JORDAN ALGEBRAS WITH MINIMUM CONDITION⁽¹⁾

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Abstract. Let J be a Jordan algebra with minimum condition on quadratic ideals over a field of characteristic not 2. We construct a maximal nil ideal R of J such that J/R is a direct sum of a finite number of ideals each of which is a simple Jordan algebra. R must have finite dimension if it is nilpotent and this is shown to be the case whenever J has "enough" connected primitive orthogonal idempotents.

Introduction. The first four sections of this paper deal with the construction of a maximal nil ideal for a Jordan algebra J with minimum condition on quadratic ideals over a field F of characteristic not 2. We call this ideal, R , the radical of J and show that J/R is semisimple in the sense of Jacobson [2].

We construct R by "building up" through the Peirce decomposition of J relative to a finite collection of primitive orthogonal idempotents. In order to facilitate computations, much of the construction is carried out under the assumption that J has an identity element. However, once R is constructed we show that this assumption is not really necessary.

In the last section we deal with questions of the nilpotence of R and use the Coordinatization Theorem as our basic tool. We show that R is nilpotent if J has enough connected primitive orthogonal idempotents. Also if R is nilpotent, then it must be finite dimensional.

1. Preliminaries. In this paper an algebra will be an algebra over a field F of characteristic not 2, which is not necessarily associative or of finite dimension. J will always denote a commutative Jordan algebra. In order to make this paper self-contained we now recall some definitions and results from [2]. If $x \in J$, U_x denotes the linear operator $2R_x^2 - R_{x^2}$ on J , and if $x, y, z \in J$ then $\{xyz\}$ will denote the trilinear product $xy \cdot z + yz \cdot x - xz \cdot y$.

$$U_x^n = U_{x^n}, \quad U_y U_x = U_x U_y U_x, \quad z U_{x+y} = z U_x + z U_y + 2\{xzy\}.$$

An element $0 \neq b \in J$ is called an absolute zero divisor of J if and only if $JU_b = 0$.

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A subspace Q of J is a quadratic ideal if and only if $JU_x \subset Q$ for all $x \in Q$.

If e is an idempotent of J , then the quadratic ideal JU_e is the Peirce space $J_1(e)$. Also any quadratic ideal (absolute zero divisor) of $J_1(e)$ or $J_0(e)$ is also a quadratic ideal (absolute zero divisor) of J .

If J has an identity element 1, then an element $x \in J$ is invertible if and only if U_x^n is invertible for any positive integer n . Also if $1 \in J$, then every quadratic ideal of J is also a subalgebra of J .

Finally, any homomorph of a quadratic ideal in J is a quadratic ideal of the homomorph of J .

LEMMA 1.1. *Let A be a nonzero quadratic ideal of J . If there exists an element $a \in A$ such that $A = AU_a$, then A contains an idempotent.*

Proof. Since $A = AU_a$ there is $c \in A$, such that $a = cU_a$. So $U_a = U_a U_c U_a$. If $z \in A$ we can find $w \in A$ with $z = wU_a = wU_a U_c U_a = zU_c U_a$, hence $U_c U_a$ is the identity map on A . In particular $c = cU_c U_a = c^3 U_a$. But then $z = zU_c U_a = zU_c^3 U_a = zU_a U_c^3 U_a^2 = zU_a U_c$. Therefore $U_a U_c = U_c U_a$ on A .

If $f = a^2 U_c$, then $zU_f = zU_a^2 U_c = zU_c U_a^2 U_a = z$. So U_f is the identity map on A . Hence $f = fU_f = f^3$. So $g = f^2 = (a^2 U_c)^2 = c^2 U_a^2 U_c$ is a nonzero idempotent of A .

We will use a special case of the following result in §3.

LEMMA 1.2. *Suppose J has an identity element 1 and that $b = b_1 + b_{12} + b_0$ is the decomposition of an absolute zero divisor b of J relative to an idempotent e . Then b_1 and b_0 are absolute zero divisors of J .*

Proof. Since $b_0 \in J_1(1-e)$ all we need show is that $b_1 \in J_1(e)$ is an absolute zero divisor of J . $0 = 1U_b = b^2 = b_1^2 + [b_{12}^2]_1 + 2(b_1 + b_0)b_{12} + b_0^2 + [b_{12}^2]_0$ so $0 = b_1^2 + [b_{12}^2]_1 = (b_1 + b_0)b_{12} = b_0^2 + [b_{12}^2]_0$.

If $x_1 \in J_1(e)$ then

$$\begin{aligned} 0 &= x_1 U_b = 2x_1 b \cdot b - x_1 b^2 = 2x_1 b \cdot b = 2[x_1(b_1 + b_{12} + b_0)](b_1 + b_{12} + b_0) \\ &= 2x_1 b_1 \cdot b_1 + 2[x_1 b_{12} \cdot b_{12}]_1 + 2[x_1 b_{12} \cdot b_1 + x_1 b_1 \cdot b_{12} + x_1 b_{12} \cdot b_0]_{12} + 2[x_1 b_{12} \cdot b_{12}]_0. \end{aligned}$$

But then $x_1 b_1 \cdot b_1 = -[x_1 b_{12} \cdot b_{12}]_1$ for all $x_1 \in J_1(e)$. In particular $eb_1 \cdot b_1 = b_1^2 = -[eb_{12} \cdot b_{12}]_1 = -\frac{1}{2}[b_{12}^2]_1$. So we have that $b_1^2 = 0$. Using the Jordan identity we get

$$\begin{aligned} 0 &= (e, b_{12}, x_1 b_{12}) + (x_1, b_{12}, eb_{12}) + (b_{12}, b_{12}, x_1 e) \\ &= \frac{1}{2}x_1 b_{12} \cdot b_{12} - [x_1 b_{12} \cdot b_{12}]_1 + \frac{1}{2}x_1 b_{12} \cdot b_{12} - \frac{1}{2}x_1 b_{12}^2 + x_1 b_{12}^2 - x_1 b_{12} \cdot b_{12} \\ &= -[x_1 b_{12} \cdot b_{12}]_1. \end{aligned}$$

Therefore $x_1 U_{b_1} = 2x_1 b_1 \cdot b_1 - x_1 b_1^2 = 0$ and b_1 is an absolute zero divisor of $J_1(e)$ hence also of J .

Henceforth, all Jordan algebras under consideration in this paper are assumed to have minimum condition on quadratic ideals. Such Jordan algebras equivalently have descending chain condition on quadratic ideals.

