DENSE SIGMA-COMPACT SUBSETS OF INFINITE-DIMENSIONAL MANIFOLDS⁽¹⁾

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Abstract. In this paper four classes of separable metric infinite-dimensional manifolds are studied; those which are locally the countable infinite product of lines, those which are locally open subsets of the Hilbert cube, and those which are locally one of two dense sigma-compact subsets of the Hilbert cube. A number of homeomorphism, product, characterization, and embedding theorems are obtained concerning these manifolds.

1. Introduction. Let s denote the countable infinite product of open intervals and regard the Hilbert cube I^∞ as the product of the closures of those intervals. A *Fréchet manifold* (or *F-manifold*) is a separable metric space having an open cover by sets each homeomorphic to an open subset of s . This definition is justified since all separable infinite-dimensional Fréchet spaces are known to be homeomorphic to s (see [7] for references). A *Hilbert cube manifold* (or *Q-manifold*) is a separable metric space having an open cover by sets each homeomorphic to an open subset of I^∞ .

Let X be a separable metric space. A closed subset K of X is said to have *Property Z* in X (more briefly we say that K is a *Z-set* in X) provided that for any nonnull homotopically trivial open subset U of X , $U \setminus K$ is nonnull and homotopically trivial. Following Anderson [6] we say that a subset M of X has the (*finite-dimensional*) *compact absorption property*, or (f-d) cap, in X if (1) $M = \bigcup_{n=1}^\infty M_n$, where each M_n is a (finite-dimensional) compact Z-set in X such that $M_n \subset M_{n+1}$, and (2) for each $\varepsilon > 0$, each integer $m > 0$, and each (finite-dimensional) compact subset K of X , there is an integer $n > 0$ and an embedding $h: K \rightarrow M_n$ such that $h|K \cap M_m = \text{id}$ and $d(h, \text{id}) < \varepsilon$. We use the convention that (f-d) cap represents two alternatives in an obvious way, one being cap and the other being f-d cap. In this sense we say that M is an (f-d) cap-set for X provided that M has the cap (or f-d cap) in X .

In [6] R. D. Anderson has established the following theorem: Let X be I^∞ or s and let M, N be subsets of X , both having the (f-d) cap in X . Then there is a

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homeomorphism of X onto itself taking M onto N . We remark that Bessaga and Pełczyński [12] and Toruńczyk [17] have also established this result by use of a device resembling the (f-d) cap.

The question of the equivalence of sets which have the (f-d) cap in an F -manifold was raised at a conference on infinite-dimensional topology held at Cornell University, January 5–7, 1969, and it appears as Problem 29 in the report of that meeting, *Problems in the topology of infinite-dimensional manifolds*. In §6 we give an affirmative answer to this question.

We will regard the Hilbert cube as a canonical compactification of s in which $I^\infty = \prod_{i=1}^\infty I_i$ and $s = \prod_{i=1}^\infty I_i^0$, where for each $i > 0$ we have $I_i = [-1, 1]$ and $I_i^0 = (-1, 1)$. The definition of the (f-d) cap characterizes two different types of subsets of I^∞ . Let σ denote the set of all points in s having at most finitely many nonzero coordinates and let Σ denote the set of all points in s having at most finitely many coordinates not in $[-\frac{1}{2}, \frac{1}{2}]$.

We remark that in real Hilbert space l^2 there are canonical versions of σ and Σ . The space σ is homeomorphic to a vector subspace of l^2 , namely, the linear span of the usual orthonormal basis. The space Σ is homeomorphic to a vector subspace of l^2 , namely, the linear span of the usual copy of the Hilbert cube there. It is shown in [6] that Σ has the cap in I^∞ and σ has the f-d cap in I^∞ .

We will be concerned with manifolds modeled on Σ and σ . A Σ -manifold (or σ -manifold) is a separable metric space having an open cover by sets each homeomorphic to an open subset of Σ (or σ). In this paper we obtain a number of homeomorphism, characterization, and embedding theorems concerning F , Q , Σ , and σ -manifolds. We summarize these results in the next section.

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2. Summary of principal results on F , Q , Σ , and σ -manifolds. In this section we list some principal results on F and Q -manifolds along with the principal results of this paper. The references refer to results of this paper unless other authorship is indicated.

I. Characterization of manifolds by homotopy type.

THEOREM 2.1 (HENDERSON [14]). *If X and Y are F -manifolds of the same homotopy type, then they are homeomorphic.*

THEOREM 2.2 (COROLLARY 10.2). *If X and Y are both σ -manifolds or both Σ -manifolds and are of the same homotopy type, then they are homeomorphic.*

CONJECTURE 2.1. *If X and Y are compact Q -manifolds of the same homotopy type, then they are homeomorphic.*

II. Open embedding theorems.

THEOREM 2.3 (HENDERSON [14]). *If X is an F -manifold, then there is an open embedding $h: X \rightarrow s$.*

THEOREM 2.4 (THEOREM 10.3). *If X is a σ (or Σ)-manifold, then X can be embedded as an open subset of σ (or Σ).*

III. Product theorems.

THEOREM 2.5 (WEST [18]). *If K is any countable locally-finite simplicial complex, then $|K| \times s$ is an F -manifold and $|K| \times I^\infty$ is a Q -manifold.*

THEOREM 2.6 (COROLLARY 8.4). *If K is any countable locally-finite simplicial complex, then $|K| \times \sigma$ is a σ -manifold and $|K| \times \Sigma$ is a Σ -manifold.*

THEOREM 2.7 (COROLLARY 10.3). *If X is a σ -manifold and Y is a Σ -manifold of the same homotopy type, then $X \times I^\infty$ and Y are homeomorphic.*

IV. Factor theorems.

THEOREM 2.8 (HENDERSON [14]). *If X is an F -manifold, then there is a countable locally-finite simplicial complex K such that X and $|K| \times s$ are homeomorphic.*

THEOREM 2.9 (COROLLARY 10.4). *If X is a σ -manifold (or Σ -manifold), then there is a countable locally-finite simplicial complex K such that X is homeomorphic to $|K| \times \sigma$ (or $|K| \times \Sigma$).*

CONJECTURE 2.2. *If X is any Q -manifold, then there is a countable locally-finite simplicial complex K such that X and $|K| \times I^\infty$ are homeomorphic.*

THEOREM 2.10 (ANDERSON AND SCHORI [9]). *If X is an F -manifold, then X , $X \times s$, and $X \times I^\infty$ are all homeomorphic.*

THEOREM 2.11 (ANDERSON AND SCHORI [9]). *If X is a Q -manifold, then X and $X \times I^\infty$ are homeomorphic.*

THEOREM 2.12 (THEOREM 8.3 AND THEOREM 10.2). *If X is a σ -manifold and I^n is any n -cell, then X , $X \times \sigma$, and $X \times I^n$ are all homeomorphic.*

THEOREM 2.13 (COROLLARY 8.5 AND THEOREM 10.2). *If X is any Σ -manifold, then X , $X \times \Sigma$, and $X \times I^\infty$ are all homeomorphic.*

V. Relationships between F , Q , Σ , and σ -manifolds.

THEOREM 2.14 (THEOREM 10.2 AND COROLLARY 10.1). *If X is a σ (or Σ)-manifold, then X can be embedded as an f -d cap (or cap)-set for an F -manifold and also for a Q -manifold.*

THEOREM 2.15 (ANDERSON [6]). *If X is I^∞ or s and M , N are (f -d) cap-sets for X , then there is a homeomorphism of X onto itself taking M onto N .*

THEOREM 2.16 (LEMMA 5.4 AND THEOREM 2.15). *If X is an F or Q -manifold and M is an f -d cap (or cap)-set for X , then M is a σ (or Σ)-manifold.*

THEOREM 2.17 (THEOREMS 6.1 AND 6.2). *If X is an F or Q -manifold and M , N are (f -d) cap-sets for X , then there is a homeomorphism of X onto itself taking M onto N .*

THEOREM 2.18 (ANDERSON [6]). *If M is an $(f-d)$ cap-set for I^∞ , then $I^\infty \setminus M$ is homeomorphic to s .*

THEOREM 2.19 (COROLLARY 8.3). *If X is any Q -manifold and M is an $(f-d)$ cap-set for X , then $X \setminus M$ is an F -manifold which is of the same homotopy type as X .*

VI. *Subsets and supersets of F , Q , Σ , and σ -manifolds.* If X is any space and $\mathcal{U}$ is any cover of X , then a function $f: X \rightarrow X$ is said to be *limited by $\mathcal{U}$* provided that for each $x \in X$, there is a member of $\mathcal{U}$ containing both x and $f(x)$.

A subset K of a space X is said to be *strongly negligible* provided that given any open cover $\mathcal{U}$ of X , there is a homeomorphism of X onto $X \setminus K$ which is limited by $\mathcal{U}$.

THEOREM 2.20 (ANDERSON, HENDERSON, AND WEST [10]). *A necessary and sufficient condition that a closed subset K of an F -manifold have Property Z is that K be strongly negligible.*

THEOREM 2.21 (THEOREMS 6.9 AND 10.2). *A necessary and sufficient condition that a closed subset K of a σ or Σ -manifold have Property Z is that K be strongly negligible.*

THEOREM 2.22 (THEOREM 6.7). *Let M be an $(f-d)$ cap-set for an F or Q -manifold X and let K be a Z -set in X . Then $M \setminus K$ is an $(f-d)$ cap-set for X .*

THEOREM 2.23 (THEOREM 6.6). *Let M be an $(f-d)$ cap-set for an F or Q -manifold X and let K be a countable union of (finite-dimensional) compact Z -sets in X . Then $M \cup K$ is an $(f-d)$ cap-set for X .*

VII. *Homeomorphism extension theorems.* If $\mathcal{U}$ is any cover of a set X , then $\text{St}(\mathcal{U})$ is defined (as usual) to be the star of $\mathcal{U}$ and for each $n > 0$ we define $\text{St}^n(\mathcal{U})$ to be the n th star of $\mathcal{U}$.

THEOREM 2.24 (ANDERSON AND McCHAREN [8]). *Let X be an F -manifold, let K_1, K_2 be Z -sets in X , let $\mathcal{U}$ be an open cover of X , and let h be a homeomorphism of K_1 onto K_2 for which there is a homotopy $H: K_1 \times I \rightarrow X$ such that for each $x \in K_1$, $H(x, 0) = h(x)$, $H(x, 1) = x$, and $H(\{x\} \times I)$ is contained in some member of $\mathcal{U}$. Then h can be extended to a homeomorphism of X onto itself which is limited by $\text{St}^4(\mathcal{U})$.*

THEOREM 2.25 (THEOREM 11.1). *Let X be a σ or Σ -manifold, let K_1, K_2 be Z -sets in X , let $\mathcal{U}$ be an open cover of X , and let h be a homeomorphism of K_1 onto K_2 for which there is a homotopy $H: K_1 \times I \rightarrow X$ such that for each $x \in K_1$, $H(x, 0) = h(x)$, $H(x, 1) = x$, and $H(\{x\} \times I)$ is contained in some member of $\mathcal{U}$. Then h can be extended to a homeomorphism of X onto itself which is limited by $\text{St}^{28}(\mathcal{U})$.*

VIII. *Complete extensions of Σ and σ -manifolds.*

THEOREM 2.26 (LEMMA 10.1 AND THEOREM 10.2). *Let X be a σ (or Σ)-manifold and let Y be a complete separable metric space containing X . Then there is an F -manifold Z such that $X \subset Z \subset Y$ and X is an $f-d$ cap (or cap)-set for Z .*

IX. *Infinite deficiency.* A subset K of I^∞ is said to have *infinite deficiency* provided that for each of infinitely many different coordinate directions, K projects onto a single interior point of $(-1, 1)$.

THEOREM 2.27 (ANDERSON [3]). *Let X be I^∞ or s and let K be a closed subset of X . A necessary and sufficient condition that K be a Z -set in X is that there exists a homeomorphism of X onto itself taking K onto a set having infinite deficiency.*

THEOREM 2.28 (CHAPMAN [13]). *Let X be an F -manifold and let K be a closed subset of X . A necessary and sufficient condition that K be a Z -set in X is that there exists a homeomorphism h of X onto $X \times s$ such that $\pi_s \circ h(K)$ has infinite deficiency.*

We remark that a separate proof of Theorem 2.28 is given in Corollary 7.1 of this paper.

THEOREM 2.29 (THEOREM 7.2). *Let X be a Q -manifold and let K be a closed subset of X . A necessary and sufficient condition that K be a Z -set in X is that there exists a homeomorphism h of X onto $X \times I^\infty$ such that $\pi_{I^\infty} \circ h(K)$ has infinite deficiency.*

3. **Preliminaries.** The metric we use for I^∞ and s is given by

$$d((x_i), (y_i)) = \left(\sum_{i=1}^{\infty} 2^{-i}(x_i - y_i)^2 \right)^{1/2},$$

where $(x_i), (y_i) \in I^\infty$. Whenever no confusion is possible we will use d to denote the metric of any space under consideration.

Let Z denote the set of positive integers and for each $i \in Z$ let τ_i be the projection of I^∞ onto I_i . For each $\alpha \subset Z$ define $I^\alpha = \prod_{i \in \alpha} I_i$, $s^\alpha = \prod_{i \in \alpha} I_i^0$, and let τ_α be the projection of I^∞ onto I^α . For each $i \in Z$ let $W_i^+ = \tau_i^{-1}(1)$, $W_i^- = \tau_i^{-1}(-1)$, and $W_i = W_i^+ \cup W_i^-$. We call W_i^+ and W_i^- the *endslices* of I^∞ in the i -direction.

