } n \equiv 0 \pmod{4}. \end{aligned}$$

We shall think of P_n as the set of framed cobordism classes $[U^n, \mathcal{F}]$ of pairs $(U^n, \mathcal{F})$, where $(U, \partial U) \subseteq (D^{n+l}, S^{n+l-1})$, $l \geq 3$, $\mathcal{F}$ is a framing of the normal disk bundle of U , and ∂U is a homotopy $(n-1)$ -sphere (see [10]).

Suppose that $\alpha_i = (F_i, V_i, f_i) \in \mathcal{S}^+(M, \nu_\phi)$ and that $\psi_0(\alpha_1) = \psi_0(\alpha_2)$. Then there is a map $H: S^{n+k} \times [1, 2] \rightarrow T(\nu_\phi)$ which is transverse regular on M such that $H|_{S^{n+k} \times i} = F_i \times i$. Set $W^{n+1} = H^{-1}(M)$ and $h = H|_W$. Let us try to make W into an h -cobordism just as in Lemma 2. The only problem occurs when we try to do surgery in the middle dimension $[(n+1)/2]$. This difficulty was circumvented in Lemma 2 by our conditions on M and n . But it follows from by now standard techniques that the obstruction to doing this surgery is a well defined element $\gamma(\alpha_1, \alpha_2) \in P_{n+1}$. In fact, we may assume that W is diffeomorphic to $V_2 \times [1, 2] \pm U^{n+1}$, where $\pm$ denotes the boundary connected sum along $V_2 \times 1$ and $(U, \partial U) \subseteq (S^{n+k} \times [1, 2], S^{n+k} \times 1)$ is a $([(n+1)/2] - 1)$ -connected π -manifold with ∂U a homotopy sphere and a framing $\mathcal{F}$ of its normal disk bundle which is induced by H (see [8, p. 20]). Then $\gamma(\alpha_1, \alpha_2) = [U, \mathcal{F}] = \gamma(U, \mathcal{F})$ (see [10, §4.5] for a definition of $\gamma(U, \mathcal{F})$).

Let $(U_1, \mathcal{F}_1)$ be a disjoint copy of $(U, \mathcal{F})$ so that $(U_1, \partial U_1) \subseteq (S^{n+k} \times [1, 2], S^{n+k} \times 1)$. Define $W \pm U_1$ in a natural manner, where we take the boundary connected sum along $V_1 \subseteq \partial W$. It is easy to obtain a map $H_1: S^{n+k} \times [1, 2] \rightarrow T(\nu_\phi)$ which is transverse regular on M such that $H_1^{-1}(M) = W \pm U_1$ and $H_1|_{S^{n+k} \times 2} = F_2 \times 2$. (H_1 is gotten by a construction similar to the one found in the definition of the connected sum of an element of $\mathcal{S}^+(M, \nu_\phi)$ with a framed homotopy sphere.) If we let $F_3 \times 1 = H_3|_{S^{n+k} \times 1}$, $V_3 = F_3^{-1}(M)$, and $f_3 = F_3|_{V_3}$, then we can also assume that $(F_3, V_3, f_3) = \alpha_1 \# -(\partial U_1, \mathcal{F}_1|_{\partial U_1})$. But there is no longer any obstruction to making $W \pm U_1$ into an h -cobordism since $\gamma((U, \mathcal{F}) \pm (U_1, \mathcal{F}_1)) = \gamma(U, \mathcal{F}) - \gamma(U_1, \mathcal{F}_1) = 0$ (see [10, §4.5]).

We summarize this discussion in a lemma. Let $\partial_1: P_{n+1} \rightarrow \theta_f^{n+k,n}$ be given by $\partial_1([U, \mathcal{F}]) = [\partial U, \mathcal{F} | \partial U]$. ∂_1 is a homomorphism. This was essentially proved in [10].