THEOREM 1.3. *J is nil or J contains an idempotent.*

Proof. Let $x \in J$ and consider the descending chain of quadratic ideals $J \supset JU_x \supset JU_{x^2} \supset \dots$. By assumption there must be a positive integer n so that $JU_{x^n} = JU_{x^{n+k}}$ for $k=1, 2, \dots$. Hence $JU_{x^n} = 0$ or JU_{x^n} is a quadratic ideal satisfying the hypothesis of Lemma 1.1. So either x is nilpotent or JU_{x^n} , and hence J , will contain an idempotent.

LEMMA 1.4. *If e is an idempotent of J then $J_1(e)$ contains a primitive idempotent of J .*

Proof. Clearly $J_1(e)$ has minimum condition on quadratic ideals since every quadratic ideal of $J_1(e)$ is also a quadratic ideal of J . Let $Q = JU_f = J_1(f)$, $f^2 = f \in J_1(e)$, be a minimal quadratic ideal of $J_1(e)$ determined by idempotents of $J_1(e)$. If f is not primitive, then Q contains an idempotent $f_1 \neq f$ and clearly $JU_{f_1} \subsetneq JU_f = Q$, which is contrary to the choice of Q .

THEOREM 1.5. *If J is not nil, then J contains a principal idempotent u where $u = \sum_{i=1}^n e_i$ where the e_i are primitive orthogonal idempotents of J .*

Proof. Since J is not nil we know that J contains an idempotent e ; which by Lemma 1.3 can be assumed to be primitive. Suppose that $e_1, e_2, \dots, e_k$ are primitive orthogonal idempotents of J . If $J_0(e_1 + \dots + e_k)$ contains an idempotent it contains a primitive one, e_{k+1} , hence $e_1, e_2, \dots, e_k, e_{k+1}$ are primitive orthogonal idempotents of J . One easily sees, however, that

$$J_0(e_1) \supset J_0(e_1 + e_2) \supset \dots \supset J_0(e_1 + e_2 + \dots + e_{k+1})$$

is a properly descending chain of quadratic ideals. Therefore this chain must come to an end after a finite number of steps, say n . So we let $u = \sum_{i=1}^n e_i$ and clearly u is principal since $J_0(u)$ contains no idempotents.

COROLLARY 1.6. *If J has an identity element 1 then $1 = \sum_{i=1}^n e_i$, where the e_i are primitive orthogonal idempotents.*

If J does not have an identity element then we may imbed J in $J' = F1 \oplus J$ in the usual way. We now prove

THEOREM 1.7. *$J' = F1 \oplus J$ also has minimum condition on quadratic ideals.*

Proof. Let $Q_1 \supset Q_2 \supset \dots \supset Q_n \supset$ be a properly descending chain of quadratic ideals of J' . We consider the descending chain of quadratic ideals

$$Q_1 \cap J \supset Q_2 \cap J \supset \dots \supset Q_n \cap J \supset \dots$$

of J . Without loss of generality we may assume that $Q_1 \cap J = Q_2 \cap J$ and that $Q_1, Q_2 \not\subset J$. Since $Q_1 \not\supseteq Q_2$, we know that there is an element $a \in J$ so that $1+a \in Q_1 \setminus Q_2$ and since $Q_2 \not\subset J$ there is an element $b \in J$ so that $1+b \in Q_2$. But then $(1+a)(1+b) = 1+a+b+ab \in Q_1$, hence $b+ab, a+ab \in Q_1 \cap J = Q_2 \cap J \supset Q_2$. But this implies that $1+a \in Q_2$, a contradiction. So with the possible exception of Q_1 , our chain is

in J , hence it has finite length and therefore J' has minimum condition on quadratic ideals.

2. J contains only one idempotent. Throughout this section we assume that $1 \in J$ and that 1 is the only idempotent of J . We show that every element in J is either invertible or nilpotent and that the nilpotent elements form an ideal.

THEOREM 2.1. *Every element of J is either invertible or nilpotent.*

Proof. By the proof of Theorem 1.3 we see that if $x \in J$ is not nilpotent then JU_x must contain an idempotent which must be 1 . Hence x is invertible.

THEOREM 2.2. *J has nilpotent elements if and only if J has absolute zero divisors.*

Proof. If J has absolute zero divisors then clearly J has nilpotent elements.

If J has no absolute zero divisors then by Theorem 2 of [2], J is either division or J contains an idempotent different from the identity. By our hypothesis, J must be division, hence contains no nilpotent elements.

We now let N be the collection of all nilpotent elements of J and we define

$$K = \{x \in J \mid x + N \subset N\}.$$

LEMMA 2.3. *$K \subset N$, K is a subspace of J , and $K \neq \{0\}$ whenever $N \neq \{0\}$.*

Proof. Clearly K is a subspace of J contained in N . By the previous lemma we know that if $N \neq \{0\}$, J contains an absolute zero divisor b . If $n \in N$ we know there is a positive integer $k > 1$ so that $n^k = 0$ and $n^{k-1} \neq 0$. Now

$$n^{k-1}U_{b+n} = n^{k-1}U_b + n^{k-1}U_n + 2\{bn^{k-1}n\} = 2[bn^{k-1} \cdot n - bn \cdot n^{k-1}] = 0$$

since $R_x^p R_x^q = R_x^q R_x^p$ in any Jordan algebra. Hence U_{b+n} is not invertible so $b+n$ is not invertible, so by Theorem 2.1, $b+n$ is nilpotent and $b \in K$.

LEMMA 2.4. *If $s \in J$ is invertible then $NU_s \subset N$.*

Proof. Since s is invertible we have that $JU_s = JU_s^{-1} = J$. If $n \in N$ and $J = JU_n U_s$ then $J = JU_s U_n U_s = JU_n U_s$. Therefore $J = JU_n$ and n is invertible a contradiction. So $nU_s \in N$.

LEMMA 2.5. *If $s \in J$ is invertible then $KU_s \subset K$.*

Proof. Let $k \in K$ and $n \in N$. Since s is invertible, U_s is invertible and so there exists $m \in N$ such that $mU_s = n$. Therefore $n + kU_s = mU_s + kU_s = (m+k)U_s \subset N$, hence $kU_s \in K$.