A subset of I^∞ of the form $\prod_{i=1}^{\infty} J_i$ is called a *basic closed set* in I^∞ provided that J_i is a closed subinterval of I_i , for each $i > 0$, and $J_i = I_i$ for all but finitely many i . The interior in I^∞ of a basic closed set is called a *basic open set* in I^∞ . A *basic open set* in s is the intersection of a basic open set in I^∞ with s . We call s the *pseudo-interior* of I^∞ and $B(I^\infty) = I^\infty \setminus s$ the *pseudo-boundary* of I^∞ .

A *core* is a set $C = \prod_{i=1}^{\infty} J_i$, where for each $i > 0$, J_i is a closed interval contained in I_i^0 . A *basic core set structured over a core* $C = \prod_{i=1}^{\infty} J_i$ is defined as the set of all points $(x_i) \in s$ such that $x_i \in J_i$, for all but finitely many i . We know from [4] that each basic core set is an apparent boundary of I^∞ and from [6] that each apparent boundary of I^∞ is a cap-set for I^∞ , where $M \subset I^\infty$ is an *apparent boundary* if there is a homeomorphism $h: I^\infty \rightarrow I^\infty$ satisfying $h(M) = B(I^\infty)$.

A homeomorphism h of I^∞ onto itself is said to be a β^* -homeomorphism provided that $h(s) = s$.

If $\{f_i\}_{i=1}^{\infty}$ is a sequence of homeomorphisms of a space X onto itself for which the sequence $\{f_i \circ f_{i-1} \circ \cdots \circ f_1\}$ converges pointwise to a homeomorphism f of X onto itself, then we call f the *infinite left product* of $\{f_i\}_{i=1}^{\infty}$ and write $f = L \prod_{i=1}^{\infty} f_i$.

We list below four convergence procedures that we will need to insure the existence of an infinite left product of homeomorphisms. The first of these is Lemma 2.1 of [3] and the second is Theorem 2 of [10]. The third and fourth are easy consequences of the apparatus used in [10] to establish the second.

CONVERGENCE PROCEDURE A. *For each homeomorphism g of a compact metric space X onto itself and each $\varepsilon > 0$ let*

$$\eta(g, \varepsilon) = \text{g.l.b. } \{d(g(x), g(y)) \mid d(x, y) \geq \varepsilon\}.$$

If $\{f_i\}_{i=1}^\infty$ is a sequence of homeomorphisms of X onto itself such that

$$d(f_i, \text{id}) < \min((3^{-i}), (3^{-i}) \cdot \eta(f_{i-1} \circ \cdots \circ f_1, 2^{-i})),$$

for all $i > 1$, then $f = L \prod_{i=1}^\infty f_i$ exists.

CONVERGENCE PROCEDURE B. *Let $\mathcal{U}$ be a countable star-finite open cover of any space X . (By star-finite cover we mean a cover such that the closure of each member of the cover intersects the closure of only finitely many other members of the cover.) There exists an ordering $\{U_i\}_{i=1}^\infty$ of the elements of $\mathcal{U}$ such that for any sequence $\{f_i\}_{i=1}^\infty$ of homeomorphisms of X onto itself, where f_i is the identity on $X \setminus U_i$ for all $i > 0$, $f = L \prod_{i=1}^\infty f_i$ exists. Moreover, we can assign a positive integer n_i to each U_i , independent of the choice of such $\{f_i\}_{i=1}^\infty$, such that $n_i \leq n_{i+1}$ and*

$$f(U_i) = (f_{n_i} \circ f_{n_i-1} \circ \cdots \circ f_1)(U_i),$$

for all $i > 0$.

CONVERGENCE PROCEDURE C. *Let $\mathcal{U}$ be a countable star-finite open cover of any space X and let $\{U_i\}_{i=1}^\infty, \{n_i\}_{i=1}^\infty$ be as in Convergence Procedure B. If Y is any space and $\{f_i\}_{i=1}^\infty$ is any sequence of homeomorphisms of $X \times Y$ onto itself such that f_i is the identity on $(X \setminus U_i) \times Y$, for all $i > 0$, then $f = L \prod_{i=1}^\infty f_i$ exists. Moreover, we have*

$$f(U_i \times Y) = (f_{n_i} \circ \cdots \circ f_1)(U_i \times Y),$$

for all $i > 0$.

CONVERGENCE PROCEDURE D. *Let $\mathcal{U}$ be a countable star-finite open cover of any space X and let $\{U_i\}_{i=1}^\infty$ be as in Convergence Procedure B. If $\{f_i\}_{i=1}^\infty$ is any sequence of homeomorphisms of X onto itself such that $f_1|_{X \setminus U_1} = \text{id}$ and $f_{i+1}|_{X \setminus f_i \circ \cdots \circ f_1(U_{i+1})} = \text{id}$, for each $i > 0$, then $f = L \prod_{i=1}^\infty f_i$ exists.*

4. Some basic results concerning (f-d) cap-sets. The following result can be found in [6].

LEMMA 4.1. *Let X be a metric space, let M be an (f-d) cap-set for X , and let f be a homeomorphism of X onto a metric space Y . Then $f(M)$ is an (f-d) cap-set for Y .*

The next result shows that (f-d) cap-sets in I^∞ are “maximal.” We will use this result to prove a corresponding property for F and Q -manifolds in §6. We will give no proof, since it is essentially Theorem 1 of [17]. We remark that Lemma 4.2 follows from Theorem II of [4] in the case that M is a cap-set for I^∞ .

LEMMA 4.2. *Let $M \subset N \subset I^\infty$, where M is an $(f-d)$ cap-set for I^∞ and N is a countable union of (finite-dimensional) Z -sets in I^∞ . Then N is an $(f-d)$ cap-set for I^∞ .*

Our next lemma is a generalization of Anderson's results on the equivalence of $(f-d)$ cap-sets in s and I^∞ [6]. We only sketch the proof.

LEMMA 4.3. *Let M and N be $(f-d)$ cap-sets for I^∞ , let $K \subset I^\infty \setminus (M \cup N)$ be a Z -set in I^∞ , and let $\varepsilon > 0$ be given. Then there is a homeomorphism h of I^∞ onto itself such that $h(M) = N$, $h|_K = \text{id}$, and $d(h, \text{id}) < \varepsilon$. Moreover, if $M \cup N \subset s$, then we can additionally require that h be a β^* -homeomorphism.*

Proof. We only treat the case in which $M \cup N \subset s$ to obtain the β^* -homeomorphism h . Write $M = \bigcup_{n=1}^\infty M_n$ and $N = \bigcup_{n=1}^\infty N_n$ so as to satisfy the definition. It is clear that there is an integer $n_1 > 0$ and an embedding $f_1: M_1 \rightarrow N_{n_1}$ such that $d(f_1, \text{id}) < \varepsilon/2$. Using Lemma 3.1 of [13], we can extend f_1 to a β^* -homeomorphism h_1 such that $h_1|_K = \text{id}$ and $d(h_1, \text{id}) < \varepsilon/2$.

Using Convergence Procedure A let $\delta = \min((3^{-2}), (3^{-2})\eta(h_1, 2^{-2}))$ and use Lemma 4.1 to get an integer $n_2 > 1$ and an embedding $f_2: N_{n_1} \rightarrow h_1(M_{n_2})$ such that $f_2|_{h_1(M_1)} = \text{id}$ and $d(f_2, \text{id}) < \min(\varepsilon/4, \delta)$. Then we can extend f_2 to a β^* -homeomorphism g_2 such that $g_2|(K \cup W_1 \cup h_1(W_1)) = \text{id}$ and $d(g_2, \text{id}) < \min(\varepsilon/4, \delta)$. We then put $h_2 = g_2^{-1}$ to complete the second stage of the construction.

It is clear that by induction we can continue this process to obtain a sequence $\{h_i\}_{i=1}^\infty$ of β^* -homeomorphisms and a sequence $\{n_i\}_{i=0}^\infty$ of integers (where $n_0 = 1$) such that the following conditions are met.

- (1) $n_0 < n_2 < n_4 < \dots$ and $n_1 < n_3 < n_5 < \dots$,
- (2) for each $i > 1$, h_i is the identity on $K \cup (\bigcup_{j=1}^{i-1} W_j) \cup (h_{i-1} \circ \dots \circ h_1)(\bigcup_{j=1}^{i-1} W_j)$,
- (3) $(h_i \circ \dots \circ h_1)(M_{n_{i-1}}) \subset N_{n_i}$, for i odd,
- (4) $(h_i \circ \dots \circ h_1)(M_{n_i}) \supset N_{n_{i-1}}$, for i even,
- (5) h_i is the identity on $N_{n_{i-2}}$, for $i \geq 3$ and odd,
- (6) h_i is the identity on $h_{i-1} \circ \dots \circ h_1(M_{n_{i-2}})$, for i even,
- (7) $d(h_1, \text{id}) < \varepsilon/2$ and $d(h_i, \text{id}) < \min(\varepsilon/2^i, 3^{-i}, (3^{-i})\eta(h_{i-1} \circ \dots \circ h_1, 2^{-i}))$, for $i > 1$.

Using Convergence Procedure A we find that $h = L \prod_{i=1}^\infty h_i$ is a β^* -homeomorphism which satisfies the required properties. $\square$

A special case of the next lemma is known in a more general setting (see Corollary 2 of [17]).

LEMMA 4.4. *Let M be an $(f-d)$ cap-set for I^∞ and let K be a Z -set in I^∞ . Then $M \setminus K$ is an $(f-d)$ cap-set for I^∞ .*

Proof. Let $M = \bigcup_{n=1}^\infty M_n$ be as in the definition and for each integer $n > 0$ let $N_n = M_n \cap \{x \in I^\infty \mid d(x, K) \geq 1/n\}$. It is then clear that $M \setminus K = \bigcup_{n=1}^\infty N_n$. It follows from Theorems 8.1 and 9.1 of [3] that Property Z and infinite deficiency are equivalent in I^∞ modulo ambient homeomorphisms. This implies that each N_n is a Z -set in I^∞ .

Now let A be a (finite-dimensional) compact subset of I^∞ , let $m > 0$ be an integer, and let $\varepsilon > 0$ be given. It follows from Theorem II of [4] that there is a cap-set M' for I^∞ in $I^\infty \setminus K$. Let $M' = \bigcup_{n=1}^\infty M'_n$ be a representation as in the definition. Without loss of generality we may assume that A is a Z -set. Let $A \setminus N_m = \bigcup_{n=1}^\infty C_n$, where each C_n is a compact set satisfying $C_n \subset C_{n+1}$. Let $n_1 > 0$ be chosen large enough so that there is an embedding $f_1: C_1 \rightarrow M'_{n_1}$ which satisfies $f_1(C_1) \cap N_m = \emptyset$ and $d(f_1, \text{id}) < \varepsilon/4$. Using Lemma 3.1 of [13] we can extend f_1 to a homeomorphism $f'_1: I^\infty \rightarrow I^\infty$ which satisfies $f'_1|_{N_m} = \text{id}$ and $d(f'_1, \text{id}) < \varepsilon/4$.

Let $\varepsilon' = \min(\varepsilon/8, 3^{-2}, (3^{-2})\eta(f'_1, 2^{-2}))$ and choose $n_2 > n_1$ large enough so that there is an embedding $f_2: f'_1(C_2) \rightarrow M'_{n_2}$ which satisfies $f_2(f'_1(C_2)) \cap N_m = \emptyset$, $f_2|_{f'_1(C_1)} = \text{id}$, and $d(f_2, \text{id}) < \varepsilon'$. Once more use Lemma 3.1 of [13] to extend f_2 to a homeomorphism $f'_2: I^\infty \rightarrow I^\infty$ which satisfies $f'_2|_{N_m} = \text{id}$ and $d(f'_2, \text{id}) < \varepsilon'$.

Using Convergence Procedure A we can continue this process in an obvious fashion to obtain a sequence $\{f'_i\}_{i=1}^\infty$ of homeomorphisms of I^∞ onto itself for which $f = L \prod_{i=1}^\infty f'_i$ exists and satisfies $f(A \setminus N_m) \subset M'$, $f|_{N_m} = \text{id}$, and $d(f, \text{id}) < \varepsilon/2$.

Let $\delta = d(f(A), K)$, which is a positive number, and let $g: f(A) \rightarrow M_n$ be an embedding, for some $n > 0$, such that $g|_{A \cap N_m} = \text{id}$ and $d(g, \text{id}) < \min(\varepsilon/2, \delta)$. Then $g \circ f: A \rightarrow M_n$ is an embedding for which $g \circ f(A) \cap K = \emptyset$ and $d(g \circ f, \text{id}) < \varepsilon$. It is clear that there is an $n' \geq n$ such that $g \circ f(A) \subset N_{n'}$, which completes the verification of the definition. $\square$

COROLLARY 4.1. *Let M be an (f-d) cap-set for s and let K be a Z -set in s . Then $M \setminus K$ is an (f-d) cap-set for s .*

Proof. We know from Theorem 8.2 of [3] that $\bar{K}$, the closure of K in I^∞ , is a Z -set in I^∞ . Now any (f-d) cap-set for s is an (f-d) cap-set for I^∞ , and any subset of s which is an (f-d) cap-set for I^∞ is an (f-d) cap-set for s . Then M is an (f-d) cap-set for I^∞ and from Lemma 4.4 we find that $M \setminus \bar{K} = M \setminus K$ is an (f-d) cap-set for s . $\square$

5. Property Z in manifolds. In this section we prove some results concerning Property Z in manifolds that will be useful in later sections. We also exhibit a technique (Theorem 5.1) by which we can relate F and Q -manifolds.