LEMMA 5. *Let $\alpha_i \in \mathcal{S}^+(M, \nu_\varphi)$ and suppose that $\psi_0(\alpha_1) = \psi_0(\alpha_2)$. Then*

$$\alpha_1 \# \partial_1(\gamma(\alpha_1, \alpha_2)) = \alpha_2.$$

Next, let

$$\pi_{n+k}^0 T(\nu_\varphi) = \text{kernel of } \deg: \pi_{n+k} T(\nu_\varphi) \rightarrow H_{n+k} T(\nu_\varphi),$$

and define $\psi: \theta(M, \nu_\varphi) \rightarrow \pi_{n+k}^0 T(\nu_\varphi)$ by $\psi([\alpha]) = \psi_0(\alpha) - (\deg \alpha)\psi_0(\varepsilon)$.

LEMMA 6. *ψ is a well defined additive map.*

Proof. Suppose that $[\alpha] = [\beta]$. Then $\alpha + r\varepsilon = \beta + s\varepsilon$ for some nonnegative integers r and s . Hence $\psi_0(\alpha) + r\psi_0(\varepsilon) = \psi_0(\alpha + r\varepsilon) = \psi_0(\beta + s\varepsilon) = \psi_0(\beta) + s\psi_0(\varepsilon)$ and $\deg \alpha + r = \deg(\alpha + r\varepsilon) = \deg(\beta + s\varepsilon) = \deg \beta + s$. It follows that

$$\begin{aligned} \psi_0(\alpha) - (\deg \alpha)\psi_0(\varepsilon) &= \psi_0(\alpha) - s\psi_0(\varepsilon) + (s - \deg \alpha)\psi_0(\varepsilon) \\ &= \psi_0(\beta) - r\psi_0(\varepsilon) + (s - \deg \alpha)\psi_0(\varepsilon) \\ &= \psi_0(\beta) - (\deg \beta)\psi_0(\varepsilon) + (\deg \beta - \deg \alpha + s - r)\psi_0(\varepsilon) \\ &= \psi_0(\beta) - (\deg \beta)\psi_0(\varepsilon), \end{aligned}$$

and so ψ is well defined. Clearly $\psi(0) = 0$.

Let $[\alpha], [\beta] \in \theta(M, \nu_\varphi)$. Then $\psi([\alpha] + [\beta]) = \psi([\alpha + \beta]) = \psi_0(\alpha + \beta) - (\deg(\alpha + \beta))\psi_0(\varepsilon) = \psi_0(\alpha) + \psi_0(\beta) - (\deg \alpha)\psi_0(\varepsilon) - (\deg \beta)\psi_0(\varepsilon) = \psi([\alpha]) + \psi([\beta])$. Thus ψ is additive and Lemma 6 is proved.

NOTE. If $\theta(M, \nu_\varphi)$ is a group, then ψ is in fact a homomorphism.

Define $\partial_0: P_{n+1} \rightarrow \mathcal{S}^+(M, \nu_\varphi)$ by $\partial_0(\gamma) = \varepsilon \# \partial_1(\gamma)$, and let $\partial: P_{n+1} \rightarrow \theta(M, \nu_\varphi)$ be the composition of ∂_0 followed by the projection of $\mathcal{S}^+(M, \nu_\varphi)$ onto $\theta(M, \nu_\varphi)$.

LEMMA 7. *∂_0 and ∂ are well defined maps. ∂ is a homomorphism.*

Proof. ∂_0 and ∂ are well defined because $\#$ is well defined. Let $\gamma_i \in P_{n+1}$. Then Lemma 3 implies that

$$\begin{aligned} \partial(\gamma_1 + \gamma_2) &= [\partial_0(\gamma_1 + \gamma_2)] = [\varepsilon \# \partial_1(\gamma_1 + \gamma_2)] = [\varepsilon \# \partial_1(\gamma_1 + \gamma_2) + \varepsilon] \\ &= [\varepsilon \# \partial_1(\gamma_1) + \varepsilon \# \partial_1(\gamma_2)] = [\varepsilon \# \partial_1(\gamma_1)] + [\varepsilon \# \partial_1(\gamma_2)] \\ &= \partial(\gamma_1) + \partial(\gamma_2), \end{aligned}$$

i.e., ∂ is a homomorphism.