LEMMA 2.6. *K is a nil ideal of J .*

Proof. Let $k \in K$ and suppose first that $x \in J$ is invertible. If $1+x$ is also invertible then $kU_{1+x} = k + kU_x + 2\{1kx\} \in K$ and so $kx \in K$. If $1+x \in N$ then $1+(1+x)$ is invertible so again $kU_{1+(1+x)} = kU_2 + kU_x + 2\{2kx\} \in K$. But this also implies that $kx \in K$. Hence if $x \in J$ is invertible $Kx \subset K$.

If $x \in N$, then $1+x$ is invertible, hence $k(1+x) = k+kx \in K$ so again $kx \subset K$. But then $KJ \subset K$ and $K \subset N$, so K is a nil ideal of J .

THEOREM 2.7. N is an ideal of J .

Proof. If $N \neq \{0\}$ then $K \neq \{0\}$ is a nil ideal of J . Let M be a maximal nil ideal of J . Since homomorphic images of invertible (nilpotent) elements are invertible (nilpotent) in the homomorph, we see that the identity of J/M is the only idempotent of J/M , and so J/M satisfies the same conditions as J does. If $M \neq N$, then J/M contains nilpotent elements and therefore has a nontrivial nil ideal. But this is a contradiction to the choice of M , and hence $N=M$ is a nil ideal of J .

COROLLARY 2.8. J/N is a division algebra.

3. Peirce decomposition relative to two primitive orthogonal idempotents. In the Peirce decomposition $J_{11} + J_{12} + J_{22}$ of J relative to two primitive orthogonal idempotents e_1 and e_2 , it is clear that both J_{11} and J_{22} have the minimum condition on quadratic ideals and the only idempotents of J_{11} and J_{22} are their respective identity elements e_1 and e_2 . So the collection, N_i , of nilpotent elements of J_{ii} is an ideal of J_{ii} for $i=1, 2$. We show in this section that $N_1 + (N_1 + N_2)J_{12} + N_2$ is a nil ideal of J . The proof is by a series of computational lemmas, so throughout this section we assume that $n_i \in N_i$, all letters or expressions with subscripts 12 and i are assumed to be elements of J_{12} and J_{ii} respectively, $i=1, 2$.

LEMMA 3.1. $J_{ii}U_{y_{12}} \subset J_{jj}$, $i, j=1, 2$; $i \neq j$.

Proof. $0 = (x_i, y_{12}, y_{12}e_j) + (y_{12}, y_{12}, x_i e_j) + (e_j, y_{12}, y_{12}x_i) = x_i y_{12} \cdot y_{12} - [x_i y_{12} \cdot y_{12}]_j - \frac{1}{2} x_i y_{12}^2$. Hence $x_i U_{y_{12}} = 2x_i y_{12} \cdot y_{12} - x_i \cdot y_{12}^2 = 2[x_i y_{12} \cdot y_{12}]_j \in J_{jj}$.

LEMMA 3.2. $x_{12}^{2k+1} = 2x_{12}([x_{12}^2]_i)^k$, $i=1, 2$; $k=1, 2, \dots$

Proof. $x_{12} \cdot e_i x_{12}^2 = x_{12} e_i \cdot x_{12}^2$. Hence $x_{12}[x_{12}^2]_i = \frac{1}{2} x_{12}^3$, and we have $2x_{12}[x_{12}^2]_i = x_{12}^3$. If $x_{12}^{2k-1} = 2x_{12}([x_{12}^2]_i^{k-1})$ then $x_{12}^{2k+1} = x_{12}^{2k-1} x_{12}^2 = 2(x_{12}[x_{12}^2]_i^{k-1}) x_{12}^2 = 2x_{12}([x_{12}^2]_i^k)$. By induction we have our lemma.

COROLLARY 3.3. Every element in J_{12} is either invertible or nilpotent.

Proof. This follows easily from the preceding lemma since $x_{12}^2 \in J_{11} + J_{22}$.

LEMMA 3.4. $[(n_i y_{12})^2]_i \in N_i$ and hence $n_i y_{12}$ is nilpotent for $i=1, 2$.

Proof.

$$0 = 2(n_i, y_{12}, n_i y_{12}) + (y_{12}, y_{12}, n_i^2) = 2(n_i y_{12})^2 - 2n_i(n_i y_{12} \cdot y_{12}) + y_{12}^2 n_i^2 - n_i^2 y_{12} \cdot y_{12}.$$

But N_i is an ideal of J_{ii} and so $[(n_i y_{12})^2]_i \in N_i$ if and only if $[n_i^2 y_{12} \cdot y_{12}]_i \in N_i$. But $[n_i y_{12} \cdot y_{12}]_i = \frac{1}{2} n_i y_{12}^2$ for any $n_i \in N_i$ by the proof of Lemma 3.1, and so $[(n_i y_{12})^2]_i \in N_i$.

LEMMA 3.5. $n_i y_{12} \cdot y_{12} \in N_1 + N_2$, $i=1, 2$.

Proof. First suppose that y_{12} is invertible. If $n_i U_{y_{12}}$ is invertible in J_{jj} then

$$J_{jj} = J_{jj} U_{n_i U_{y_{12}}} = J_{jj} U_{y_{12}} U_{n_i} U_{y_{12}} \subset J_{ii} U_{n_i} U_{y_{12}} \subset N_i U_{y_{12}} \subset J_{jj}$$

and so $J_{jj} = N_i U_{y_{12}}$. Then $J_{jj} U_{y_{12}} = N_i U_{y_{12}}^2 = N_i U_{y_{12}^2} \subset N_i$. But then $e_j U_{y_{12}^2} = [y_{12}^2]_i \in N_i$ and thus y_{12} is nilpotent, a contradiction. So $n_i U_{y_{12}} = 2[n_i y_{12} \cdot y_{12}]_j \in N_j$ and thus $n_i y_{12} \cdot y_{12} \in N_1 + N_2$.

If y_{12} is nilpotent then $(n_i y_{12} + y_{12})^2 = (n_i y_{12})^2 + 2n_i y_{12} \cdot y_{12} + y_{12}^2 = (n_i y_{12})^2 + 2[n_i y_{12} \cdot y_{12}]_i + 2[n_i y_{12} \cdot y_{12}]_j + [y_{12}^2]_i + [y_{12}^2]_j$. But $[(n_i y_{12})^2]_i$, $[n_i y_{12} \cdot y_{12}]_i$, $[y_{12}^2]_i$ are all elements of N_i , so $n_i y_{12} + y_{12}$ is nilpotent and $n_i y_{12} \cdot y_{12} \in N_1 + N_2$.