In Lemma 1 of [10] some techniques are given for establishing Property Z in normal, locally homotopically trivial spaces. Lemma 5.1 is essentially a restatement of this result and Lemma 5.2 is a consequence of the proof that is given there.

LEMMA 5.1. *Let X be normal and locally homotopically trivial. If A is a closed subset of X such that each point of A lies in an open set U for which $A \cap U$ has Property Z in U , then A has Property Z in X .*

LEMMA 5.2. *Let X be normal and locally homotopically trivial and let A be a closed subset of X . If, for each point $p \in A$ and each open set U containing p , there is a homotopically trivial open set V containing p such that $V \subset U$, $V \setminus A \neq \emptyset$, and $V \setminus A$ is homotopically trivial, then A has Property Z in X .*

In Lemma 2 of [10] it is shown that if F is any Z -set in an F -manifold X , then any closed subset of F is also a Z -set in X . Lemma 5.3 establishes a similar property for Q -manifolds. We remark that for F -manifolds all compact subsets are Z -sets. This follows from Theorem 2.1 and the fact that any compact subset of s is a Z -set (see Theorem 3.5 of [2] and Theorem 9.1 of [3]).

LEMMA 5.3. *Let F be a Z -set in any Q -manifold X . If A is any closed subset of F , then A is also a Z -set in X .*

Proof. Using Lemma 5.1 let U be any open subset of X which is homeomorphic to a basic open subset of I^∞ . Then there is a homeomorphism h of U onto $I^\infty \setminus W$, where W is a finite union of endslices of I^∞ . Since Property Z obviously carries over to open subsets we find that $h(F \cap U)$ is a Z -set in $I^\infty \setminus W$.

Define $K = h(F \cap U) \cup W$, which is a compact subset of I^∞ . Let V be any nonnull, homotopically trivial open subset of I^∞ and note that $V \setminus K = (V \setminus W) \setminus h(F \cap U)$. Since W obviously has Property Z in I^∞ it follows that $V \setminus W$ is a nonnull, homotopically trivial open subset of $I^\infty \setminus W$, hence $V \setminus K$ is a nonnull, homotopically trivial open set. This means that K is a Z -set in I^∞ .

Using Theorem 8.1 of [3] there is a homeomorphism g of I^∞ onto itself such that $g(K)$ has infinite deficiency. Using techniques similar to those of Theorem 9.1 of [3] it now easily follows that $g \circ h(A \cap U)$ is a Z -set in $g \circ h(U)$. $\square$

The following result was pointed out to the author by William Barit.

LEMMA 5.4. *Let X be an F or Q -manifold and let M be an $(f-d)$ cap-set for X . If U is an open subset of X , then $M \cap U$ is an $(f-d)$ cap-set for U .*

Proof. Let $M = \bigcup_{n=1}^\infty M_n$ be a representation as given in the definition and for each $n > 0$, let $A_n = \{x \in X \mid d(x, X \setminus U) \geq 1/n\}$. If we put $N = M \cap U$ and $N_n = M_n \cap A_n$, for each $n > 0$, then $N = \bigcup_{n=1}^\infty N_n$. It follows from Lemma 5.3 and the comments preceding Lemma 5.3 that each N_n is a Z -set in U . Then $N = \bigcup_{n=1}^\infty N_n$ is clearly a representation which satisfies the definition for N to be an $(f-d)$ cap-set for U . $\square$

Since $(f-d)$ cap-sets for s exist, we can use Theorem 2.1 and Lemma 5.4 to prove the existence of $(f-d)$ cap-sets for F -manifolds. In Lemma 5.6 we prove that cap-sets for Q -manifolds exist. The existence of $f-d$ cap-sets for Q -manifolds will be established in §6. We will first need a preliminary result.

LEMMA 5.5. *Let X, Y be Q -manifolds and let A be a Z -set in X . Then $A \times Y$ is a Z -set in $X \times Y$.*

Proof. Using Lemma 5.2 let U be a nonnull, homotopically trivial open subset of X and let V be a nonnull, homotopically trivial open subset of Y . All we have to do is prove that $(U \times V) \setminus (A \times Y)$ is nonnull and homotopically trivial. But $(U \times V) \setminus (A \times Y) = (U \setminus A) \times V$ is clearly nonnull and homotopically trivial. $\square$

LEMMA 5.6. *If X is any Q -manifold, then X has a cap-set.*

Proof. Using Theorem 2.11 we know that there is a homeomorphism of X onto $X \times I^\infty$. It will then suffice to prove that $X \times \Sigma$ has the cap in $X \times I^\infty$. Since Σ has the cap in I^∞ we can write $\Sigma = \bigcup_{n=1}^\infty M_n$ so as to satisfy the definition. Write $X = \bigcup_{n=1}^\infty X_n$, where each X_n is a compact subset of X lying in the interior of X_{n+1} . Then $X \times \Sigma = \bigcup_{n=1}^\infty (X_n \times M_n)$ which we now show to satisfy the definition of a cap-set for $X \times I^\infty$.

It follows from Lemmas 5.3 and 5.5 that each $X_n \times M_n$ is a Z -set in $X \times I^\infty$. Now let $m > 0$ be an integer, let K be a compact subset of $X \times I^\infty$, and let $\varepsilon > 0$ be given. Since $\pi_{I^\infty}(K)$ is a compact subset of I^∞ there is an integer $n_1 > 0$ and an embedding $h_1: \pi_{I^\infty}(K) \rightarrow M_{n_1}$ such that $h_1|_{\pi_{I^\infty}(K) \cap M_m} = \text{id}$ and $d(h_1, \text{id}) < \varepsilon$. It is clear that there is an integer $n_2 > 0$ such that $\pi_X(K) \subset X_{n_2}$. Put $n = \max(n_1, n_2)$ and we obviously have an embedding h of K into $X_n \times M_n$ such that $h|_{K \cap (X_m \times M_m)} = \text{id}$ and $d(h, \text{id}) < \varepsilon$. $\square$

With the existence of cap-sets in Q -manifolds established we can now prove a result which relates F and Q -manifolds.

THEOREM 5.1. *If X is any F -manifold, then X can be regarded as a subset of a Q -manifold Y such that $Y \setminus X$ is a cap-set for Y . On the other hand, if Y is any Q -manifold, then there is an F -manifold $X \subset Y$ such that $Y \setminus X$ is a cap-set for Y .*

Proof. If X is any F -manifold, then Theorem 2.1 gives an open embedding $h: X \rightarrow s$. Let Y be an open subset of I^∞ such that $Y \cap s = h(X)$. Then Y is obviously a Q -manifold and it follows from Lemma 5.4, and the fact that $B(I^\infty)$ is a cap-set for I^∞ , that $Y \setminus h(X)$ is a cap-set for Y .

If Y is any Q -manifold, it follows from Lemma 5.6 that there is a cap-set M for Y . To show that $Y \setminus M$ is an F -manifold let U be any open subset of Y which is homeomorphic to a basic open subset of I^∞ . Let $h: U \rightarrow I^\infty$ be an embedding such that $I^\infty \setminus h(U) = W$, where W is a finite union of endslices of I^∞ . It is obvious that $h(U \cap M)$ is a cap-set for I^∞ and we know that Σ is a cap-set for I^∞ . Using Lemma 4.3 there is a homeomorphism g of I^∞ onto itself such that $g(\Sigma) = h(U \cap M)$ and $g|_W = \text{id}$. This means that $h(U \setminus M)$ is homeomorphic to $I^\infty \setminus (\Sigma \cup W)$. But Theorem 2.18 implies that $I^\infty \setminus (\Sigma \cup W)$ is homeomorphic to s . $\square$

COROLLARY 5.1. *Let Y be any Q -manifold and let M be an $(f-d)$ cap-set for Y . Then $Y \setminus M$ is an F -manifold.*

Proof. For M a cap-set the proof is given above. For M an $f-d$ cap-set we use the outline given above. $\square$

In Theorem 8.2 of [3] it is shown that if F is a Z -set in s , then $\bar{F}$ (closure taken in I^∞) is a Z -set in I^∞ . The next result generalizes this to Q -manifolds.

LEMMA 5.7. *Let X be a Q -manifold and let M be a cap-set for X . If F is a Z -set in $X \setminus M$, then $\bar{F}$ (closure taken in X) is a Z -set in X .*

Proof. Using Lemma 5.1 let U be an open subset of X and let $h: U \rightarrow I^\infty$ be an embedding such that $I^\infty \setminus h(U) = W$, where W is a finite union of endslices of I^∞ .

Using the proof of Theorem 5.1 we can additionally require that $h(U \cap M) = B(I^\infty) \setminus W$. Then $h(U \setminus M) = s$ and hence $K = h(F \cap U)$ is a Z -set in s . This means that $\bar{K}$ is a Z -set in I^∞ . Then $h(\bar{F} \cap U) = \bar{K} \cap h(U)$ is a Z -set in $h(U)$, which is what we needed. $\square$

The next result will be used in proving an open embedding theorem for σ and Σ -manifolds.

LEMMA 5.8. *Let X be an F -manifold, let M be an $(f-d)$ cap-set for X , and let F be a closed subset of X contained in $X \setminus M$. Then F is a Z -set in X .*

Proof. Using Lemma 5.1 it is sufficient to prove that for any open subset U of X which is homeomorphic to s , $U \cap F$ is a Z -set in U . Thus, our problem is reduced to the following: if N is an $(f-d)$ cap-set for s and K is a closed subset of s contained in $s \setminus N$, then K is a Z -set in s .

To show that K has Property Z in s let V be a nonnull, homotopically trivial open subset of s . Let $f: S^{n-1} \rightarrow V \setminus K$ be continuous and let $g: B^n \rightarrow V$ be a continuous extension of f , where B^n is an n -ball and S^{n-1} is the boundary of B^n . If we think of a canonical version of N (i.e. $N = \sigma$ or $N = \Sigma$) then it is clear that there is a continuous function $h: g(B^n) \rightarrow s$ such that $h(g(B^n) \setminus g(S^{n-1})) \subset N$, $h|_{g(S^{n-1})} = \text{id}$, and $d(h, \text{id}) < \varepsilon$, where $\varepsilon = d(g(B^n), s \setminus V)$. Note that $h(g(B^n)) \cap K = \emptyset$, $h(g(B^n)) \subset V$, and $h \circ g$ is a continuous extension of f . This proves that $V \setminus K$ is homotopically trivial. Since $V \cap N \neq \emptyset$ we have $V \setminus K \neq \emptyset$. $\square$

The following two results will be used in the last section of the paper.

LEMMA 5.9. *Let X be any F -manifold, let F be a Z -set in X , and for each $n > 0$ let F_n be a Z -set in X . Then $F \cap (X \setminus \bigcup_{n=1}^\infty F_n)$ is a Z -set in $X \setminus \bigcup_{n=1}^\infty F_n$.*

Proof. Using Lemma 5.2 let U be any nonnull, homotopically trivial open subset of X . Then $U \cap (X \setminus \bigcup_{n=1}^\infty F_n) = U \setminus \bigcup_{n=1}^\infty (F_n \cap U)$, and using the negligibility result of [4] we know that $U \setminus \bigcup_{n=1}^\infty (F_n \cap U)$ is homeomorphic to U . Thus $U \cap (X \setminus \bigcup_{n=1}^\infty F_n)$ is homotopically trivial. We then have $U \cap (X \setminus \bigcup_{n=1}^\infty F_n) \setminus F$ nonnull and homotopically trivial by the same argument. Lemma 5.2 then implies that $F \cap (X \setminus \bigcup_{n=1}^\infty F_n)$ is a Z -set in $X \setminus \bigcup_{n=1}^\infty F_n$. $\square$

LEMMA 5.10. *Let X be an F -manifold, let $M \subset X$ be an $(f-d)$ cap-set for X , and let F be a Z -set in M . Then $\bar{F}$ (closure in X) is a Z -set in X .*

Proof. Using Lemma 5.1 it is clear that our problem reduces to the following: If N is an $(f-d)$ cap-set for s and K is a Z -set in N , then $\bar{K}$ (closure in s) is a Z -set in s .

Let U be a nonnull, homotopically trivial open subset of s , let $f: S^{n-1} \rightarrow U \setminus \bar{K}$ be a continuous function, and let $g: B^n \rightarrow U$ be a continuous extension of f . Using the idea contained in the proof of Lemma 5.8 there is a continuous function $h: B^n \rightarrow s$ such that $h(g(B^n) \setminus g(S^{n-1})) \subset N$, $h|_{g(S^{n-1})} = \text{id}$, and $h \circ g(B^n) \subset U$. This argument has already shown that $U \cap N$ is homotopically trivial.

It is clear that there is an r , $0 < r < 1$, such that $h \circ g(\{x \in B^n \mid r \leq \|x\| \leq 1\}) \cap \bar{K} = \emptyset$. Put $S_r^{n-1} = \{x \in B^n \mid \|x\| = r\}$ and use the fact that K has Property Z in N to extend the map $h \circ g|_{S_r^{n-1}}: S_r^{n-1} \rightarrow (U \cap N) \setminus K$ to a continuous function $\varphi: B_r^n \rightarrow (U \cap N) \setminus K$, where $B_r^n = \{x \in B^n \mid \|x\| \leq r\}$. Piecing these together we then have a continuous function $\psi: B^n \rightarrow U \setminus \bar{K}$ which extends f . $\square$

6. Properties of (f-d) cap-sets in F and Q -manifolds. The main results of this section are Theorems 6.1 and 6.2, in which we establish the equivalence of (f-d) cap-sets in F and Q -manifolds. We also use similar techniques to obtain a number of other results.

