MAXIMAL ORDERS OVER REGULAR LOCAL RINGS⁽¹⁾

BY
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Abstract. In this paper various sufficient conditions are given for the maximality of an R -order in a finite-dimensional central simple K -algebra, where R is a regular local ring whose quotient field is K . Stronger results are obtained when we assume the dimension of R to be three. This work depends upon earlier results of this author [5] for regular local rings of dimension two, and the fundamental work of Auslander and Goldman [1] for dimension one.

Introduction. Let $(R, \mathfrak{M})$ be a regular local ring with maximal ideal $\mathfrak{M}$, and K the quotient field of R . Let Σ be a central simple K -algebra, finite dimensional over K , and Λ an R -order in Σ . Under what conditions is Λ maximal? Auslander and Goldman [1] have shown that if $\text{gl.dim. } R=1$ then Λ is maximal if and only if $\text{gl.dim. } \Lambda=1$ and Λ is quasi-local (i.e., Λ has a unique maximal two-sided ideal, necessarily $\text{Rad } \Lambda$, the Jacobson radical of Λ). In [5] this author has shown that when $\text{gl.dim. } R=2$, a sufficient condition for maximality is that (a) $\text{gl.dim. } \Lambda < \infty$ and (b) Λ is quasi-local [5, Theorem 5.4]. (Under these circumstances $\text{gl.dim. } \Lambda=2$.) Examples were given to show that neither (a) nor (b) is necessary. The natural generalization of this theorem is the following:

If $\text{gl.dim. } R=n$, $\text{gl.dim. } \Lambda < \infty$, and Λ is quasi-local, then Λ is maximal. (*Note.* These conditions ensure that $\text{gl.dim. } \Lambda=n$ and Λ is R -free [5, Corollary 2.17].)

In this paper we prove a weaker result. Namely, we prove the above statement under the additional hypothesis that Λ is contained in an R -free maximal order. It is still an open question whether Σ must contain *any* R -free maximal order (see [1, p. 20]). We obtain a slightly stronger result in dimension three. There we show that if Σ does contain an R -free maximal order Γ (not necessarily containing Λ) then Λ is isomorphic to Γ . Technical difficulties prevent us from extending this result to higher dimensions.

Two other theorems are proved for dimension three which do not hypothesize the existence of R -free maximal orders and which are therefore of more practical value in determining the maximality of a given Λ . The first requires instead that $\mathfrak{M} \not\subseteq N^2$, where $N = \text{Rad } \Lambda$. The second requires the existence of an element $x \in \mathfrak{M} - \mathfrak{M}^2$ such that the center of $\Lambda/x\Lambda$ is R/x and $\text{gl.dim. } \Lambda/x\Lambda < \infty$. An example

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is given in which the second, but not the first theorem, is applicable. Both these theorems follow from a more general one, valid for regular local rings of any dimension: Suppose Λ is an R -reflexive order and $x \in \mathfrak{M} - \mathfrak{M}^2$ is such that $\Lambda_{(x)}$ is $R_{(x)}$ -separable. If $\Lambda/x\Lambda$ is a maximal R/x -order in $\Lambda_{(x)}/x\Lambda_{(x)}$, then Λ is a maximal R -order.

In a sense, the maximality of Λ really boils down to the question of whether $\Lambda \otimes_R R_{\mathfrak{P}}$ is quasi-local for every height one prime $\mathfrak{P}$ in R . For Λ is maximal if and only if Λ is R -reflexive and $\Lambda \otimes_R R_{\mathfrak{P}}$ is a maximal $R_{\mathfrak{P}}$ -order for every height one prime $\mathfrak{P}$ [1, Theorem 1.5]. If $\text{gl.dim. } \Lambda < \infty$, then by [5, Corollary 2.17] Λ is R -free (and hence R -reflexive) and $\text{gl.dim. } \Lambda = \text{gl.dim. } R$. Furthermore, since $\text{gl.dim. } \Lambda = \text{gl.dim. } R$ and Λ is torsion-free over R , for any prime ideal $\mathfrak{Q}$ of R ,

$$\text{gl.dim. } \Lambda \otimes_R R_{\mathfrak{Q}} = \text{gl.dim. } R_{\mathfrak{Q}}$$

[5, Corollary 3.4]. Thus our assertion is justified by the Auslander-Goldman criterion in dimension one. A direct proof that $\Lambda \otimes_R R_{\mathfrak{P}}$ is quasi-local seems difficult, and an example is provided which may suggest why. We exhibit a Λ which has global dimension three and is quasi-local. Furthermore, it is maximal. But for a certain height two prime $\mathfrak{P}$ of R , $\Lambda \otimes_R R_{\mathfrak{P}}$ is *not* quasi-local.