LEMMA 3.6. (a) $[n_i y_{12} \cdot z_{12}]_j \in N_j$, $i, j = 1, 2$, $i \neq j$;

(b) $[n_i y_{12} \cdot z_{12}]_i + [n_i z_{12} \cdot y_{12}]_i \in N_i$.

Proof. Linearizing the result in Lemma 3.5 we see that $n_i y_{12} \cdot z_{12} + n_i z_{12} \cdot y_{12} \in N_1 + N_2$ and clearly (b) holds.

$$\begin{aligned} 0 &= (n_i, y_{12}, y_{12} e_i) + (e_i, y_{12}, n_i z_{12}) + (z_{12}, y_{12}, n_i e_i) \\ &= \frac{1}{2}(z_{12}, y_{12}, n_i) + (e_i, y_{12}, n_i z_{12}) \\ &= \frac{1}{2} n_i \cdot y_{12} z_{12} - \frac{1}{2} n_i y_{12} \cdot z_{12} + \frac{1}{2} n_i z_{12} \cdot y_{12} - [n_i z_{12} \cdot y_{12}]_i. \end{aligned}$$

Multiplying through by e_j we see that $[n_i y_{12} \cdot z_{12}]_j = [n_i z_{12} \cdot y_{12}]_j$ and hence

$$[n_i y_{12} \cdot z_{12}]_j \in N_j.$$

LEMMA 3.7. $n_i y_{12} \cdot z_{12} \in N_1 + N_2$, $i = 1, 2$.

Proof. We have already shown that $[n_i y_{12} \cdot z_{12}]_j \in N_j$ so we will now show that $[n_i y_{12} \cdot z_{12}]_i \in N_i$ by looking at $([n_i y_{12} \cdot z_{12}]_i)^2 = [(n_i y_{12} \cdot z_{12})^2]_i$.

$$\begin{aligned} 0 &= 2(n_i y_{12}, z_{12}, n_i y_{12} \cdot z_{12}) + (z_{12}, z_{12}, (n_i y_{12})^2) \\ &= 2(n_i y_{12} \cdot z_{12})^2 - 2(n_i y_{12})((n_i y_{12} \cdot z_{12}) z_{12}) + (z_{12}, z_{12}, (n_i y_{12})^2). \end{aligned}$$

By Lemmas 3.4 and 3.5 $(z_{12}, z_{12}, (n_i y_{12})^2) \in N_1 + N_2$, so $[(n_i y_{12} \cdot z_{12})^2]_i \in N_i$ if and only if $[(n_i y_{12})((n_i y_{12} \cdot z_{12}) z_{12})]_i \in N_i$. If we now let $w_{12} = n_i y_{12}$,

$$\begin{aligned} 0 &= 2(z_{12}, e_i, w_{12} z_{12}) + (w_{12}, e_i, z_{12}^2) \\ &= [w_{12} z_{12}]_j z_{12} - [w_{12} z_{12}]_i z_{12} - \frac{1}{2} w_{12} [z_{12}^2]_i + \frac{1}{2} w_{12} [z_{12}^2]_j. \end{aligned}$$

And so $[w_{12} z_{12}]_i z_{12} = [w_{12} z_{12}]_j z_{12} - \frac{1}{2} w_{12} [z_{12}^2]_i + \frac{1}{2} w_{12} [z_{12}^2]_j$. Hence

$$\begin{aligned} [w_{12}(w_{12} z_{12} \cdot z_{12})]_i &= [w_{12}(z_{12}[w_{12} z_{12}]_i + z_{12}[w_{12} z_{12}]_j)]_i \\ &= [w_{12}(z_{12}[w_{12} z_{12}]_j)]_i + \frac{1}{2} [w_{12} \cdot w_{12} [z_{12}^2]_i]_i - \frac{1}{2} [w_{12} \cdot w_{12} [z_{12}^2]_j]_i. \end{aligned}$$

Now $[w_{12} z_{12}]_j = [n_i y_{12} \cdot z_{12}]_j \in N_j$, therefore by Lemma 3.6(a), $[w_{12}(z_{12}[w_{12} z_{12}]_j)]_i \in N_i$. We now consider $[w_{12} \cdot w_{12} [z_{12}^2]_j]_i$ and $[w_{12} \cdot w_{12} [z_{12}^2]_i]_i$. Clearly

$$w_{12} \cdot w_{12} [z_{12}^2]_j = w_{12}(n_i y_{12} \cdot [z_{12}^2]_j) = w_{12}(n_i \cdot y_{12} [z_{12}^2]_j) = n_i y_{12} \cdot n_i x_{12}$$

with $x_{12} = y_{12} [z_{12}^2]_j$. By Lemma 3.4

$$[n_i(y_{12} + x_{12})^2] = (n_i y_{12})^2 + 2n_i y_{12} \cdot n_i x_{12} + (n_i x_{12})^2 \in N_1 + N_2$$

and so $[w_{12} \cdot w_{12}[z_{12}^2]_j]_i \in N_i$. Also $[w_{12} \cdot w_{12}[z_{12}^2]_i]_i$ is in N_i since again as in the proof of Lemma 3.1, $[w_{12} \cdot w_{12}[z_{12}^2]_i]_i = \frac{1}{2}w_{12}^2[z_{12}^2]_i$ which is in N_i because $w_{12}^2 \in N_i$. So finally we have proven that $n_i y_{12} \cdot z_{12} = [n_i y_{12} \cdot z_{12}]_i + [n_i y_{12} \cdot z_{12}]_j \in N_1 + N_2$.

We can now easily prove the following:

THEOREM 3.7. $N_1 + (N_1 + N_2)J_{12} + N_2$ is a proper nil ideal of J .

Proof. Clearly $N = N_1 + (N_1 + N_2)J_{12} + N_2$ is a proper ideal of J by using the well-known multiplication properties of the Peirce subspaces J_{ij} of J , and the fact that N_i is an ideal of J_{ii} .

Using the proof of Theorem 1.3 we see that N is either nil or N contains an idempotent $e = n_1 + x_{12} + n_2$. But then since $e^2 = e$ we have $\frac{1}{2}x_{12} = x_{12}(n_1 + n_2)$. Letting $n = n_1 + n_2$ we see that n and hence R_n are nilpotent where $xR_n = xn$. But clearly then $x_{12} = 0$ which contradicts e being an idempotent. Hence N is a nil ideal of J .

We will need the next two lemmas in §4.