We first establish the equivalence of (f-d) cap-sets in F -manifolds.

THEOREM 6.1. *Let X be an F -manifold, let $\mathcal{U}$ be an open cover of X , and let M, N be (f-d) cap-sets for X . Then there is a homeomorphism h of X onto itself such that $h(M) = N$ and h is limited by $\mathcal{U}$.*

Proof. Using Theorem 2.1 we may, without loss of generality, assume that X is an open subset of s . Applying Theorem 1 of [10] there is a star-finite open cover $\{U_i\}_{i=1}^\infty$ of X such that for each $i > 0$, U_i is a basic open subset of s and there is a $V_i \in \mathcal{U}$ such that $d(U_i, s \setminus V_i) = \varepsilon_i > 0$. Assume that $\{U_i\}_{i=1}^\infty$ is ordered as in Convergence Procedure B.

We may regard each $\text{Cl}(U_i)$ (closure taken in s) as a copy of s together with a finite union of "endslices." Since $M \cap U_1$ and $N \cap U_1$ are (f-d) cap-sets for U_1 we can use Lemma 4.3 to get a homeomorphism h_1 of X onto itself such that $h_1(M \cap U_1) = N \cap U_1$, $h_1|_{X \setminus U_1} = \text{id}$, and $d(h_1, \text{id}) < \varepsilon_1/n_1$, where $\{n_i\}_{i=1}^\infty$ is the sequence of Convergence Procedure B that is associated with $\{U_i\}_{i=1}^\infty$.

Proceeding by induction assume that homeomorphisms $\{h_i\}_{i=1}^j$ of X onto itself have been defined so that $h_i|_{X \setminus U_i} = \text{id}$, for $1 \leq i \leq j$, $h_{i+1}(h_i \circ \dots \circ h_1(M) \cap U_i) = N \cap U_i$, for $1 \leq i < j$, and $d(h_i, \text{id}) < (1/n_i) \cdot \min(\varepsilon_1, \dots, \varepsilon_i)$, for $1 \leq i \leq j$. Then let h_{j+1} be a homeomorphism of X onto itself such that $h_{j+1}|_{X \setminus U_{j+1}} = \text{id}$, $h_{j+1}(h_j \circ \dots \circ h_1(M) \cap U_{j+1}) = N \cap U_{j+1}$, and $d(h_{j+1}, \text{id}) < (1/n_{j+1}) \cdot \min(\varepsilon_1, \dots, \varepsilon_{j+1})$.

We can use Convergence Procedure B to conclude that $h = L \bigcap_{i=1}^\infty h_i$ is a homeomorphism of X onto itself such that $h(M) = N$. Moreover we note that if $x \in U_j \setminus \bigcup_{i=1}^{j-1} U_i$, then $d(h(x), x) < \varepsilon_j$. This implies that h is limited by $\mathcal{U}$. $\square$

We now prove the corresponding result for Q -manifolds. The following lemma takes the place of Theorem 1 of [10] in this case.

LEMMA 6.1. *Let X be a Q -manifold and let $\mathcal{U}$ be an open cover of X . Then there is a star-finite open cover $\{U_i\}_{i=1}^\infty$ of X which refines $\mathcal{U}$ and for which each U_i is homeomorphic to a basic open subset of I^∞ .*

Proof. Since X is a separable, locally compact metric space, it can be written as the union of an increasing sequence $\{X_n\}_{n=1}^\infty$ of compacta, each lying in the interior of its successor.

It is clear that there is a collection $\{U_i^1\}_{i=1}^\infty$ of open subsets of X , each homeomorphic to a basic open subset of I^∞ , such that $\{U_i^1\}_{i=1}^\infty$ covers X_1 , and for each i , $U_i^1 \subset X_2^0$ (the interior of X_2). For each $n > 1$ let $\{U_i^n\}_{i=1}^\infty$ be an open cover of $X_n \setminus X_{n-1}^0$, where each U_i^n is homeomorphic to a basic open subset of I^∞ , such that for each i , $U_i^n \subset X_{n+1}^0 \setminus X_{n-2}$ (or $U_i^2 \subset X_3^0$ for $n=2$). Then put $\{U_i\}_{i=1}^\infty = \{U_i^n \mid 1 \leq i \leq j_n, n \geq 1\}$ for our required star-finite open cover of X . $\square$

Since we do not have an open embedding theorem for Q -manifolds we will first need a preliminary result for open subsets of I^∞ . With the aid of Lemma 6.1 a proof of the following result can be given that is almost identical to the proof of Theorem 6.1.

LEMMA 6.2. *Let G be an open subset of I^∞ , let $\mathcal{U}$ be an open cover of G , and let M, N be (f-d) cap-sets for G . Then there is a homeomorphism h of G onto itself such that $h(M) = N$ and h is limited by $\mathcal{U}$.*

THEOREM 6.2. *Let X be a Q -manifold, let $\mathcal{U}$ be an open cover of X , and let M, N be (f-d) cap-sets for X . Then there is a homeomorphism h of X onto itself such that $h(M) = N$ and h is limited by $\mathcal{U}$.*

Proof. Using Lemma 6.1 it is clear that there is a star-finite open cover $\{U_i\}_{i=1}^\infty$ of X such that for each $i > 0$, U_i is homeomorphic to an open subset of I^∞ and there is a $V_i \in \mathcal{U}$ such that $d(U_i, X \setminus V_i) = \varepsilon_i > 0$.

Using the notation of Theorem 6.1 we need a sequence $\{h_i\}_{i=1}^\infty$ of homeomorphisms of X onto itself such that for each $i > 0$, $h_i|_{X \setminus U_i} = \text{id}$, $h_1(M \cap U_1) = N \cap U_1$, $h_{i+1}(h_i \circ \dots \circ h_1(M) \cap U_i) = N \cap U_i$, and $d(h_i, \text{id}) < (1/n_i) \cdot \min(\varepsilon_1, \dots, \varepsilon_i)$. All of these except the last can be automatically met. The last condition can clearly be met by choosing an appropriate cover of each U_i , transferring it over to an open cover in I^∞ , and applying Lemma 6.2. Then $h = L \bigcap_{i=1}^\infty h_i$ is our required homeomorphism. $\square$

Although it is obvious that any (f-d) cap-set for s is also an (f-d) cap-set for I^∞ , the corresponding property for Q -manifolds is not quite so obvious. This is our next theorem.

THEOREM 6.3. *Let X be a Q -manifold, let $M \subset X$ be a cap-set for X , and let N be an (f-d) cap-set for the F -manifold $X \setminus M$. Then N is an (f-d) cap-set for X .*

Proof. Using the proof of Lemma 5.6 we know that $X \times B(I^\infty)$ is a cap-set for $X \times I^\infty$. Combining the fact that X is homeomorphic to $X \times I^\infty$ with Theorem 6.2, there is a homeomorphism h of X onto $X \times I^\infty$ such that $h(M) = X \times B(I^\infty)$.

Let $N = \bigcup_{n=1}^\infty N_n$ be a representation as given in the definition, let $m > 0$ be given, let K be a (finite-dimensional) compact subset of X , and let $\varepsilon > 0$ be given. Using h to transfer the problem to $X \times I^\infty$, there is an embedding $f: K \rightarrow X \setminus M$ such that $f|_{K \cap N_m} = \text{id}$ and $d(f, \text{id}) < \varepsilon/2$. Using the fact that N is an (f-d) cap-set for $X \setminus M$ there is an integer $n > 0$ and an embedding $g: f(K) \rightarrow N_n$ such that $g|_{K \cap N_m} = \text{id}$ and $d(g, \text{id}) < \varepsilon/2$. Then $g \circ f$ is our required embedding. $\square$

The next result shows that we can piece together certain portions of (f-d) cap-sets to obtain (f-d) cap-sets.

THEOREM 6.4. *Let X be an F or Q -manifold and let U be an open subset of X . Then the following are true:*

(1) *If M is an (f-d) cap-set for U , then there is an (f-d) cap-set N for X such that $N \cap U = M$.*

(2) *If M, N are (f-d) cap-sets for X , then $(M \setminus U) \cup (N \cap U)$ is an (f-d) cap-set for X .*

Proof. For the proof of (1) let N' be an (f-d) cap-set for X . (From Theorem 6.3 we now know that f-d cap-sets exist in Q -manifolds.) Using Theorems 6.1 and 6.2 we can obtain a homeomorphism h of X onto itself such that $h(N' \cap U) = M$ and, by a proper choice of a cover of U , $h|_{X \setminus U} = \text{id}$. Then $N = h(N')$ satisfies $N \cap U = M$.

For the proof of (2) let h be a homeomorphism of X onto itself such that $h(M \cap U) = N \cap U$ and $h|_{X \setminus U} = \text{id}$. Then $h(M) = (M \setminus U) \cup (N \cap U)$ is an (f-d) cap-set for X . $\square$

In the next theorem we prove that (f-d) cap-sets can be determined locally.

THEOREM 6.5. *Let X be an F or Q -manifold, let $\{U_i\}_{i=1}^\infty$ be an open cover of X , and for each $i > 0$ let M_i be an (f-d) cap-set for U_i . Then $M = \bigcup_{i=1}^\infty M_i$ is an (f-d) cap-set for X .*

Proof. We only sketch the proof for Q -manifolds. The proof for F -manifolds is similar. Let $\{V_i\}_{i=1}^\infty$ be a star-finite open cover of X which is a refinement of $\{U_i\}_{i=1}^\infty$ and which is ordered as in Convergence Procedure B. Let N have the (f-d) cap in X and use the techniques of Theorem 6.1 to inductively define a sequence $\{h_i\}_{i=1}^\infty$ of homeomorphisms of X onto itself for which $h_1(M \cap V_1) = N \cap V_1$, $h_{i+1}(h_i \circ \dots \circ h_1(M \cap V_{i+1})) = N \cap h_i \circ \dots \circ h_1(V_{i+1})$, for all $i > 0$, $h_1|_{X \setminus V_1} = \text{id}$, and $h_{i+1}|_{X \setminus h_i \circ \dots \circ h_1(V_{i+1})} = \text{id}$, for all $i > 0$. We remark that in order to achieve the first and second conditions we have to make use of Lemma 4.1. Then Convergence Procedure D assures us that $h = \bigcap_{i=1}^\infty h_i$ is a homeomorphism of X onto itself which obviously satisfies $h(M) = N$. $\square$

We can easily use this result to obtain a generalization of Lemma 4.2 to F and Q -manifolds.

THEOREM 6.6. *Let X be an F or Q -manifold, let M be an (f-d) cap-set for X , and let K be a countable union of (finite-dimensional) compact Z -sets in X . Then $M \cup K$ is an (f-d) cap-set for X .*

Proof. In case X is an F -manifold we use Theorem 2.1 to get an open embedding $h: X \rightarrow s$. Using Theorem 6.4 there is an (f-d) cap-set N for s such that $N \cap h(X) = h(M)$. By Lemma 4.2 we know that $N \cup h(K)$ is an (f-d) cap-set for s , hence $(N \cup h(K)) \cap h(X) = h(M) \cup h(K)$ is an (f-d) cap-set for $h(X)$.

If X is a Q -manifold let U be any open subset of X which is homeomorphic to a basic open subset of I^∞ . Then there is a homeomorphism f of U onto $I^\infty \setminus W$, where

W is a finite union of endslices. It is clear that $f(M \cap U)$ is an (f-d) cap-set for I^∞ and $f(K \cap U)$ is a countable union of (finite-dimensional) compact Z -sets in I^∞ . From Theorem 4.2 we know that $f(M \cap U) \cup f(K \cap U)$ is an (f-d) cap-set for I^∞ , hence for $f(U)$. Thus Theorem 6.5 implies that $M \cup K$ is an (f-d) cap-set for X . $\square$

We can also use Theorem 6.5 to prove that we can remove certain portions of an (f-d) cap-set and still have an (f-d) cap-set.

THEOREM 6.7. *Let X be an F or Q -manifold, let M be an (f-d) cap-set for X , and let F be a Z -set in X . Then $M \setminus F$ is an (f-d) cap-set for X .*

Proof. In case X is an F -manifold let U be an open subset of X which is homeomorphic to s . Then $F \cap U$ is a Z -set in U . Using Corollary 4.1 we find that $(M \cap U) \setminus (F \cap U)$ has the (f-d) cap in U . Then Theorem 6.5 implies that $M \setminus F$ is an (f-d) cap-set for X .

If X is a Q -manifold let U be an open subset of X for which there is a homeomorphism h of U onto $I^\infty \setminus W$, where W is a finite union of endslices. We observed in Lemma 5.3 that $f(F \cap U) \cup W$ is a compact Z -set in I^∞ . Using Lemma 4.4 we find that $h(M \cap U) \setminus [h(F \cap U) \cup W] = h(M \cap U) \setminus h(F \cap U)$ is an (f-d) cap-set for I^∞ , and hence for $h(U)$. Once more Theorem 6.5 implies that $M \setminus F$ is an (f-d) cap-set for X . $\square$

In our next theorem we show how an (f-d) cap-set of a product relates to the (f-d) cap-sets of the factors.