Finally, let $\mu: \pi_{n+k}^0 T(\nu_\varphi) \rightarrow P_n$ be the well-known mapping which assigns to every $x \in \pi_{n+k}^0 T(\nu_\varphi)$ the surgery obstruction to finding a representative $F: S^{n+k} \rightarrow T(\nu_\varphi)$ for $x + \psi_0(\varepsilon) \in \pi_{n+k} T(\nu_\varphi)$ such that F is transverse regular on M and $F|F^{-1}(M): F^{-1}(M) \rightarrow M$ is a homotopy equivalence. If $n \not\equiv 2 \pmod{4}$, then $\mu = 0$. If $n \equiv 2 \pmod{4}$, then our knowledge of μ is in general limited (see [3]); however,

with our restrictions on the homology of M , μ is a well defined homomorphism.

Consider the diagram

$$\begin{array}{ccccccc}
 & & \mathcal{S}^+(M, \nu_\varphi) & \xrightarrow{\psi_0} & \pi_{n+k}T(\nu_\varphi) & & \\
 & \nearrow \partial_0 & \downarrow & & & & \\
 P_{n+1} & \xrightarrow{\psi} & \theta(M, \nu_\varphi) & \xrightarrow{\psi} & \pi_{n+k}^0T(\nu_\varphi) & \xrightarrow{\mu} & P_n
 \end{array}$$

THEOREM 1. $\theta(M, \nu_\varphi)$ is an abelian group and the bottom row is exact.

Proof. It is easy to see that $\psi_0\partial_0=0$. From this it follows immediately that $\psi\partial=0$, i.e. $(\text{image } \partial) \subseteq (\text{kernel } \psi)$. Let $[\alpha] \in (\text{kernel } \psi)$ and let $\deg \alpha = d$. Then $\psi_0(\alpha) = d\psi_0(\varepsilon) = \psi_0(d\varepsilon)$. By Lemma 5,

$$(d-1)\varepsilon + \partial_0(\gamma(d\varepsilon, \alpha)) = (d-1)\varepsilon + (\varepsilon \# \partial_1(\gamma(d\varepsilon, \alpha))) = d\varepsilon \# \partial_1(\gamma(d\varepsilon, \alpha)) = \alpha.$$

Therefore, $\partial(\gamma(d\varepsilon, \alpha)) = [\alpha]$, and we have shown that $(\text{kernel } \psi) \subseteq (\text{image } \partial)$. This proves that $(\text{image } \partial) = (\text{kernel } \psi)$.

Next, let $x \in \pi_{n+k}^0T(\nu_\varphi)$. Suppose that $\mu(x) = 0$. Then $x + \psi_0(\varepsilon) \in \pi_{n+k}T(\nu_\varphi)$ belongs to the image of ψ_0 , i.e., there is an $\alpha \in \mathcal{S}^+(M, \nu_\varphi)$ with $\psi_0(\alpha) = x + \psi_0(\varepsilon)$. Hence, $\psi([\alpha]) = \psi_0(\alpha) - (\deg \alpha)\psi_0(\varepsilon) = x$, so that $(\text{kernel } \mu) \subseteq (\text{image } \psi)$. Conversely, let $[\alpha] \in \theta(M, \nu_\varphi)$ and set $y = \psi_0(\alpha) - (\deg \alpha - 1)\psi_0(\varepsilon) \in \pi_{n+k}T(\nu_\varphi)$. Then $\deg y = 1$. Assume $n \equiv 2 \pmod{4}$. By definition, $\mu(\psi([\alpha]))$ is the obstruction to finding a representative $F: S^{n+k} \rightarrow T(\nu_\varphi)$ for y such that F is transverse regular on M and $F|F^{-1}(M): F^{-1}(M) \rightarrow M$ is a homotopy equivalence. But using our hypothesis on the homology of M , we see that for this particular y we can start with a representative F such that $H_t(F^{-1}(M)) = 0$ and $H_{t-1}(F^{-1}(M))$ is torsion-free, where $t = [n/2]$. Therefore, we shall never have to do surgery in the middle dimension; and so $\mu(\psi([\alpha])) = 0$. Since $\mu = 0$ when $n \not\equiv 2 \pmod{4}$, we have shown that $(\text{image } \psi) = (\text{kernel } \mu)$.