LEMMA 3.8. Suppose that J_{11} and J_{22} are division algebras. Then $b_{12} \in J_{12}$ is an absolute zero divisor of J if and only if $b_{12}J_{12} = 0$.

Proof. $0 = 2(y_{12}, b_{12}, y_{12}b_{12}) + (b_{12}, b_{12}, y_{12}^2) = 2(b_{12}y_{12})^2 - 2(y_{12}b_{12} \cdot b_{12})y_{12} + b_{12}^2y_{12}^2 - y_{12}^2b_{12} \cdot b_{12} = 2(b_{12}y_{12})^2$. But $b_{12}y_{12} \in J_{11} + J_{22}$ and so $b_{12}y_{12} = 0$, i.e. $b_{12}J_{12} = 0$.

If $c_{12}J_{12} = 0$, then $xU_{c_{12}} = 2xc_{12} \cdot c_{12} - xc_{12}^2 = 2(x_1 + x_{12} + x_2)c_{12} \cdot c_{12} = 2x_1c_{12} \cdot c_{12} + 2x_{12}c_{12} \cdot c_{12} + x_2c_{12} \cdot c_{12} = 0$. Since $x_i c_{12} \in J_{12}$, $i = 1, 2$. So c_{12} is an absolute zero divisor of J .

LEMMA 3.9. If both J_{11} and J_{22} are division algebras, then the collection B_{12} of absolute zero divisors of J is an ideal of J .

Proof. By the previous lemma it is clear that B_{12} is a subspace of J_{12} . If $x_i \in J_{ii}$, $i = 1, 2$ and $b_{12} \in B_{12}$, then $0 = (x_i, b_{12}, y_{12}e_j) + (y_{12}, b_{12}, e_jx_i) + (e_j, b_{12}, y_{12}x_i) = \frac{1}{2}x_i b_{12} \cdot y_{12}$. Hence $x_i b_{12} \in B_{12}$ and $J_{ii}B_{12} \subset B_{12}$ which implies that B_{12} is an ideal of J .

4. The Peirce decomposition in general. If J is not nil, we recall that J must contain a principal idempotent u which is the sum of finite number of primitive orthogonal idempotents $e_1, \dots, e_n$. We show that the $N_i = \{x \in J_{ii} \mid x \text{ is nilpotent}\}$ generate an ideal N so that the Peirce one spaces in J/N are division algebras. If J/N has absolute zero divisors then we show there is a nil ideal R of J with $N \subset R$, such that J/R has no absolute zero divisors, and J/R is semisimple.

LEMMA 4.1. If N_i is the collection of nilpotent elements of J_{ii} in the Peirce decomposition of J then

- (1) $N_i J_{ij} \cdot J_{ij} \subset N_i + N_j$, $N_i J_{ij} \cdot J_{ii} \subset N_i J_{ij}$, $N_i J_{ij} \cdot J_{jj} \subset N_i J_{ij}$;
- (2) $N_i J_{ij} \cdot J_{jk} \subset N_i J_{ik}$, $i, j, k \neq$;
- (3) $N_i J_{ij} \cdot J_{ik} \subset N_i J_{ik} \cdot J_{ij}$, $i, j, k \neq$;
- (4) $(N_i J_{ij} \cdot J_{ik})J_{ik} \subset N_i J_{ij}$, $i, j, k \neq$;

- (5) $(N_i J_{ij} \cdot J_{ik}) J_{jk} \subset N_{jj} + N_{kk}$, $i, j, k \neq$;
 (6) $(N_i J_{ij} \cdot J_{ik}) J_{kk} \subset N_i J_{ij} \cdot J_{ik}$, $i, j, k \neq$;
 (7) $(N_i J_{ij} \cdot J_{ik}) J_{kl} \subset N_i J_{ij} \cdot J_{il}$, $i, j, k, l \neq$.

Using this lemma we easily prove

THEOREM 4.2. *If $J = \sum_{i=1}^n J_{ij}$ is the Peirce decomposition of J relative to the primitive orthogonal idempotents $e_1, \dots, e_n$, where $1 = \sum_{i=1}^n e_i$ and $N_i = \{x \in J_{ii} \mid x \text{ is nilpotent}\}$, then $N = \sum N_i + \sum N_i J_{ij} + \sum N_i J_{ij} \cdot J_{ik}$ is a proper ideal of J .*

If we let φ_i be the restriction of the natural homomorphism $\varphi: J \rightarrow J/N$ to J_{ii} we set that $\varphi_i: J_{ii} \rightarrow J_{ii}/N_i$ and so $\varphi_i(J_{ii})$ is a division algebra. J/N may have absolute zero divisors. However, they are easily handled by the following lemma and theorem.

LEMMA 4.3. *If $J = \sum J_{ij}$ is the Peirce decomposition of J relative to the orthogonal idempotents $e_1, \dots, e_n$, where $1 = \sum e_i$ is the identity element of J , and if J_{ii} is a division algebra for $i = 1, 2, \dots, n$ then if b is an absolute zero divisor of J , $b = \sum_{i < j} b_{ij}$ where each $b_{ij} \in J_{ij}$ is an absolute zero divisor of J .*

Proof. Let $b = b_1 + b_{12} + b_0$ be the decomposition of b relative to e_i . Then by Lemma 1.2, b_1 is an absolute zero divisor of J_{ii} and so $b_1 = 0$. Hence $b = \sum_{i < j} b_{ij}$. If $f = e_i + e_j$, and if $b = c_1 + c_{12} + c_0$ is the decomposition of b relative to f we know that c_1 is an absolute zero divisor of J , $c_1 \in J_1(f) = J_{ii} + J_{ij} + J_{jj}$. But the only component of b in $J_{ii} + J_{ij} + J_{jj}$ is b_{ij} so $b_{ij} = c_1$ is an absolute zero divisor of J .

THEOREM 4.4. *If J_{ii} is a division algebra for $i = 1, \dots, n$ in the Peirce decomposition of J relative to the orthogonal idempotents $e_1, \dots, e_n$ with $1 = \sum e_i$, then*

- (1) $A_0 = \sum_{i < j} B_{ij}$ is an ideal of J where $B_{ij} = \{x \in J_{ij} \mid x J_{ij} = 0\}$,
 (2) J/A_0 has no absolute zero divisors.