THEOREM 6.8. *Let X, Y both be F -manifolds or both be Q -manifolds and let $M \subset X, N \subset Y$ be (f-d) cap-sets. Then $M \times N$ is an (f-d) cap-set for $X \times Y$.*

Proof. First assume that X and Y are F -manifolds. Let $f_1: X \rightarrow s^{\alpha_1}$ and $f_2: Y \rightarrow s^{\alpha_2}$ be open embeddings, where α_1 is the set of odd positive integers and $\alpha_2 = \mathbb{Z} \setminus \alpha_1$. Then $f_1(X) \times f_2(Y)$ is an open subset of $s = s^{\alpha_1} \times s^{\alpha_2}$. We know that there is an (f-d) cap-set P_1 for s^{α_1} such that $P_1 \cap f_1(X) = f_1(M)$ and an (f-d) cap-set P_2 for s^{α_2} such that $P_2 \cap f_2(Y) = f_2(N)$.

It is obvious that there exist canonical (f-d) cap-sets R_1 for s^{α_1} and R_2 for s^{α_2} such that $R_1 \times R_2$ is an (f-d) cap-set for s . This implies that $P_1 \times P_2$ is an (f-d) cap-set for s and thus $(P_1 \times P_2) \cap (f_1(X) \times f_2(Y)) = f_1(M) \times f_2(N)$ is an (f-d) cap-set for $f_1(X) \times f_2(Y)$.

Now assume that X and Y are Q -manifolds. Using Lemmas 5.3 and 5.5 we find that if $K_1 \subset X$ is a compact Z -set in X and $K_2 \subset Y$ is a compact Z -set in Y , then $K_1 \times K_2$ is a Z -set in $X \times Y$. If we use the techniques involved in the proof of Lemma 5.6 it follows that if M and N are cap-sets, then $M \times N$ is a cap-set for $X \times Y$. Thus all we have left to do is the case in which M and N are f-d cap-sets.

Let M' be a cap-set for X and let N' be a cap-set for Y . It clearly follows from the techniques involved in the proof of Lemma 5.6 that $X \times N'$ and $M' \times Y$ are cap-sets for $X \times Y$. Using Theorem 6.6 we find that $(X \times N') \cup (M' \times Y)$ is a cap-set for $X \times Y$.

Now choose f-d cap-sets M'' for $X \setminus M'$ and N'' for $Y \setminus N'$. Using what we proved above we know that $M'' \times N''$ is an f-d cap-set for $(X \setminus M') \times (Y \setminus N')$. Since $(X \times Y) \setminus [(X \times N') \cup (M' \times Y)] = (X \setminus M') \times (Y \setminus N')$ it follows from Theorem 6.3 that $M'' \times N''$ is an f-d cap-set for $X \times Y$. Then we can clearly use Theorem 6.2 to relate this to M and N . $\square$

We can use Theorem 6.7 to characterize Z-sets in (f-d) cap-sets.

THEOREM 6.9. *Let X be an F-manifold and let M be an (f-d) cap-set for X . A necessary and sufficient condition that a closed subset K of M has Property Z in M is that K be strongly negligible in M .*

Proof. If K is strongly negligible in M , then we can use the proof of Theorem 2.20 to prove that K is a Z-set in M .

Thus assume that K is a Z-set in M and let $\{U_i\}_{i=1}^\infty$ be an open cover of M . For each $i > 0$ let V_i be an open subset of X satisfying $V_i \cap M = U_i$. Without loss of generality we may assume that $X = \bigcup_{i=1}^\infty V_i$.

By Lemma 5.10, $\bar{K}$ (closure in X) is a Z-set in X . Using Theorem 6.7 it follows that $M \setminus \bar{K}$ is an (f-d) cap-set for X . By Theorem 6.1 there exists a homeomorphism h of X onto itself such that $h(M) = M \setminus \bar{K}$ and h is limited by $\{V_i\}_{i=1}^\infty$. Thus K is strongly negligible in M . $\square$

7. Infinite-deficiency in manifolds. In [13] a proof of the following theorem is given.

THEOREM 7.1. *Let X be an F-manifold and let F be a closed subset of X . A necessary and sufficient condition that F have Property Z in X is that there exists a homeomorphism h of X onto $X \times s$ such that $\pi_s(h(F))$ has infinite deficiency.*

The technique involved in the proof of Theorem 7.1 is to use Theorem 2.10 and then to modify the copy of F in $X \times s$ so that its projection in s has infinite deficiency.

In our next theorem we give a similar characterization of Property Z in Q -manifolds. As a corollary we get an easy proof of Theorem 7.1.

THEOREM 7.2. *Let X be a Q -manifold and let F be a closed subset of $X \times I^\infty$. A necessary and sufficient condition that F have Property Z in $X \times I^\infty$ is that there exists a homeomorphism h of $X \times I^\infty$ onto itself taking F onto a set whose projection in I^∞ has infinite deficiency. Moreover, we can construct h so that $h(X \times B(I^\infty)) = X \times B(I^\infty)$.*

Proof. The step from infinite deficiency to Property Z is easy and is similar to the proof of Theorem 9.1 of [3]. Thus assume that F is a Z-set in $X \times I^\infty$. Since $X \times B(I^\infty)$ is a cap-set for $X \times I^\infty$, it follows from Theorem 6.7 that $(X \times B(I^\infty)) \setminus F$ is also a cap-set for $X \times I^\infty$. Using Theorem 6.2 there is a homeomorphism f_1 of $X \times I^\infty$ onto itself such that $f_1((X \times B(I^\infty)) \setminus F) = X \times B(I^\infty)$. This gives us

$$f_1(F) \cap (X \times B(I^\infty)) = \emptyset.$$

Let $j > 0$ be an integer and for each $x \in X \times \prod_{i \neq j} I_i$ let $U_x \times (\delta_x, 1]$ be an open subset of $(X \times \prod_{i \neq j} I_i) \times I_j$ containing $(x, 1)$ such that $(U_x \times (\delta_x, 1]) \cap f_1(F) = \emptyset$. Note that $\{U_x \mid x \in X \times \prod_{i \neq j} I_i\}$ is an open cover of $X \times \prod_{i \neq j} I_i$. Thus we can use Lemma 6.1 to get a star-finite open refinement $\{U_i\}_{i=1}^\infty$ of $\{U_x \mid x \in X \times \prod_{i \neq j} I_i\}$ which covers $X \times \prod_{i \neq j} I_i$. Moreover we may assume that the U_i 's are indexed as in Convergence Procedure B. It is clear that for each $i > 0$ there is a number $\delta_i > 0$ such that $\frac{1}{2} \leq \delta_i < 1$ and $(U_i \times (\delta_i, 1]) \cap f_1(F) = \emptyset$.

Now construct a closed cover $\{C_i\}_{i=1}^\infty$ of $X \times \prod_{i \neq j} I_i$ such that $C_i \subset U_i$, for all $i > 0$. For each $i > 0$ let $\varphi_i: X \times \prod_{i \neq j} I_i \rightarrow [0, 1]$ be a continuous function such that $\varphi_i(x) = 1$ for $x \in C_i$ and $\varphi_i(x) = 0$ for $x \in X \times \prod_{i \neq j} I_i \setminus U_i$. For each $i > 0$ we now construct a homeomorphism g_i of $(X \times \prod_{i \neq j} I_i) \times I_j$ onto itself which slides points linearly in the I_j -direction as follows: if $x \in X \times \prod_{i \neq j} I_i$, then $\{x\} \times I_j$ is taken linearly onto itself such that $g_i(x, \delta_i) = (x, \delta_i(1 - \varphi_i(x)) + (\frac{1}{2})\varphi_i(x))$. Applying Convergence Procedure C to $\{g_i\}_{i=1}^\infty$ let $g = L \prod_{i=1}^\infty g_i$, which is a homeomorphism of $X \times I^\infty$ onto itself satisfying $g \circ f_1(F) \subset (X \times \prod_{i \neq j} I_i) \times [-\frac{1}{2}, \frac{1}{2}]$. Using similar techniques we can construct a homeomorphism g' of $X \times I^\infty$ onto itself which slides points only in the I_j -direction such that $g' \circ g \circ f_1(F) \subset X \times \prod_{i \neq j} I_i \times [-\frac{1}{2}, \frac{1}{2}]$. Then put $g^j = g' \circ g$. We note that g^j is a homeomorphism of $X \times I^\infty$ onto itself which satisfies $g^j \circ f_1(F) \subset X \times \prod_{i \neq j} I_i \times [-\frac{1}{2}, \frac{1}{2}]$, $\pi_X \circ g^j(x, t) = x$, and

$$\tau_i \circ \pi_{I^\infty} \circ g^j(x, t) = \tau_i(t),$$

for all $i \neq j$ and $(x, t) \in X \times I^\infty$.

Exercising some control on the "size" of the g^j 's and using Theorem 4.2 of [7], it is now clear that $f_2 = L \prod_{j=1}^\infty g^j$ gives a homeomorphism f_2 of $X \times I^\infty$ onto itself such that $f_2(f_1(F)) \subset X \times \prod_{i=1}^\infty [-\frac{1}{2}, \frac{1}{2}]$ and $f_2(X \times B(I^\infty)) = X \times B(I^\infty)$. Using Theorem 3.5 of [2] there is a β^* -homeomorphism $\psi: I^\infty \rightarrow I^\infty$ such that $\psi(\prod_{i=1}^\infty [-\frac{1}{2}, \frac{1}{2}])$ has infinite deficiency. Then $f_3 = \text{id} \times \psi$ gives a homeomorphism of $X \times I^\infty$ onto itself such that $\pi_{I^\infty} \circ f_3(X \times \prod_{i=1}^\infty [-\frac{1}{2}, \frac{1}{2}])$ has infinite deficiency and $f_3(X \times B(I^\infty)) = X \times B(I^\infty)$.

Now $f_3 \circ f_2 \circ f_1$ is a homeomorphism of $X \times I^\infty$ onto itself such that $\pi_{I^\infty} \circ f_3 \circ f_2 \circ f_1(F)$ has infinite deficiency. We also note that

$$f_3 \circ f_2 \circ f_1[F \cap (X \times B(I^\infty))] \subset X \times s$$

and is a countable union of compact sets. Using Corollary 5.7 of [2] for pushing a σ -compact subset of s to $B(I^\infty)$ there is a homeomorphism f_4 of $X \times I^\infty$ onto itself such that $\pi_{I^\infty} \circ f_4 \circ f_3 \circ f_2 \circ f_1(F)$ has infinite deficiency and $f_4 \circ f_3 \circ f_2 \circ f_1(X \times B(I^\infty)) = X \times B(I^\infty)$. Then $h = f_4 \circ f_3 \circ f_2 \circ f_1$ satisfies our requirements. $\square$

COROLLARY 7.1. *If X is an F -manifold and F is a Z -set in $X \times s$, then there is a homeomorphism h of $X \times s$ onto itself such that $\pi_s \circ h(F)$ has infinite deficiency.*

Proof. Using Theorem 5.1 there is a Q -manifold Y containing $X \times s$ such that $Y \setminus (X \times s)$ is a cap-set for Y . This means that there is a homeomorphism f of Y onto

$Y \times I^\infty$ such that $f(X \times s) = Y \times s$. Then $f(F)$ is a Z -set in $Y \times s$. By Lemma 5.7 we know that $\text{Cl}(f(F))$ (closure taken in $Y \times I^\infty$) is a Z -set in $Y \times I^\infty$. Thus there is a homeomorphism g of $Y \times I^\infty$ onto itself such that $g(Y \times s) = Y \times s$ and $g(\text{Cl}(f(F)))$ has infinite deficiency. We can write $s = s_1 \times s_2$, where s_1, s_2 are copies of s and $g(\text{Cl}(f(F)))$ is infinitely deficient in the s_2 factor. Then the following diagram, with appropriate restrictions on the homeomorphisms involved, gives our desired homeomorphism.

$$X \times s \xrightarrow{f} Y \times s \xrightarrow{g} Y \times s = (Y \times s_1) \times s_2 \xrightarrow{f^{-1} \times \text{id}} X \times s. \quad \square$$

8. Product and factor theorems. The first result we establish is a relationship between cap-sets and f -d cap-sets in F -manifolds. For its proof we will require two preliminary lemmas.

The first lemma gives sufficient conditions in order that a subset of I^∞ be a cap-set for I^∞ . These conditions are also necessary, and in [5] Anderson used them for a definition of cap-set.

LEMMA 8.1. *Let $M \subset I^\infty$ be a countable union of Z -sets in I^∞ which satisfies the property that for each $\varepsilon > 0$ and each pair A, B of compact subsets of I^∞ , with $A \subset B \cap M$, there is an embedding $h: B \rightarrow M$ such that $h|_A = \text{id}$ and $d(h, \text{id}) < \varepsilon$. Then M is a cap-set for I^∞ .*

Proof. Let N be a cap-set for I^∞ and write $N = \bigcup_{n=1}^\infty N_n$, as in the definition. Then there is an embedding $f_1: N_1 \rightarrow M$. Since $f_1(N_1)$ obviously has Property Z in I^∞ we can use Lemma 2.1 of [13] to extend f_1 to a homeomorphism g_1 of I^∞ onto itself.

Using the above properties of M we can inductively obtain a sequence $\{g_i\}_{i=1}^\infty$ of homeomorphisms of I^∞ onto itself such that

- (1) for each $i > 0$, $g_i \circ \dots \circ g_1(N_i) \subset M$,
- (2) for each $i > 0$, $g_{i+1}|_{g_i \circ \dots \circ g_1(N_i)} = \text{id}$, and
- (3) for each $i > 1$, $d(g_i, \text{id}) < \min((3^{-i}), (3^{-i}) \cdot \eta(g_{i-1} \circ \dots \circ g_1, 2^{-i}))$.