Preliminaries. All rings in this paper have units, and every module is unital and finitely generated. In particular, every algebra is a finitely generated module over its center.

We shall use the abbreviations gl.dim. for global dimension, p.d. for projective dimension, and codim for codimension. (If A is a commutative local ring and E is a nonzero A -module, then $\text{codim}_A E$ is the length of the longest possible sequence of nonunits $x_1, \dots, x_t$ in A such that x_1 is not a zero-divisor on E and if $2 \leq i \leq t$, x_i is not a zero-divisor on $E/(x_1, \dots, x_{i-1})E$.) ($R, \mathfrak{M}$) will always be a regular local ring with maximal ideal $\mathfrak{M}$, and K will denote its quotient field. Σ is a central simple K -algebra.

1. R has arbitrary (finite) global dimension.

PROPOSITION 1.1. *Suppose Γ is a central R -algebra. If $\text{gl.dim. } \Gamma < \infty$ and Γ is quasi-local, then $\Gamma \otimes_R K$ is a central simple K -algebra.*

Proof. By [5, Corollary 2.17], $\text{gl.dim. } \Gamma = \text{gl.dim. } R$ and Γ is R -free. Thus by [5, Corollary 3.4], $\text{gl.dim. } \Gamma \otimes_R K = \text{gl.dim. } K = 0$. Hence $\Gamma \otimes_R K$ is semisimple, say $\Gamma \otimes_R K = \Sigma_1 \oplus \dots \oplus \Sigma_s$ where each Σ_i is a simple K -algebra. Since the center of Γ is R , the center of $\Gamma \otimes_R K$ is K . If K_i is the center of Σ_i , then $K = K_1 \oplus \dots \oplus K_s$. But K is a field, so $s = 1$. Thus $\Gamma \otimes_R K = \Sigma_1$, which is a central simple K -algebra. ■

PROPOSITION 1.2. *Let $\Omega \subset \Gamma$ be R -orders in Σ . Suppose $\text{gl.dim. } \Omega = \text{gl.dim. } R = n$ and Γ is R -free. Then $\text{gl.dim. } \Gamma = n$.*

Proof. Since Γ is R -free, $\text{gl.dim. } \Gamma = \text{gl.dim. } R$ if and only if every R -free Γ -module E is Γ -projective [5, Proposition 3.5]. Since $\text{gl.dim. } \Omega = n$,

$$n \geq \text{p.d.}_\Omega E / \mathfrak{M}E = \text{codim}_R E + \text{p.d.}_\Omega E = n + \text{p.d.}_\Omega E.$$

($\text{codim}_R E = \text{codim}_R R = n$ since E is R -free.) It follows that $\text{p.d.}_\Omega E = 0$, i.e., E is Ω -projective. Now E is Γ -projective by [4, Lemma 1.3]. ■

COROLLARY 1.3. *Suppose $\text{gl.dim. } R = 2 = \text{gl.dim. } \Omega$ where Ω is an R -order in Σ . Then if Γ is any maximal order containing Ω , $\text{gl.dim. } \Gamma = 2$.*

Proof. Any maximal order is R -reflexive [1, Theorem 1.5]. Hence Γ is R -free, since $\text{gl.dim. } R = 2$. Proposition 1.2 now applies. ■

THEOREM 1.4. *Suppose Λ is an R -order in Σ such that $\text{gl.dim. } \Lambda < \infty$ and Λ is quasi-local. If Γ is any R -free order in Σ containing Λ , then $\Gamma = \Lambda$. Consequently if Λ is contained in an R -free maximal order, then Λ is maximal.*

Proof. By [5, Corollary 2.17] $\text{gl.dim. } \Lambda = n = \text{gl.dim. } R$. Let