Proof. We will show that $b_{ij} y_{jk} \in B_{ik}$ where $b_{ij} \in B_{ij}$ and $y_{jk} \in J_{jk}$. By Lemma 3.8, b_{ij} is an absolute zero divisor of J , so $0 = 2(y_{jk}, b_{ij}, y_{jk} b_{ij}) + (b_{ij}, b_{ij}, y_{jk}^2) = 2(b_{ij} y_{jk})^2$, also $0 = (y_{jk}, b_{ij}, x_{ik} e_i) + (e_i, b_{ij}, y_{jk} x_{ik}) + (x_{ik}, b_{ij}, y_{jk} e_i) = \frac{1}{2}(b_{ij} y_{jk} \cdot x_{ik} - b_{ij} x_{ik} \cdot y_{jk})$ so $b_{ij} y_{jk} \cdot x_{ik} = b_{ij} x_{ik} \cdot y_{jk}$. By comparing components we see that $b_{ij} y_{jk} \cdot x_{ik} = [b_{ij} y_{jk} \cdot x_{ik}]_k$. If we let $z_{ik} = b_{ij} y_{jk}$, then $z_{ik}^2 = 0$ and $z_{ik} x_{ik} = [z_{ik} x_{ik}]_k \in J_{kk}$.

$$0 = 2(z_{ik}, e_k, x_{ik} z_{ik}) + (x_{ik}, e_k, z_{ik}^2) = z_{ik} \cdot x_{ik} z_{ik} - [2z_{ik} x_{ik}]_k z_{ik} = -z_{ik} x_{ik} \cdot z_{ik}.$$

So $x_{ik} U_{z_{ik}} = 0$. If $x_{ii} \in J_{ii}$, then $x_{ii} U_{z_{ik}} = w_{kk} \in J_{kk}$. But $w_{kk}^2 = (x_{ii} U_{z_{ik}})^2 = z_{ik}^2 U_{x_{ii}} U_{z_{ik}} = 0$. But J_{kk} is division, hence $w_{kk} = 0$. So $J_{ii} U_{z_{ik}} = 0$. In the same way $J_{kk} U_{z_{ik}} = 0$, and so $z_{ik} = b_{ij} y_{jk}$ is an absolute zero divisor of $J_{ii} + J_{ik} + J_{kk}$. Hence $B_{ij} J_{jk} \subset B_{ik}$, and $A_0 = \sum_{i < j} B_{ij}$ is a proper ideal of J .

Let α_{ij} be the restriction of the natural map $J \rightarrow J/A_0$ to $J_1(e_i + e_j) = J_{ii} + J_{ij} + J_{jj}$. So $\alpha_{ij}: J_1(e_i + e_j) \rightarrow J_1(e_i + e_j)/B_{ij}$. Clearly $\alpha_{ij}(J_{ii})$ and $\alpha_{ij}(J_{jj})$ are division algebras so if $\alpha_{ij}(J_1(e_i + e_j))$ contains an absolute zero divisor $c_{ij} + B_{ij}$, we must have that $(c_{ij} + B_{ij})(J_{ij} + B_{ij}) = c_{ij} J_{ij} \subset B_{ij}$. But then $c_{ij} J_{ij} = 0$ and $c_{ij} \in B_{ij}$ and $\alpha_{ij}(J_1(e_i + e_j))$ cannot contain any absolute zero divisors. So by Lemma 4.3 J/A_0 contains no absolute zero divisors.

COROLLARY 4.5. *The ideal A_0 in Theorem 4.3 is finite dimensional.*

Proof. $A_0 = \sum_{i < j} B_{ij}$ and every element of B_{ij} is an absolute zero divisor of J . Hence every subspace of B_{ij} is a quadratic ideal of J , and so B_{ij} has finite dimension $i, j = 1, 2, \dots, n, i < j$. Therefore A_0 has finite dimension.

THEOREM 4.6. *Let J have identity element 1. Then there exists an ideal R of J so that*

- (1) R is a maximal nil ideal of J ,
- (2) J/R has no absolute zero divisors,
- (3) J/R is the direct sum of a finite number of ideals which are simple Jordan algebras.

Proof. Let S be any ideal of J such that J/S has no absolute zero divisors. Then the ideal N of Theorem 4.2 must be contained in S , since if not, J/S would contain absolute zero divisors.

J/N clearly satisfies the hypothesis of Theorem 4.4, so if $\bar{J} = J/N$ contains any absolute zero divisors it has an ideal $\bar{A}_0$ such that $\bar{J}/\bar{A}_0$ contains no absolute zero divisors. Let R be the complete inverse image $A_0 + N$ of $\bar{A}_0$ under the mapping $J \rightarrow J/N$. Clearly $R \subset S$. Now J/R satisfies axioms (i), (ii), and (iii) of [2] and hence is a direct sum of ideals which are simple Jordan algebras also satisfying axioms (i), (ii), and (iii). If $R \neq S$ then S must contain an idempotent. If S contains no idempotents then $S = R$. Hence R is a nil ideal (again by the proof of Theorem 1.3) which is clearly maximal.

DEFINITION. The ideal R is called the radical of J .

We now will consider the situation in which J does not contain an identity element.

THEOREM 4.7. *If J is not nil, then there exists an ideal R of J such that*

- (1) R is a maximal nil ideal,
- (2) J/R has no absolute zero divisors,
- (3) J/R is the direct sum of a finite number of ideals which are simple Jordan algebras,
- (4) J/R has an identity element.

Proof. By Theorem 1.5 we know that J contains a principal idempotent $u = \sum_{i=1}^n e_i$ where the e_i are primitive orthogonal idempotents of J . So $J = J_1(u) + J_{1/2}(u) + J_0(u)$ and $J_0(u)$ is nil. Let $J' = F1 \oplus J$ be the algebra obtained by adjoining an identity element 1 to J . If $g = 1 - u$, then we know (see [1]): $gu = 0$, $J'_1(u) = J'_0(g) = J_1(u)$, $J'_{1/2}(u) = J_{1/2}(u)$; $J'_0(u) = J'_1(g) = J_0(u) + Fg$, and clearly g is primitive. If R' is the radical of J' , then by the construction of R' we know that $J_0(u) \subset R'$.

Now $J' = \sum_{i \leq j \leq 1} J_{ij} + \sum_{i=1}^n J_{i0} + J_{00} + Fg$ where

$$J_{ii} = \{x \mid xe_i = x\}, \quad J_{ij} = J_{1/2}(e_i) \cap J_{1/2}(e_j), \quad i, j = 1, 2, \dots, n, \quad i \leq j;$$

$$J_{i0} = J_{1/2}(e_i) \cap J_{1/2}(u) = J_{1/2}(e_i) \cap J_{1/2}(g), \quad J_{00} = J_0(u).$$

Note that $J_{00} + Fg = J_1(g)$.