Applying Convergence Procedure A we find that $g = L \prod_{i=1}^\infty g_i$ gives a homeomorphism of I^∞ onto itself such that $g(N) \subset M$. Since $g(N)$ is a cap-set for I^∞ it follows from Lemma 4.2 that M is also. $\square$

LEMMA 8.2. *For each $n > 0$ define*

$$M_n = \{x \in I^\infty \mid \tau_i(x) = 0, \text{ for } i \geq 2n-1 \text{ and odd}\}.$$

Then $M = \bigcup_{n=1}^\infty M_n$ has the cap in I^∞ .

Proof. We will prove that M has the properties of Lemma 8.1. Thus let A be a compact subset of M and let $\varepsilon > 0$ be given. It will clearly be sufficient to prove the existence of an embedding $h: I^\infty \rightarrow M$ satisfying $h|_A = \text{id}$ and $d(h, \text{id}) < \varepsilon$.

Let $\mathcal{U}$ be an open cover of $I^\infty \setminus A$ such that for elements $U \in \mathcal{U}$, $\text{diam}(U)$ "becomes small" as $d(U, A)$ "becomes small." Then let $\{U_i\}_{i=1}^\infty$ be a star-finite

refinement of $\mathcal{U}$ such that each U_i is a basic open set in I^∞ and assume that $\{U_i\}_{i=1}^\infty$ is ordered as in Convergence Procedure B. Choose an odd integer m such that $(\sum_{i=m}^\infty 2^{2^{-i}})^{1/2} < \varepsilon$. We will construct h so that h does not affect the first $m-1$ coordinates of any point.

Using the simplified geometry of the basic open sets U_i , we can rotate coordinates in an obvious manner to obtain embeddings $h_i: I^\infty \rightarrow I^\infty$ such that for each $i > 0$, $h_i(U_i) \subset M$, $h_i|_{I^\infty \setminus U_i} = \text{id}$, and h_i does not affect the first $m-1$ coordinates of any point. Each point x of I^∞ is moved only a finite number of times in the sequence $h_1(x)$, $h_2 \circ h_1(x)$, $h_3 \circ h_2 \circ h_1(x)$, $\dots$. Thus $h(x) = \lim_{i \rightarrow \infty} h_i \circ \dots \circ h_1(x)$ defines an embedding I^∞ into M such that $h|_A = \text{id}$ and $d(h, \text{id}) < \varepsilon$. $\square$

THEOREM 8.1. *Let X be an F -manifold and let $M \subset X$ be an f -d cap-set for X . Then $M \times I^\infty$ is a cap-set for $X \times I^\infty$.*

Proof. We will use Theorem 6.5 by showing that if U is an open subset of X which is homeomorphic to s , then $(M \cap U) \times I^\infty$ has the cap in $U \times I^\infty$.

We know that there is a homeomorphism f of U onto s satisfying $f(M \cap s) = \sigma$. Thus it will suffice to prove that $\sigma \times I^\infty$ has the cap in $s \times I^\infty$. Let $I_1^\infty = \prod \{I_i \mid i \text{ odd}\}$ and $I_2^\infty = \prod \{I_i \mid i \text{ even}\}$. Let A be the set of all elements of I_1^∞ having at most finitely many nonzero coordinates. Using Lemma 4.2 we know that A has the f -d cap in I_1^∞ . Lemma 8.2 shows that $A \times I_2^\infty$ has the cap on $I_1^\infty \times I_2^\infty$. There is a homeomorphism g of I_1^∞ onto itself such that $g(A) = \sigma_1$, where σ_1 is the set of all points in $s_1 = \prod \{I_i^0 \mid i \text{ odd}\}$ having at most finitely many nonzero coordinates. Thus $\sigma_1 \times I_2^\infty$ has the cap in $I_1^\infty \times I_2^\infty$ and hence in $s_1 \times I_2^\infty$. This is what we wished to prove. $\square$

COROLLARY 8.1. *Let X be an F -manifold, let $M \subset X$ be an f -d cap-set for X , and let $N \subset X$ be a cap-set for X . Then $M \times I^\infty$ is homeomorphic to N .*

Proof. This follows immediately from Theorems 6.1 and 8.1. $\square$

We know from Theorem 2.8 that given any F -manifold X , there is a countable locally-finite simplicial complex K such that X is homeomorphic to $|K| \times s$. We use this in our next theorem.

THEOREM 8.2. *Let X be an F -manifold and let K be a complex as above. If $M \subset X$ is a cap-set (or f -d cap-set) for X , then M is homeomorphic to $|K| \times \Sigma$ (or $|K| \times \sigma$).*

Proof. All that we need to do is prove that $|K| \times \sigma$ has the f -d cap in $|K| \times s$, since the proof that $|K| \times \Sigma$ has the cap in $|K| \times s$ is easy and resembles the proof of Lemma 5.6.

Write $|K| = \bigcup_{n=1}^\infty |K_n|$, where $|K_n|$ is a finite complex contained in the interior of $|K_{n+1}|$, and write $\sigma = \bigcup_{n=1}^\infty M_n$ so as to satisfy the definition of an f -d cap-set for s . We will show that the representation $|K| \times \sigma = \bigcup_{n=1}^\infty (|K_n| \times M_n)$ satisfies the definition for $|K| \times \sigma$ to have the f -d cap in $|K| \times s$. To this end let $m > 0$ be an integer, let F be a finite-dimensional compact subset of $|K| \times s$, and let $\varepsilon > 0$ be given.

If C is any compact subset of s , then there is a homeomorphism of s onto

$s \times [0, 1)$ taking C into $s \times \{0\}$. This means that there is an isotopy $G: s \times I \rightarrow s$ such that $G_0 = \text{id}$, $d(x, G_t(x)) < \varepsilon/3$, for all $t \in I$ and $x \in s$, and $G_t(s) \cap M_m = \emptyset$, for all $t > 0$. For each $x \in F$ define $f(x) = G_t \circ \pi_s(x)$, where

$$t = d(x, |K_m| \times M_m) / (1 + d(x, |K_m| \times M_m)).$$

Then $f: F \rightarrow s$ is a continuous function such that $d(f, \pi_s) < \varepsilon/3$ and $f(F \setminus (|K_m| \times M_m)) \cap M_m = \emptyset$.

In [8] it is shown that if X is any F -manifold, A is a topologically complete separable metric space, $\mathcal{U}$ is an open cover of X , and $g: A \rightarrow X$ is a continuous function, then there is an embedding $\tilde{g}: A \rightarrow X$ such that $\tilde{g}$ is $\mathcal{U}$ -close to g (i.e. for each $a \in A$ there exists an element of $\mathcal{U}$ containing both $\tilde{g}(a)$ and $g(a)$). If we let $X = s \setminus M_m$ and choose an appropriate cover of X , then we can use this result to obtain a continuous function $\tilde{f}: F \rightarrow s$ such that $\tilde{f}|_{|K_m| \times M_m} = \pi_s|_{|K_m| \times M_m}$, $\tilde{f}|_{F \setminus (|K_m| \times M_m)}$ is an embedding, and $d(\tilde{f}, f) < \varepsilon/3$.

Note that $\tilde{f}(F)$ is a finite-dimensional compact subset of s . Thus there exists an integer $n > 0$ and an embedding $g: \tilde{f}(F) \rightarrow M_n$ such that $g|_{\tilde{f}(F) \cap M_m} = \text{id}$ and $d(g, \text{id}) < \varepsilon/3$. Now let $n' \geq n$ be an integer such that $\pi_{|K|}(F) \subset |K_{n'}|$. Define $h: F \rightarrow |K_{n'}| \times M_{n'}$ by $h(x) = (\pi_{|K|}(x), g \circ \tilde{f}(x))$, for all $x \in F$. Then h is an embedding, $h|_{F \cap (|K_m| \times M_m)} = \text{id}$, and $d(h, \text{id}) < \varepsilon$, as we wanted. $\square$

COROLLARY 8.2. *Let X be an F or Q -manifold, let $M \subset X$ have the f -d cap, and let $N \subset X$ have the cap. Then X , M , and N are all of the same homotopy type.*

Proof. For X an F -manifold the proof follows from Theorem 8.2. If X is a Q -manifold we have already shown (Lemma 5.6 and Theorem 6.2) that N is homeomorphic to $X \times \Sigma$, and hence has the same homotopy type as X . Using Corollary 5.1 we find that M has the same homotopy type as N . $\square$

COROLLARY 8.3. *Let X be a Q -manifold and let $M \subset X$ have the (f) -d cap. Then $X \setminus M$ is an F -manifold of the same homotopy type as X .*

Proof. If N is an (f) -d cap-set for $X \setminus M$, then Corollary 8.2 implies that N and $X \setminus M$ have the same homotopy type. From Theorem 6.2 it follows that N is homeomorphic to M , and from Corollary 8.2 it follows that M and X have the same homotopy type. $\square$

From Theorem 2.5 we know that if K is any countable locally-finite simplicial complex, then $|K| \times s$ is an F -manifold. We use this to prove a corresponding property for σ and Σ -manifolds.

COROLLARY 8.4. *If K is any countable locally-finite simplicial complex, then $|K| \times \sigma$ is a σ -manifold and $|K| \times \Sigma$ is a Σ -manifold.*

Proof. From Theorem 8.2 we know that $|K| \times \sigma$ is an f -d cap-set for $|K| \times s$ and $|K| \times \Sigma$ is a cap-set for $|K| \times s$. Then Lemma 5.4 implies that $|K| \times \sigma$ is a σ -manifold and $|K| \times \Sigma$ is a Σ -manifold. $\square$

We can easily establish some factor theorems for (f-d) cap-sets in F -manifolds.

THEOREM 8.3. *Let X be an F -manifold, let M be an f-d cap-set for X , and let I^n be any n -cell. Then M , $M \times \sigma$, and $M \times I^n$ are all homeomorphic.*

Proof. Using Theorem 6.8 it follows that $M \times \sigma$ is an f-d cap-set for $X \times s$. This implies that M and $M \times \sigma$ are homeomorphic.

Since X and $X \times I^\infty$ are homeomorphic it follows that X and $X \times I^n$ are also homeomorphic. Thus to show that M and $M \times I^n$ are homeomorphic all we have to do is prove that $M \times I^n$ is an f-d cap-set for $X \times I^n$. Using Theorem 6.5 it is clearly sufficient to prove that if U is any open subset of X which is homeomorphic to s , then $(M \cap U) \times I^n$ is an f-d cap-set for $U \times I^n$. Thus the problem reduces to the following: Show that $\sigma \times I^n$ is an f-d cap-set for $s \times I^n$.

Let $s_1 = \prod \{I_i^0 \mid i > n\}$ and $\sigma_1 = \{x \in s_1 \mid x \text{ has at most finitely many nonzero coordinates}\}$. Then Lemma 4.2 implies $\sigma_1 \times I^n$ is an f-d cap-set for $I^\infty = I_1^\infty \times I^n$, where $I_1^\infty = \prod \{I_i \mid i > n\}$. Thus $\sigma_1 \times I^n$ is an f-d cap-set for $s_1 \times I^n$, which is what we wished to prove. $\square$

COROLLARY 8.5. *Let X be an F -manifold and let M be a cap-set for X . Then M , $M \times \Sigma$, and $M \times I^\infty$ are all homeomorphic.*

Proof. It follows from Corollary 8.1 that M is homeomorphic to $N \times I^\infty$, where N is an f-d cap-set for X . Thus M and $M \times I^\infty$ are homeomorphic. It then follows that Σ and $\sigma \times I^\infty$ are homeomorphic. Thus (with “ $\simeq$ ” meaning “is homeomorphic to”)

$$M \times \Sigma \simeq (N \times I^\infty) \times (\sigma \times I^\infty) \simeq (N \times \sigma) \times I^\infty \simeq N \times I^\infty \simeq M. \quad \square$$

9. Permuting sets which have the (f-d) cap. The following result is a generalization of Lemma 4.3. Since its proof is similar, except that we are handling several subsets of I^∞ having the (f-d) cap at one time, we omit its proof.

LEMMA 9.1. *Let $\{M_i\}_{i=1}^n$ (n finite or ∞) be a collection of disjoint subsets of I^∞ which have the (f-d) cap in I^∞ and let $\{N_i\}_{i=1}^n$ be a collection of disjoint subsets of I^∞ which have the (f-d) cap in I^∞ . If $K \subset B(I^\infty) \setminus [\bigcup_{i=1}^n (M_i \cup N_i)]$ is compact, then there is a homeomorphism h of I^∞ onto itself such that $h|K = \text{id}$ and $h(M_i) = N_i$, for all i . If $\bigcup_{i=1}^n (M_i \cup N_i) \subset s$, then we may additionally require that h be a β^* -homeomorphism.*

We can easily apply the techniques of §6 to prove the following theorem.

THEOREM 9.1. *Let $\{M_i\}_{i=1}^n$ (n finite or ∞) be a collection of disjoint subsets of an F or Q -manifold X which have the (f-d) cap in X and let $\{N_i\}_{i=1}^n$ be a collection of disjoint subsets of X which have the (f-d) cap in X . Then there is a homeomorphism h of X onto itself such that $h(M_i) = N_i$, for all i .*

In our next theorem we prove that any finite collection of n disjoint (f-d) cap-sets for an F or Q -manifold can be permuted with a period n homeomorphism.

THEOREM 9.2. *Let $\{M_i\}_{i=1}^n$ be a finite collection of disjoint subsets of an F or Q -manifold X , with each M_i having the $(f-d)$ cap in X . Then there is a period n homeomorphism h of X onto itself such that $h(M_i) = M_{i+1}$, for $1 \leq i \leq n-1$, and $h(M_n) = M_1$.*

Proof. We only treat the case in which X is an F -manifold and each M_i is a cap-set for X . The other cases are similar.