It remains to prove that $\theta(M, \nu_\varphi)$ is an abelian group. As was observed earlier, it suffices to show that every $a \in \theta(M, \nu_\varphi)$ has an inverse. Choose a $b \in \psi^{-1}(-\psi(a))$. Such a b exists because $(\text{image } \psi) = (\text{kernel } \mu)$ is a subgroup of $\pi_{n+k}^0T(\nu_\varphi)$. Then $\psi(a+b) = \psi(a) + \psi(b) = \psi(a) - \psi(a) = 0$. By exactness we can now find a $\gamma \in P_{n+1}$ such that $\partial(\gamma) = a+b$. Hence $0 = \partial(0) = \partial(\gamma - \gamma) = \partial(\gamma) + \partial(-\gamma) = a + (b + \partial(-\gamma))$. This finishes the proof of Theorem 1.

Let us show that Theorem 1 is a generalization of a well known exact sequence. Suppose that $M^n = S^n$ and $\varphi: S^n \rightarrow S^{n+k}$ is the standard inclusion. Define $\lambda_0: \mathcal{S}^+(S^n, \nu_\varphi) \rightarrow \theta_f^{n+k,n}$ by $\lambda_0((F, V, f)) = [V, \mathcal{F}_F]$, where $\mathcal{F}_F$ is the framing of V induced from the framing of S^n in S^{n+k} . (Note that V is indeed a homotopy sphere.) Clearly λ_0 is well defined. Also, $\lambda_0(\alpha + r\varepsilon) = \lambda_0(\alpha)$ for each $\alpha \in \mathcal{S}^+(S^n, \nu_\varphi)$ because if $\mathcal{F}_0$ is the standard framing of ν_φ , then $(\Sigma^n, \mathcal{F}) \# (S^n, \mathcal{F}_0) = (\Sigma^n, \mathcal{F})$ for every framed homotopy sphere $(\Sigma^n, \mathcal{F})$ in S^{n+k} . (Here $\#$ denotes the operation

of framed connected sum which induces the addition in $\theta_f^{n+k,n}$.) Therefore, λ_0 induces a well-defined map $\lambda: \theta(S^n, \nu_\varphi) \rightarrow \theta_f^{n+k,n}$ given by $\lambda([\alpha]) = \lambda_0(\alpha)$.

If $[\Sigma, \mathcal{F}] \in \theta_f^{n+k,n}$, let $f: \Sigma \rightarrow S^n$ be a homotopy equivalence with $\deg f = 1$. Let $g: \nu_\Sigma \rightarrow \nu_\varphi$ be given by $g(\mathcal{F}(y, u)) = (h(y), u) \in S^n \times D^k \subseteq S^{n+k}$ for $(y, u) \in \Sigma \times D^k$. Then g induces a map $F: S^{n+k} \rightarrow T(\nu_\varphi)$, and $\lambda_0((F, \Sigma, f)) = [\Sigma, \mathcal{F}]$. Thus λ_0 , and hence λ is onto.

Next, let $\alpha, \beta \in \mathcal{S}^+(S^n, \nu_\varphi)$ and suppose that $\lambda_0(\alpha) = \lambda_0(\beta)$. Without loss of generality assume that $\deg \beta - \deg \alpha = r \geq 0$. Then it is not hard to show that $\alpha + r\varepsilon = \beta$. (Observe that if $\alpha_i \in \mathcal{S}^+(S^n, \nu_\varphi)$, then $\alpha_1 + \alpha_2 = \alpha_1 \# \alpha_2$.) It follows that λ is one-to-one. But λ_0 , and hence λ , is additive since the addition in $\mathcal{S}^+(S^n, \nu_\varphi)$ and $\theta_f^{n+k,n}$ both come from a connected sum operation, and so we have proved

LEMMA 8. λ is an isomorphism.