Since J'/R' contains no absolute zero divisors we see that the image of $J_{i1} + J_{i0} + J_1(g)$ is a Jordan algebra with identity element being the sum of two primitive orthogonal idempotents whose one spaces are division. Since every element in J_{i0} is clearly nilpotent, we see that the image of J_{i0} is 0. So if we let $R = J \cap R'$, we see that J/R contains no absolute zero divisors, and that J/R has an identity element (the image of u). Hence our theorem.

COROLLARY 4.8. $\text{Rad } J' = \text{Rad } (F1 \oplus J) = \text{Rad } J$.

5. Properties of the radical. If A is an algebra over a field of characteristic not 2, then we may define an algebra A^+ having the same vector space as A but with multiplication " $\circ$ " defined by $x \circ y = \frac{1}{2}(xy + yx)$. If A has an involution then $H(A)$, the collection of symmetric elements of A , forms a subalgebra of A^+ . No confusion should arise if we denote this subalgebra also by the symbol $H(A)$.

We use Jacobson's Coordinatization Theorem and some results on isotopy for Jordan algebras as tools in this section. See [3].

LEMMA 5.1. *Let D be an algebra with identity element 1 over F and let D have an involution " $-$ " so that $H(D)$ is in the nucleus of D . Also let L be a left ideal of D . Then*

$$Q = \{ |a_{ij}| \mid a_{11} \in H(D), a_{12} \in L, a_{21} = \bar{a}_{12}, b_{22} \in H(D) \cap L, a_{ij} = 0 \text{ otherwise} \}$$

is a quadratic ideal of the Jordan algebra $H(D_n)$, $n > 1$, where D is alternative if $n = 2, 3$, and associative if $n \geq 4$.

Proof. It is well known [3] that $H(D_n)$ is a Jordan algebra when D is alternative if $n = 2, 3$ or D is associative if $n \geq 4$. Let $J = H(D_n)$, and let E_{ij} be the matrix units of D_n . $E_{11}, E_{22}, \dots, E_{nn}$ are orthogonal idempotents of J such that $I = \sum_{i=1}^n E_{ii}$. Clearly Q is isomorphic to the subalgebra

$$\left\{ \begin{vmatrix} a & x \\ \bar{x} & b \end{vmatrix} \mid a \in H(D), x \in L, b \in H(D) \cap L \right\}$$

of $H(D_2)$. We identify Q with this subalgebra and let

$$A = \begin{vmatrix} a & x \\ \bar{x} & b \end{vmatrix} \in Q \quad \text{and} \quad B = \begin{vmatrix} c & y \\ \bar{y} & d \end{vmatrix} \in H(D_2).$$

One easily sees by computation that $BU_A = ABA \in H(D_2)$. Also $(ABA)_{12} = acx + x\bar{y}x + ayb + xdb \in L$ and $(ABA)_{22} = \bar{x}cx + b\bar{y}x + \bar{x}yb + bdb \in H(D) \cap L$. Hence Q is a quadratic ideal of $H(D_2)$, hence of $J_1(E_{11} + E_{22})$, and so finally of $H(D_n)$.

THEOREM 5.2. *Let $1 = \sum_{i=1}^n e_i$ in J where the e_i are primitive orthogonal idempotents. Suppose that $n \geq 3$ and the e_i are connected. Then $R = \text{Rad } J$ is nilpotent.*

Proof. Since $n \geq 3$ and the e_i are connected, the Coordinatization Theorem [3] tells us that there exists an algebra D with an identity element 1 and an involution " $-$ " such that D is associative if $n \geq 4$, and is alternative with its selfadjoint elements in the nucleus if $n \geq 3$, so that J is isomorphic to $H(D_n, \gamma)$.

First of all we assume that $\gamma=1$, i.e. $H(D_n, \gamma)=H(D_n)$. Let $L_1 \supset L_2 \supset \dots \supset L_n \supset \dots$ be a descending chain of left ideals of D . Corresponding to each L_k we form a quadratic ideal:

$$Q_k = \{ |a_{ij}| \mid a_{11} \in H(D), a_{12} \in L_k, a_{21} = \bar{a}_{12}, a_{22} \in L_k \cap H(D), a_{ij} = 0 \text{ otherwise} \}$$

of $H(D_n)$. So we have $Q_1 \supset Q_2 \supset \dots \supset Q_n$ is a descending chain of quadratic ideals of $H(D_n)$. Since J has minimum condition on quadratic ideals, $H(D_n)$ does, hence D has minimum condition on left ideals.

If D is associative, then the Jacobson radical, $\text{Rad } D$, is nilpotent. Hence $\text{Rad } (D_n) = (\text{Rad } D)_n$ is nilpotent.

If D is alternative, then in [8] K. A. Ževlakov has shown that the Smiley radical [7], $\text{Rad } D$, is nilpotent if D is of characteristic not 2 or 3. Using results of M. Slater [6], this is true without the restriction of characteristic not 3. So when D is alternative $(\text{Rad } D)_n$ is an ideal of D_n and is certainly nilpotent since $(\text{Rad } D)^2$ is an ideal of D properly contained in $\text{Rad } D$.

Since $J \cong H(D_n)$ we have that $J_{ii} \cong H(D)$. If $H(D)/(H(D) \cap \text{Rad } D)$ contains nilpotent elements it must also contain absolute zero divisors, i.e. there is an element $b \in H(D)$ so that $aU_b \in H(D) \cap \text{Rad } D$ for all $a \in H(D)$. Suppose such an element $b \in H(D)$ exists, and let $x=h+k$ be an arbitrary element of D where $h \in H(D)$ and $\bar{k} = -k$. Then $(xb)^3 = (h+k)bhb(h+k)b + (h+k)bkbhb + (h+k)b(kbk)b$ is in $\text{Rad } D$ since bhb and $b(kbk)b$ are. Therefore xb is nilpotent for all x in D , and thus Db is a nil left ideal of D since b is in the nucleus of D . Hence b must be in $\text{Rad } D$ and so finally $H(D)/(H(D) \cap \text{Rad } D)$ contains no absolute zero divisors. So the isomorph of $N_i = \{x \in J_{ii} \mid x \text{ is nilpotent}\}$ is contained in $H(D) \cap \text{Rad } D$. Therefore the isomorph of the ideal N of Theorem 4.2 is contained in

$$H(D_n \cap (\text{Rad } D)_n).$$

Hence N is nilpotent. Now $R = \text{Rad } J$ is $A_0 + N$. But R is nil and $\bar{A}_0 + A_0 + N/N$ is finite dimensional, hence $\bar{A}_0$ is nilpotent and therefore R is also.