Let $\{[a_i, b_i] \times [c_i, d_i]\}_{i=1}^n$ be a collection of disjoint rectangles in $(-1, 1) \times (-1, 1)$ and let f be a homeomorphism of $(-1, 1) \times (-1, 1)$ onto itself of period n such that $f([a_i, b_i] \times [c_i, d_i]) = [a_{i+1}, b_{i+1}] \times [c_{i+1}, d_{i+1}]$, for $1 \leq i \leq n-1$, and

$$f([a_n, b_n] \times [c_n, d_n]) = [a_1, b_1] \times [c_1, d_1].$$

For $1 \leq i \leq n$ let C_i be the subset of s which consists of all points $x \in s$ such that $\tau_{2j-1}(x) \in [a_i, b_i]$ and $\tau_{2j}(x) \in [c_i, d_i]$, for all $j > 0$.

For $1 \leq i \leq n$ let N_i be the basic core set in s structured over C_i . Then $\{N_i\}_{i=1}^n$ is a collection of disjoint cap-sets for s . Let g be the homeomorphism of s onto itself such that $\tau_{(2j-1, 2j)} \circ g(x) = f(\tau_{(2j-1, 2j)}(x))$, for all $j > 0$. Then g is a homeomorphism of s onto itself of period n such that $g(N_i) = N_{i+1}$, for $1 \leq i \leq n-1$, and $g(N_n) = N_1$.

Now let M be a cap-set for X and use Theorem 6.8 to conclude that $\{M \times N_i\}_{i=1}^n$ is a collection of disjoint cap-sets for $X \times s$. Also $\text{id} \times g: X \times s \rightarrow X \times s$ gives a homeomorphism of period n which permutes the collection $\{M \times N_i\}_{i=1}^n$. Since X is homeomorphic to $X \times s$ this means that there is a collection $\{N'_i\}_{i=1}^n$ of disjoint cap-sets for X and a period n homeomorphism h' of X onto itself which permutes $\{N'_i\}_{i=1}^n$.

Using Theorem 9.1 there is a homeomorphism φ of X onto itself such that $\varphi(M_i) = N'_i$, for $1 \leq i \leq n$. Then $h = \varphi^{-1} \circ h' \circ \varphi$ satisfies our required properties. $\square$

Using similar devices we can also permute a countably infinite collection of disjoint $(f-d)$ cap-sets for an F or Q -manifold.

THEOREM 9.3. *Let $\{M_i\}_{i=-\infty}^{\infty}$ be a collection of disjoint subsets of an F or Q -manifold X , each M_i having the $(f-d)$ cap in X . Then there is a homeomorphism h of X onto itself such that $h(M_i) = M_{i+1}$, for all i .*

Proof. Once more we only treat the case in which X is an F -manifold and each M_i is a cap-set for X .

Let $\{[a_i, b_i]\}_{i=1}^{\infty}$ be a collection of disjoint subintervals of $(-1, 1)$ such that $b_i < a_{i+1}$, for all i , and let f be a homeomorphism of $(-1, 1)$ onto itself such that $f([a_i, b_i]) = [a_{i+1}, b_{i+1}]$, for all i . For each i let C_i be the subset of s consisting of all points $x \in s$ for which $\tau_j(x) \in [a_i, b_i]$, for all $j > 0$. As in Theorem 9.2 let N_i be the basic core set in s structured over C_i . It is now clear that we can use techniques like those used in the proof of Theorem 9.2 to complete the proof of this theorem. $\square$

10. Open embedding theorems for σ and Σ -manifolds. The main result of this section is Theorem 10.3 in which we prove that any σ (or Σ)-manifold can be embedded as an open subset of σ (or Σ). Our proof makes use of Henderson's open

embedding theorem for F -manifolds (Theorem 2.3). Our strategy is to first embed the σ (or Σ)-manifold as an f -d cap (or cap)-set for an F -manifold (Theorem 10.2), and then apply Henderson's result.

In this and the next section we will need the following well-known result which is due to Lavrent'ev (see [16, p. 235]).

THEOREM 10.1. *Each homeomorphism between subspaces A and B of topologically complete metric spaces X and Y , respectively, can be extended to a homeomorphism between G_δ -subsets of X and Y .*

The following result gives a technique for finding F -manifolds in topologically complete metric spaces. We will also use this in the next section.

LEMMA 10.1. *Let X be an F -manifold, let M be an $(f$ -d) cap-set for X , let Y be a topologically complete metric space, and let $h: M \rightarrow Y$ be an embedding. Then there is a copy X' of X such that $h(M) \subset X' \subset Y$ and $h(M)$ is an $(f$ -d) cap-set for X' .*

Proof. Using Theorem 10.1 we can extend h to a homeomorphism $H: X'' \rightarrow H(X'')$, where X'' and $H(X'')$ are G_δ -sets. Using Lemma 5.8 we find that $X \setminus X'' = \bigcup_{i=1}^{\infty} F_i$, where each F_i is a Z -set in X . Using Theorem 1 of [4] there is a homeomorphism h of X onto X'' . Then X'' is a copy of X and it is obvious that M is an $(f$ -d) cap-set for X'' . We set $X' = H(X'')$ and note that $h(M)$ is an $(f$ -d) cap-set for X' . $\square$

We are now ready for the major step in establishing our open embedding theorems. This will also be used in the next section.

THEOREM 10.2. *If X is a σ (or Σ)-manifold, then there is an F -manifold Y and an embedding $h: X \rightarrow Y$ such that $h(X)$ is an f -d cap (or cap)-set for Y .*

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be an open cover of X such that each U_i is homeomorphic to an open subset of σ (or Σ). Without loss of generality assume that $X \subset Z$, where Z is a topologically complete separable metric space. For each $i > 0$ let V_i be an open subset of Z such that $V_i \cap X = U_i$ and let G_i be an open subset of s such that $G_i \cap \sigma$ (or $G_i \cap \Sigma$) is homeomorphic to U_i . Then $G_i \cap \sigma$ (or $G_i \cap \Sigma$) is an f -d cap (or cap)-set for G_i , and making use of Lemma 10.1 there is an embedding $h_i: G_i \rightarrow V_i$ such that $U_i \subset h_i(G_i)$ and U_i is an f -d cap (or cap)-set for $h_i(G_i)$.

We note that each $h_i(G_i)$ is a topologically complete metric space and hence it must be a G_δ in V_i . Thus for each $i > 0$ we have $V_i \setminus h_i(G_i) = \bigcup_{n=1}^{\infty} A_i^n$, with each A_i^n closed in Z . We also note that for each i and n , $A_i^n \cap X = \emptyset$. This means that $A_i^n \cap h_j(G_j)$ is a Z -set in $h_j(G_j)$, for all n, i , and j . Now put $G'_i = h_i(G_i) \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} A_i^n$, and from Theorem 1 of [4] we know that each G'_i is homeomorphic to $h_i(G_i)$. It is also true that each U_i is an f -d cap (or cap)-set for G'_i .

We note that for each $j > 0$, $V_j \cap (\bigcup_{i=1}^{\infty} G'_i) = G'_j$. This means that $Y = \bigcup_{i=1}^{\infty} G'_i$ is an F -manifold. Since X has the f -d cap (or cap) locally in Y , we find that X is an f -d cap (or cap)-set for Y . $\square$

We can also prove a corresponding property for Q -manifolds.

COROLLARY 10.1. *If X is a σ (or Σ)-manifold, then there is a Q -manifold Y and an embedding $h: X \rightarrow Y$ such that $h(X)$ is an f-d cap (or cap)-set for Y .*

Proof. Let Z be an F -manifold and let $f: X \rightarrow Z$ be an embedding such that $f(X)$ is an f-d cap (or cap)-set for Z . Using Theorem 5.1 there is a Q -manifold Y and an embedding $g: Z \rightarrow Y$ such that $Y \setminus g(Z)$ is a cap-set for Y . Then Theorem 6.3 implies that $g \circ f(X)$ is an f-d cap (or cap)-set for Y . $\square$

We can easily use Theorem 10.2 to classify σ and Σ -manifolds according to homotopy type.

COROLLARY 10.2. *If X and Y are both σ -manifolds (or both Σ -manifolds) and are of the same homotopy type, then they are homeomorphic.*

Proof. Using Theorem 10.2 there are F -manifolds X' , Y' and embeddings $f: X \rightarrow X'$, $g: Y \rightarrow Y'$ such that $f(X)$ is an f-d cap (or cap)-set for X' and $g(Y)$ is an f-d cap (or cap)-set for Y' . Since X and Y have the same homotopy type, it follows from Corollary 8.2 that X' and Y' have the same homotopy type. Using Theorem 2.1 we have X' homeomorphic to Y' . Then Theorem 6.1 implies that $f(X)$ is homeomorphic to $g(Y)$. $\square$

We can also use these techniques to sharpen Corollary 8.1.

COROLLARY 10.3. *Let X be a σ -manifold and let Y be a Σ -manifold of the same homotopy type. Then $X \times I^\infty$ and Y are homeomorphic.*

Proof. Using the techniques of the proof of Corollary 10.2 there is an F -manifold Z and embeddings $f: X \rightarrow Z$, $g: Y \rightarrow Z$ such that $f(X)$ is an f-d cap-set for Z and $g(Y)$ is a cap-set for Z . Then Corollary 8.1 implies that $f(X) \times I^\infty$ and $g(Y)$ are homeomorphic. $\square$

Finally we sharpen Theorem 8.2.

COROLLARY 10.4. *Let X be a σ (or Σ)-manifold. Then there is a locally-finite simplicial complex K such that X is homeomorphic to $|K| \times \sigma$ (or $|K| \times \Sigma$).*

Proof. This is an immediate application of Theorem 10.2. $\square$

We now use Theorem 10.2 to establish open embedding theorems for σ and Σ -manifolds.

THEOREM 10.3. *Let X be a σ (or Σ)-manifold. Then X can be embedded as an open subset of σ (or Σ).*

Proof. Using Theorem 10.2 there is an embedding $h: X \rightarrow Y$, where Y is an F -manifold and $h(X)$ is an f-d cap (or cap)-set for Y . Using Henderson's open embedding theorem there is an open embedding $g: Y \rightarrow s$. Then $g \circ h(X)$ is an f-d cap (or cap)-set for $g(Y)$, which is an open subset of s . Also $g(Y) \cap \sigma$ (or $g(Y) \cap \Sigma$) is an open subset of σ (or Σ) and has the f-d cap (or cap) in $g(Y)$. By Theorem 6.1 we have $g \circ h(X)$ and $g(Y) \cap \sigma$ (or $g(Y) \cap \Sigma$) homeomorphic, which is what we wished to prove. $\square$

11. Extending homeomorphisms in σ and Σ -manifolds. The main result of this section is Theorem 11.1 in which we establish a homeomorphism extension theorem for σ and Σ -manifolds. Our strategy is to use techniques similar to those used in §10 to transfer the problem to an F -manifold and then apply Theorem 2.24.

We will need a number of preliminary results. The first two of these follow routinely from a close examination of Lavrent'ev's theorem on p. 335 of [16].

LEMMA 11.1 *Let X be a topological space, let Y be a topologically complete metric space, and let $f: A \rightarrow Y$ be a continuous function, where $A \subset X$. Then there is a G_δ -subset B of X such that $A \subset B \subset \bar{A}$ and an extension of f to a continuous function $g: B \rightarrow Y$.*

LEMMA 11.2. *Let X and Y be topologically complete metric spaces, let A be a subset of X , and let $f: A \rightarrow Y$ be an embedding. If B is a G_δ -subset of X and $g: B \rightarrow Y$ is a continuous extension of f , then there is a G_δ -subset B' of X such that $A \subset B' \subset \bar{A}$ and $g|B'$ is an embedding.*

The proof of our next lemma is elementary.

LEMMA 11.3. *Let X be a metric space and let Y be a compact metric space. Then the projection π_X is a closed map.*

We now combine these results to show that we can extend homotopies.

LEMMA 11.4. *Let X be a topologically complete metric space and let A be a subset of X . Let $h: A \times I \rightarrow Y$, where Y is a topologically complete metric space, be a homotopy such that $h|A \times \{0\}$ and $h|A \times \{1\}$ are embeddings. Then there is a G_δ -subset B of X such that $A \subset B \subset \bar{A}$ and an extension of h to a homotopy $H: B \times I \rightarrow Y$ such that $H|B \times \{0\}$ and $H|B \times \{1\}$ are embeddings.*

Proof. Using Lemma 11.1 there is a G_δ -subset C of $X \times I$ such that $A \times I \subset C \subset \text{Cl}(A \times I)$ (closure taken in $X \times I$) and an extension of h to a continuous function $f: C \rightarrow Y$. Using Lemma 11.3 we find that $\pi_X((X \times I) \setminus C) = D$ is an F_σ -subset of X . Then $B_1 = X \setminus D$ is a G_δ -subset of X such that $B_1 \times I \subset C$, which means that $g = f|B_1 \times I$ is an extension of h .

Applying Lemma 11.2 to $g|B_1 \times \{0\}$ and $g|B_1 \times \{1\}$ we find that there are G_δ -subsets B_2 and B_3 of X such that $A \subset B_2 \subset B_1$, $A \subset B_3 \subset B_1$, $g|B_2 \times \{0\}$ is an embedding, and $g|B_3 \times \{0\}$ is an embedding. Then put $B = B_2 \cap B_3$ and $H = g|B \times I$ to fulfill the conditions of the lemma. $\square$

If we examine the proof of Lemma 11.4 we can easily sharpen it.