$$C_\Lambda^\Gamma(\Gamma) = \{x \in \Sigma \mid \Gamma x \subset \Lambda\},$$

the right conductor of Γ in Λ . Since $\Lambda \subset \Gamma$, $C_\Lambda^\Gamma(\Gamma) = \text{Hom}_\Lambda(\Gamma, \Lambda)$. Since Γ is R -free and $\text{gl.dim. } \Lambda = \text{gl.dim. } R$, Γ is Λ -projective by [5, Proposition 3.5] and therefore $\text{Hom}_\Lambda(\Gamma, \Lambda)$ is Λ -projective. Thus $C_\Lambda^\Gamma(\Gamma)$ is a projective right ideal in Λ , and since Λ is quasi-local, $C_\Lambda^\Gamma(\Gamma)$ is principal (see [1, Proposition 3.3]). Say $C_\Lambda^\Gamma(\Gamma) = t\Lambda$. $\Gamma t \subset C_\Lambda^\Gamma(\Gamma)$ so $\Gamma t \subset t\Lambda$. But t is a unit in Σ since $C_\Lambda^\Gamma(\Gamma) \cap R \neq (0)$. Hence $\Gamma \subset t\Lambda t^{-1}$. Since $\Lambda \subset \Gamma$, we have $\Lambda \subset t\Lambda t^{-1}$. We shall show that $\Lambda = t\Lambda t^{-1}$, from which it follows that $\Lambda = \Gamma$.

Let $\sigma: \Sigma \rightarrow \Sigma$ be the inner automorphism $x \mapsto txt^{-1}$. Thus $\Lambda \subset \sigma(\Lambda)$, which implies that $\sigma(\Lambda) \subset \sigma^2(\Lambda)$, and so on. We obtain an increasing chain of orders $\Lambda \subset \sigma(\Lambda) \subset \sigma^2(\Lambda) \subset \dots \subset \sigma^i(\Lambda) \subset \dots$. But there can be no infinite ascending chain of R -orders in Σ . For the union of such a chain would be a subring of Σ integral over R , and hence finitely generated as an R -module (see [3, p. 70, Satz 6]). But R is noetherian, so a finitely generated R -module cannot contain an infinite ascending chain of submodules. Thus for some $m \geq 1$, $\sigma^m(\Lambda) = \sigma^{m+1}(\Lambda)$. Therefore $\sigma(\Lambda) = \Lambda$ and we are done. ■

REMARK. By Proposition 1.1 we could have replaced the hypothesis that Λ is an R -order in Σ with the hypothesis that R is the center of Λ . The same is true for Theorem 2.2 in the next section.

2. The global dimension of R is three. We begin by stating a lemma which we have been able to prove only for regular local rings of dimensions at most three.

LEMMA 2.1. *Let Λ be an R -algebra which as an R -module is R -free. Suppose $\text{gl.dim. } \Lambda = \text{gl.dim. } R (= 3)$. Let E be a left Λ -module which is R -reflexive, and suppose that $\text{End}_\Lambda(E)$ is R -free. Then E is Λ -projective.*

Before proving this lemma let us state and demonstrate the main theorem.

THEOREM 2.2. *R is a regular local ring of dimension three. Let Λ be a quasi-local R -order in Σ with $\text{gl.dim. } \Lambda < \infty$. If there exists an R -free maximal order Γ in Σ then $\Lambda = t^{-1}\Gamma t$ for some unit t in Σ . Thus Λ is maximal and all R -free maximal orders in Σ are isomorphic.*

Proof. Since Γ is a maximal order and R is an integrally closed noetherian domain, for any R -order Ω in Σ , $\Gamma = \text{End}_{\Omega}(C_{\Omega}^{\lambda}(\Gamma))$ [5, Theorem 6.2]. So $\Gamma = \text{End}_{\Lambda}(C_{\Lambda}^{\lambda}(\Gamma))$. Since Γ is R -free (by [5, Corollary 2.17]) and hence R -reflexive, $C_{\Lambda}^{\lambda}(\Gamma)$ is also R -reflexive [5, Proposition 5.3]. But Γ is R -free, so $C_{\Lambda}^{\lambda}(\Gamma)$ is Λ -projective by Lemma 2.1. Just as in the proof of Theorem 1.4 we have $C_{\Lambda}^{\lambda}(\Gamma) = t\Lambda$ for some t which is invertible in Σ . Again, since $C_{\Lambda}^{\lambda}(\Gamma)$ is a left Γ -module, $\Gamma t \subset t\Lambda$ and so $\Gamma \subset t\Lambda t^{-1}$. But Γ is maximal, so $\Gamma = t\Lambda t^{-1}$ and thus $\Lambda = t^{-1}\Gamma t$. ■

COROLLARY 2.3. *If Σ is a full matrix ring $M_n(K)$ and Λ is a quasi-local R -order in Σ and $\text{gl.dim. } \Lambda < \infty$ then Λ is maximal and $\Lambda \simeq M_n(R)$.*

Proof. $M_n(R)$ is an R -free maximal order in Σ .