Now ν_φ is trivial, and so $T(\nu_\varphi) = S^{n+k} \vee S^k$ by [11]. $\pi_{n+k}T(\nu_\varphi) = \pi_{n+k}S^{n+k} \oplus \pi_{n+k}S^k$, where $\deg$ maps the first factor isomorphically onto $H_{n+k}T(\nu_\varphi) = \mathbb{Z}$. Therefore, we can identify $\pi_{n+k}^0T(\nu_\varphi)$ in a natural way with $\pi_{n+k}S^k$. (In fact, one can make this identification in the case of any imbedding $\varphi: S^n \rightarrow S^{n+k}$ with $2k \geq n+3$ because ν_φ will then be trivial by [6].) It follows from Lemma 8 that the sequence

$$P_{n+1} \xrightarrow{\partial} \theta(S^n, \nu_\varphi) \xrightarrow{\psi} \pi_{n+k}^0T(\nu_\varphi) \xrightarrow{\mu} P_n$$

can be identified with the Milnor-Kervaire sequence

$$P_{n+1} \xrightarrow{\partial_1} \theta_f^{n+k,n} \xrightarrow{\psi_1} \pi_{n+k}S^k \xrightarrow{\mu_1} P_n,$$

where ψ_1 is defined via the Pontrjagin-Thom construction and μ_1 , like μ , is the usual surgery obstruction.

These observations lead us to another definition of $\theta(M, \nu_\varphi)$. Briefly, it is possible to define $\theta(M, \nu_\varphi)$ to be the h -cobordism classes of (F, V, f) for which f is a homotopy equivalence. The sum of $[(F_1, V_1, f_1)]$ and $[(F_2, V_2, f_2)]$ is defined to be the class of that triple (F_3, V_3, f_3) which is obtained from $(F_1, V_1, f_1) \# (F_2, V_2, f_2) \# (T_\varphi, -M, \text{identity})$ by surgery for which f_3 is a homotopy equivalence. In order that this addition is well defined and that we get a group we have to be able to do the necessary surgery. This is why we need some conditions on n and the homology of M . Our condition, that either $n \equiv 0 \pmod{4}$ or $H_t M = 0$ and $H_{t-1} M$ is torsion-free, can probably be weakened. The reason that we did not give this straightforward definition of $\theta(M, \nu_\varphi)$ at the beginning and proceeded in a roundabout fashion to define $\mathcal{S}^+(M, \nu_\varphi)$ first is that $\mathcal{S}^+(M, \nu_\varphi)$ and ψ_0 are interesting in their own right (see the next section).

Next, observe that the inclusion $i: S^{n+k} \subseteq S^{n+k+1}$ induces natural maps $\mathcal{S}_0^+(M, \nu_\varphi) \rightarrow \mathcal{S}_0^+(M, \nu_{i\varphi})$, $\mathcal{S}_0^+(M, \nu_\varphi) \rightarrow \mathcal{S}_0^+(M, \nu_{i\varphi})$, $\mathcal{S}^+(M, \nu_\varphi) \rightarrow \mathcal{S}^+(M, \nu_{i\varphi})$, $\mathcal{S}^+(M, \nu_\varphi) \rightarrow \mathcal{S}^+(M, \nu_{i\varphi})$, and $\theta(M, \nu_\varphi) \rightarrow \theta(M, \nu_{i\varphi})$. These "suspension" maps

are clearly additive and will all be denoted by s . Let θ^k be the trivial k -disk bundle over M . Then $\nu_{i\varphi} = \nu_\varphi \oplus \theta^1$, and the following diagrams commute:

$$\begin{array}{ccccc} P_{n+1} & \xrightarrow{\partial_0} & \mathcal{S}^+(M, \nu_\varphi \oplus \theta^1) = \mathcal{S}^+(M, \nu_{i\varphi}) & \xrightarrow{\psi_0} & \pi_{n+k+1} T(\nu_{i\varphi}) = \pi_{n+k+1} S(T(\nu_\varphi)) \\ & \searrow \partial_0 & \uparrow s & & \uparrow s_\# \\ & & \mathcal{S}^+(M, \nu_\varphi) & \xrightarrow{\psi_0} & \pi_{n+k} T(\nu_\varphi) \end{array}$$