If J is not isomorphic to $H(D_n)$ but to $H(D_n, \gamma)$, $\gamma \neq 1$, then we know that the isomorph $\mathcal{J}$ of R in $H(D_n, \gamma)$ is an ideal not only of $H(D_n, \gamma)$ but also of $H(D_n)$. In fact it is precisely the radical of $H(D_n)$. So the subalgebra of the enveloping algebra of $H(D_n)$ generated by $\mathcal{J}$ is nilpotent. But this subalgebra is identical with the subalgebra generated by $\mathcal{J}$ in the enveloping algebra of $H(D_n, \gamma)$. Therefore $\mathcal{J}$ is nilpotent in $H(D_n, \gamma)$ and hence R is nilpotent.

THEOREM 5.3. *Suppose N is an ideal of J , such that $N^3=0$. Then N has finite dimension.*

Proof. If $N^2=0$ then every element of N is an absolute zero divisor and so every subspace of N is a quadratic ideal and so N has finite dimension.

If $N^2 \neq 0$, we consider the ideal $N^2 + N^2J$ of J which is contained in N . $N^3=0$ implies that every element of N^2 is an absolute zero divisor and so N^2 has finite dimension since every subspace of N^2 is a quadratic ideal of J .

Now let $a, x \in J$ and let $n \in N^2$. Using the Jordan identity we have $0 = 2(n, a, na) + (a, a, n^2) = 2(na)^2$ since $na \cdot a \in N$. Hence $(na)^2 = 0$. Linearizing, we see that $na \cdot nb = 0$ for any $b \in J$. Also $0 = (x, na, na) + (a, na, nx) + (n, na, ax) = (x \cdot na)na + (na \cdot a)nx$. Again $0 = (n, x, na \cdot a) + (a, x, na \cdot n) + (na, x, na) = (na \cdot a)nx$. Hence $(x \cdot na)na = 0$, so we finally have $xU_{na} = 2(x \cdot na)na - x(na)^2 = 0$ for all $n \in N^2$ and for all $x, a \in J$. Let $n_1, n_2, \dots, n_k$ be a basis for N^2 over F . Then $N^2J = n_1J + n_2J + \dots + n_kJ$. Every element in n_iJ is an absolute zero divisor, and so every subspace of n_iJ is a quadratic ideal. Hence n_iJ is finite dimensional, $i = 1, 2, \dots, k$, and so $N^2 + N^2J$ has finite dimension.

If $N = N^2 + N^2J$ we are done. If $N \neq N^2 + N^2J$ then under the natural homomorphism $J \rightarrow J/(N^2 + N^2J)$, N goes onto $\bar{N} = N/(N^2 + N^2J)$. But then $\bar{N}^2 = \bar{0}$. Hence again $\bar{N} = N + N^2 + N^2J$ has finite dimension and thus N has finite dimension.

THEOREM 5.4. *If N is a nilpotent ideal of J , then N has finite dimension.*

Proof. For any ideal N we construct the descending chain of ideals $M_1 \supset M_2 \supset \dots$ by defining $M_1 = N$, and $M_{k+1} = M_k^3$, $k = 1, 2, \dots$. Since in our case N is nilpotent, this chain properly descends and has finite length, say t . If $t = 2$, i.e. $M_2 = N^3 = 0$, we are done by the preceding theorem. Assume that the theorem is true for all nilpotent ideals with chains of length less than t . In our chain $M_{t-1}^3 = M_t = 0$ hence M_{t-1} has finite dimension. Under the natural homomorphism $J \rightarrow J/M_{t-1}$, N goes to $\bar{N} = N/M_{t-1}$. But the chain for $\bar{N}$ in J/M_{t-1} has length less than t , so by assumption $\bar{N} = N + M_{t-1}$ has finite dimension. Therefore N has finite dimension and we have our theorem.

COROLLARY 5.5. *If $1 = \sum_{i=1}^n e_i$ is the identity element of J and the e_i are connected with $n \geq 3$, then $R = \text{Rad } J$ has finite dimension.*

THEOREM 5.6. *Let $J = J_{11} + J_{12} + J_{22}$ be the Peirce decomposition of a Jordan algebra J relative to the orthogonal idempotents e_1 and e_2 . Suppose that $J_{12}^2 \subset N_1 + N_2$ where N_i is a nilpotent ideal of J_{ii} , $i = 1, 2$. Then $K = N_1 + J_{12} + N_2$ is a nilpotent ideal of J .*

Proof. Clearly K is a subalgebra of J . Also $B = N_1 + N_2$ is a finite-dimensional subalgebra of J . Therefore if B^* denotes the subalgebra of the multiplication algebra generated by the right multiplications by elements of B , B^* is nilpotent (see [5, p. 95]). If $J_{12} = BJ_{12}$, then $J_{12} = B(B(\dots(BJ_{12})\dots)) = (B^*)^k J_{12}$ for each k and so $J_{12} = 0$ and in this case K is a nilpotent ideal of J .

If BJ_{12} is strictly contained in J_{12} , we see that $K^2 \subset B + BJ_{12} \subsetneq K$. Define $K^{(1)} = K_{(1)} = K$, and $K^{(j+1)} = [K^{(j)}]^2$, $K_{(j+1)} = [K_{(j)}]^3$. Then clearly $K_{(j)} \subset K^{(j)}$ and $K^{(j)} \subset B + (B^*)^{(j-1)} J_{12} \cdot K$. K is a solvable ideal and there is an ideal $I \neq 0$ of J contained in K so that $I^3 = 0$. Therefore by an argument similar to that used in the proof of Theorem 5.4 we see that K is finite dimensional and hence nilpotent.

COROLLARY 5.7. *Let $1 = \sum_{i=1}^n e_i$ with the e_i primitive orthogonal idempotents, and let $P_1, P_2, \dots, P_k$ be the partition of $\{e_1, \dots, e_n\}$ determined by the relation "connected." Then if $|P_i| \geq 3$, $i = 1, \dots, k$, the radical of J is nilpotent.*

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