COROLLARY 11.1. *Using the notation of Lemma 11.4 let $\mathcal{U}$ be an open cover of Y such that for each $x \in A$, $h(\{x\} \times I)$ is contained in some member of $\mathcal{U}$. Then we can construct H so that for each $x \in B$, $H(\{x\} \times I)$ is contained in some member of $\text{St}(\mathcal{U})$.*

We are now ready to prove our main result.

THEOREM 11.1. *Let X be a Σ (or σ)-manifold, let $\mathcal{U}$ be an open cover of X , let A be a separable metric space, and let $f: A \times I \rightarrow X$ be a homotopy such that $f|_{A \times \{0\}}$ and $f|_{A \times \{1\}}$ are embeddings, $f(A \times \{0\})$ and $f(A \times \{1\})$ are Z -sets in X and for each $x \in A$, $f(\{x\} \times I)$ is contained in some member of $\mathcal{U}$. Then the induced homeomorphism of $f(A \times \{0\})$ onto $f(A \times \{1\})$ can be extended to a homeomorphism of X onto itself which is limited by $\text{St}^{28}(\mathcal{U})$.*

Proof. Without loss of generality assume that $X \subset Y$, where Y is an F -manifold and X is an (f-d) cap-set for Y . Moreover we can assume that for each $U \in \mathcal{U}$, there is an open subset V_U of Y such that $V_U \cap X = U$ and $Y = \bigcup \{V_U \mid U \in \mathcal{U}\}$.

Let Z be a topologically complete metric space containing A and use Lemma 11.4 to get a G_δ -subset B of Z such that $A \subset B \subset \bar{A}$ and an extension of f to a homotopy $f_1: B \times I \rightarrow Y$ such that $f_1|_{B \times \{0\}}$ and $f_1|_{B \times \{1\}}$ are embeddings. Using Corollary 11.1 we can additionally assume that for each $x \in B$, $f_1(\{x\} \times I)$ is contained in some member of $\text{St}(\mathcal{V})$, where $\mathcal{V} = \{V_U \mid U \in \mathcal{U}\}$.

Now put $F_1 = \text{Cl}(f(A \times \{0\}))$ and $F_2 = \text{Cl}(f(A \times \{1\}))$, where the closure is taken in Y , and note that Lemma 5.10 implies that F_1 and F_2 are Z -sets in Y .

Since $Y \times B(I^\infty)$ is a cap-set for $Y \times I^\infty$, there is a homeomorphism $h_1: Y \rightarrow Y \times I^\infty$ such that $h_1(X) = Y \times B(I^\infty)$. Using Theorem 7.2 there is a homeomorphism $h_2: Y \times I^\infty \rightarrow Y \times I^\infty$ such that $\pi_{I^\infty} \circ h_2 \circ h_1(F_1 \cup F_2)$ has infinite deficiency and $h_2(Y \times B(I^\infty)) = Y \times B(I^\infty)$. It is obvious that by adjusting one of the coordinates in which $\pi_{I^\infty} \circ h_2 \circ h_1(F_1 \cup F_2)$ is deficient we can get a homeomorphism $h_3: Y \times I^\infty \rightarrow Y \times I^\infty$ such that $h_3 \circ h_2 \circ h_1(F_1 \cup F_2) \cap h_2 \circ h_1(F_1 \cup F_2) = \emptyset$, $h_3(Y \times B(I^\infty)) = Y \times B(I^\infty)$, and h_3 is limited by $h_2 \circ h_1(\mathcal{V})$. Then $h = (h_2 \circ h_1)^{-1} \circ h_3 \circ h_2 \circ h_1$ is a homeomorphism of Y onto itself such that $h(X) = X$, $h(F_2) \cap F_1 = \emptyset$, and h is limited by $\mathcal{V}$.

Using the construction of h_2 we can easily get a homotopy $F: Y \times I \rightarrow Y$ such that for each $x \in Y$, $F(x, 0) = x$, $F(x, 1) = h(x)$, and $F(\{x\} \times I)$ is contained in some member of $\mathcal{V}$. We can clearly use F to modify f_1 to a homotopy $f_2: B \times I \rightarrow Y$ such that $f_2|_{B \times \{0\}} = f_1|_{B \times \{0\}}$, $f_2|_{B \times \{1\}} = h \circ f_1|_{B \times \{1\}}$, and for each $x \in B$, $f_2(\{x\} \times I)$ is contained in some member of $\text{St}^2(\mathcal{V})$.

Now let g_1 be a homeomorphism of Y onto $Y \times I^\infty$ such that $\pi_{I^\infty} \circ g_1(F_1 \cup h(F_2))$ has infinite deficiency. Without loss of generality we may assume that $\tau_1 \circ \pi_{I^\infty} \circ g_1(F_1 \cup h(F_2)) = \{0\}$. Now apply Theorem 2.24 to obtain a homeomorphism g_2 of $Y \times I^\infty$ onto itself such that $\tau_1 \circ \pi_{I^\infty} \circ g_2 \circ g_1(F_1 \cup h(F_2)) = \{-1\}$.

Using this fact we can easily get a homotopy $G: Y \times I \rightarrow Y$ such that for $x \in Y$ and $t \in (0, 1)$, $G(x, 0) = G(x, 1) = x$, $G(x, t) \notin F_1 \cup h(F_2)$, and $G(\{x\} \times I)$ is contained in some member of $\mathcal{V}$. We can obviously use G to modify f_2 to a homotopy $f_3: B \times I \rightarrow Y$ such that $f_3|_{B \times \{0\}} = f_1|_{B \times \{0\}}$, $f_3|_{B \times \{1\}} = h \circ f_1|_{B \times \{1\}}$, $f_3(B \times (0, 1)) \cap (F_1 \cup h(F_2)) = \emptyset$, and for each $x \in B$, $f_3(\{x\} \times I)$ is contained in some member of $\text{St}^3(\mathcal{V})$.

Put $C_1 = f_3(B \times \{0\})$, which is a G_δ -subset of F_1 , and $C_2 = f_3(B \times \{1\})$, which is a G_δ -subset of $h(F_2)$. Let $F_1 \setminus C_1 = \bigcup_{n=1}^\infty A_n$ and $h(F_2) \setminus C_2 = \bigcup_{n=1}^\infty B_n$, where each A_n and each B_n is a closed subset of Y . It is then clear that $(\bigcup_{n=1}^\infty A_n) \cap (\bigcup_{n=1}^\infty B_n) = \emptyset$ and for each $n > 0$, A_n, B_n are Z -sets in Y .

Let $Y' = Y \setminus \bigcup_{n=1}^\infty (A_n \cup B_n)$, which is homeomorphic to Y by Theorem I of [4], and let $\mathcal{V}' = \{V \cap Y' \mid V \in \mathcal{V}\}$. We note that C_1 and C_2 are Z -sets in Y' and $f_3(B \times I) \subset Y'$. If we let φ be the homeomorphism of C_1 onto C_2 induced by f_3 , then there is a homotopy $\Phi: C_1 \times I \rightarrow Y$ such that for each $x \in C_1$, $\Phi(x, 0) = x$, $\Phi(x, 1) = \varphi(x)$, and $\Phi(\{x\} \times I)$ is contained in some member of $\text{St}^3(\mathcal{V}')$.

Define $\varphi_1: C_1 \cup C_2 \rightarrow C_1 \cup C_2$ by $\varphi_1|_{C_1} = \varphi$ and $\varphi_1|_{C_2} = \varphi^{-1}$. Clearly there is a homotopy $\Phi_1: (C_1 \cup C_2) \times I \rightarrow Y$ such that for each $x \in C_1 \cup C_2$, $\Phi_1(x, 0) = x$, $\Phi_1(x, 1) = \varphi_1(x)$, and $\Phi_1(\{x\} \times I)$ is contained in some member of $\text{St}^3(\mathcal{V}')$. Moreover, we note that $\varphi_1((C_1 \cup C_2) \cap X) = (C_1 \cup C_2) \cap X$.

Since $X \setminus (C_1 \cup C_2)$ is an (f-d) cap-set for Y' , there is a homeomorphism ψ_1 of Y' onto itself such that $\psi_1(X \setminus (C_1 \cup C_2)) = X$ and ψ_1 is limited by $\mathcal{V}'$. Using Theorem 2.24 there is a homeomorphism ψ_2 of Y' onto itself which extends the homeomorphism $\psi_1 \circ \varphi_1 \circ \psi_1^{-1}: \psi_1(C_1 \cup C_2) \rightarrow \psi_1(C_1 \cup C_2)$ and is limited by $\text{St}^{24}(\mathcal{V}')$. Using Theorem 6.1 there is a homeomorphism ψ_3 of Y' onto itself such that $\psi_3(\psi_2(X)) = X$, $\psi_3|_{\psi_1(C_1 \cup C_2)} = \text{id}$, and ψ_3 is limited by $\mathcal{V}'$. Then $\psi = \psi_1^{-1} \circ \psi_3 \circ \psi_2 \circ \psi_1$ is a homeomorphism of Y' onto itself such that $\psi(X) = X$, ψ extends φ , and ψ is limited by $\text{St}^{27}(\mathcal{V}')$.

We note now that $g = (h^{-1}|_X) \circ (\psi|_X)$ is a homeomorphism of X onto itself which fulfills our requirements. $\square$

By examining the proof of Theorem 11.1 we can state a stronger result.

COROLLARY 11.2. *Let the hypotheses be as in Theorem 11.1. Then there is an F -manifold Z containing X and an open cover $\mathcal{V}$ of Z such that $\{V \cap X \mid V \in \mathcal{V}\} = \mathcal{U}$ and for which the induced homeomorphism of $f(A \times \{0\})$ onto $f(A \times \{1\})$ can be extended to a homeomorphism of Z onto itself which takes X onto itself and is limited by $\text{St}^{28}(\mathcal{V})$.*

If we apply Theorem 11.1 to σ or Σ we can easily obtain the following homeomorphism extension theorem, where the homotopy takes place along straight line segments joining each point to its image.

COROLLARY 11.3. *Let $X = \sigma$ or Σ , let K_1 and K_2 be Z -sets in X , let $\varepsilon > 0$ be given, and let h be a homeomorphism of K_1 onto K_2 such that $d(h, \text{id}) < \varepsilon$. Then h can be extended to a homeomorphism H of X onto itself such that $d(H, \text{id}) < 29\varepsilon$.*

We remark that by exercising a bit more care the estimate $\text{St}^{28}(\mathcal{U})$ in Theorem 11.1 can be improved upon, however it does not seem worth the effort. In regard to Corollary 11.3 it is possible to give a proof independent of Theorem 11.1 and obtain the estimate 2ε , where Theorem 11.1 would be replaced by the homeomorphism extension theorem announced by Barit [11].

REFERENCES

1. R. D. Anderson, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519. MR **32** #8298.
2. ———, *Topological properties of the Hilbert cube and the infinite product of open intervals*, Trans. Amer. Math. Soc. **126** (1967), 200–216. MR **34** #5045.
3. ———, *On topological infinite deficiency*, Michigan Math. J. **14** (1967), 365–383. MR **35** #4893.
4. ———, *Strongly negligible sets in Fréchet manifolds*, Bull. Amer. Math. Soc. **75** (1969), 64–67. MR **38** #6634.
5. ———, *A characterization of apparent boundaries of the Hilbert cube*, Notices Amer. Math. Soc. **16** (1969), 429. Abstract #697–G17.
6. ———, *On sigma-compact subsets of infinite-dimensional spaces*, Trans. Amer. Math. Soc. (submitted).
7. R. D. Anderson and R. H. Bing, *A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **74** (1968), 771–792. MR **37** #5847.
8. R. D. Anderson and John D. McCharen, *On extending homeomorphisms to Fréchet manifolds*, Proc. Amer. Math. Soc. **25** (1970), 283–289.
9. R. D. Anderson and R. Schori, *Factors of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. **142** (1969), 315–330.
10. R. D. Anderson, David W. Henderson and James E. West, *Negligible subsets of infinite-dimensional manifolds*, Compositio Math. **21** (1969), 143–150.
11. William Barit, *Small extensions of small homeomorphisms*, Notices Amer. Math. Soc. **16** (1969), 295. Abstract #663–715.
12. C. Bessaga and A. Pełczyński, *Estimated extension theorem, homogeneous collections and skeletons, and their applications to topological classifications of linear metric spaces and convex sets*, Fund. Math. (submitted).
13. T. A. Chapman, *Infinite deficiency in Fréchet manifolds*, Trans. Amer. Math. Soc. **148** (1970), 137–146.
14. David W. Henderson, *Infinite-dimensional manifolds are open subsets of Hilbert space*, Bull. Amer. Math. Soc. **75** (1969), 759–762.
15. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Math. Series, vol. 4, Princeton Univ. Press, Princeton, N. J., 1941. MR **3**, 312.
16. K. Kuratowski, *Topologie*. Vol. 1: *Espaces métrisables, espaces complets*, 2nd ed., Monografie Mat., Tom 20, PWN, Warsaw, 1948. MR **10**, 389.
17. H. Toruńczyk, *Skeletonized sets in complete metric spaces and homeomorphisms of the Hilbert cube*, Bull. Acad. Polon. Sci. Cl. III **18** (1970), 119–126.
18. James E. West, *Infinite products which are Hilbert cubes*, Trans. Amer. Math. Soc. **150** (1970), 1–25.

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