$$\begin{array}{ccccc} P_{n+1} & \xrightarrow{\partial} & \theta(M, \nu_\varphi \oplus \theta^1) = \theta(M, \nu_{i\varphi}) & \xrightarrow{\psi} & \pi_{n+k+1}^0 T(\nu_{i\varphi}) = \pi_{n+k+1}^0 S(T(\nu_\varphi)) \\ & \searrow \partial & \uparrow s & & \uparrow s_\# \\ & & \theta(M, \nu_\varphi) & \xrightarrow{\psi} & \pi_{n+k}^0 T(\nu_\varphi) \end{array}$$

($S(T(\nu_\varphi))$ is the reduced suspension of $T(\nu_\varphi)$ and $s_\#$ is the usual suspension map on homotopy.)

Finally, define

$$\theta(M) = \lim_t \theta(M, \nu_\varphi \oplus \theta^t).$$

It follows from the above remarks and Theorem 2 that $\theta(M)$ is a well-defined abelian group. In fact, $\theta(M)$ is isomorphic to $\theta(M, \nu_\varphi \oplus \theta^k)$ whenever $k+t \geq n+3$. $\theta(M)$ is the group of manifolds which are "framed" homotopy equivalent to M .

4. Conclusion. We would like to conclude with some unanswered questions which arise naturally in the context of this paper:

1. $\theta(M, \nu_\varphi)$ corresponds to $\theta_f^{n+k,n}$. A natural analogue of $\theta^{n+k,n}$, the h -cobordism classes of (unframed) homotopy n -spheres in S^{n+k} , would seem to be the set, $\mathcal{S}_k(M)$, of h -cobordism classes of homotopy smoothings of M which are imbedded in S^{n+k} . The set $\mathcal{S}_k(M)$, for large k , was considered in [12] and fit into an exact sequence. There is a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{S}_k(M) & \longrightarrow & [M, F_{k-1}/O_{k-1}] \\ & \nearrow & \uparrow & & \uparrow \\ P_{n+1} & & \theta(M, \nu_\varphi) & \xrightarrow{\psi} & \pi_{n+k}^0 T(\nu_\varphi) \\ & \searrow \partial & & & \nearrow \mu \\ & & & & P_n \end{array}$$

The top row was defined in [12] for large k and shown to be exact. It reduces to the Milnor-Kervaire sequence when $M^n = S^n$. Can one generalize other sequences in [5] and [10]?

2. When is an element $x \in \pi_{n+k} T(\nu_\varphi)$ in the image of ψ_0 ? This question is partially answered in [2] and involves the study of Browder-Novikov theory for maps of degree $d > 1$. Note how much easier it is to determine the image of ψ .

3. Let $(F, V, f) \in \mathcal{S}^+(M, \nu_\phi)$. Is V homotopy equivalent to M ? (The homotopy equivalence may have no relation to f .) This question in conjunction with question 2 has bearing on the problem of whether manifolds imbed in the metastable range. Since $f_*(f^*(a) \cap [V]) = (\deg f)(a \cap [M])$ for $a \in H^i M$, it follows that $f_*: H_i V \rightarrow H_i M$ is an isomorphism for $0 \leq i < n$ whenever $H_i M$ is finite and the order of $H_i M$ is relatively prime to $\deg f$ for $0 < i < n$. This suggests a somewhat weaker question: Is V homotopy equivalent to M if $f_*: H_i V \rightarrow H_i M$ is an isomorphism for $0 \leq i < n$?

4. When does a manifold M^n admit a map $f: M \rightarrow M$ of degree $d > 0$? Are there some more or less simple conditions on the homology or homotopy groups which will guarantee the existence of f ? This problem fits into our context because it is related to the previous questions about $\mathcal{S}^+(M, \nu_\phi)$ and $\pi_{n+k} T(\nu_\phi)$.

5. Would it be useful to study manifolds M^n which have the property that $d\varepsilon = (F_d, \varphi(M^n), f_d)$? For example, $M^n = S^n$ has this property. Do products of spheres $S^i \times S^{n-i}$ behave similarly?

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