PROPOSITION 2.4. *Let R be regular local of dimension three. Suppose Λ is an R -free maximal order and $\text{gl.dim. } \Lambda = 3$. Let I be a two-sided ideal in Λ which is R -reflexive. Then I is R -free (and hence Λ -projective).*

Proof. Since I is two-sided, $\text{End}_{\Lambda}(I)$ is an R -order containing Λ . By the maximality of Λ , $\text{End}_{\Lambda}(I) = \Lambda$ and is thus R -free. Since I is R -reflexive and $\text{gl.dim. } \Lambda = 3$, the desired result follows from Lemma 2.1. ■

COROLLARY 2.5. *Let R and Λ satisfy the hypotheses of the preceding proposition. Suppose for $i = 1, 2$ that I_i is a two-sided ideal in Λ and is Λ -projective. Then $I_1 \cap I_2$ is Λ -projective.*

Proof. Since I_i is Λ -projective it is R -free and hence R -reflexive. Therefore $I_1 \cap I_2$ is R -reflexive, since over an integrally closed noetherian domain the double dual of a finitely generated torsion-free module is the intersection of its localizations at all the height one primes, and for any prime $\mathfrak{P}$, we have $(I_1 \cap I_2)_{\mathfrak{P}} = (I_1)_{\mathfrak{P}} \cap (I_2)_{\mathfrak{P}}$. Clearly $I_1 \cap I_2$ is two-sided. By Proposition 2.4 $I_1 \cap I_2$ is Λ -projective. ■

We return now to the proof of Lemma 2.1. This lemma is a generalization of [1, Theorem 4.4 (b) $\Rightarrow$ (a)]. The idea is to ape that proof, which consists of a sequence of propositions. We shall state each of those propositions in its generalized version. For the first two we omit proofs since the proofs of their counterparts in [1] carry over nearly verbatim.

Throughout this proof $(S, \mathfrak{M})$ will be a commutative noetherian local ring with maximal ideal $\mathfrak{M}$ and Γ will be an S -algebra which is a finitely generated S -module.

PROPOSITION 2.1 A (cf. [1, PROPOSITION 4.7]). *Let A and B be Γ -modules such that $\text{Hom}_{\Gamma}(A, B) \neq 0$. If $\text{codim}_S B \geq i$ for $i = 1, 2$ then $\text{codim}_S \text{Hom}_{\Gamma}(A, B) \geq i$.*

PROPOSITION 2.1 B (CF. [1, LEMMA 4.8]). *Let S be regular local of dimension at least three and let B be a Γ -module such that $\text{codim}_S B \geq 2$. If A is a Γ -module such that $\text{Hom}_\Gamma(A, B)$ is S -projective and $\text{Ext}_\Gamma^1(A, B) \neq 0$, then $\text{codim}_S \text{Ext}_\Gamma^1(A, B) > 0$.*

PROPOSITION 2.1 C (CF. [1, PROPOSITION 4.9]). *If S is a regular local ring, Γ is S -free, $\text{gl.dim. } \Gamma_{\mathfrak{P}} = \text{gl.dim. } S_{\mathfrak{P}}$ for all primes $\mathfrak{P}$ of height at most two, E a Γ -module which is S -reflexive and $\text{Hom}_\Gamma(E, E)$ is S -free, then $\text{Ext}_\Gamma^1(E, E) = 0$.*

Proof. By induction on $\text{gl.dim. } S$. If $\text{gl.dim. } S \leq 2$, then since E is S -reflexive it is S -projective. By hypothesis Γ is S -free and $\text{gl.dim. } \Gamma = \text{gl.dim. } S$ (since we are now assuming that the height of $\mathfrak{M}$ is at most two). Thus E is Γ -projective by [5, Proposition 3.5], and so $\text{Ext}_\Gamma^1(E, E) = 0$.

Now suppose that $\text{gl.dim. } S = k \geq 3$ and the proposition is true for rings of dimension less than k . Let $\mathfrak{P}$ be a nonmaximal prime ideal of S . Then $\text{gl.dim. } S_{\mathfrak{P}} < k$. $E_{\mathfrak{P}}$ is $S_{\mathfrak{P}}$ -reflexive and $\text{Hom}_{S_{\mathfrak{P}}}(E_{\mathfrak{P}}, E_{\mathfrak{P}}) = \text{Hom}_S(E, E) \otimes_S S_{\mathfrak{P}}$ is $S_{\mathfrak{P}}$ -free. Any prime ideal of $S_{\mathfrak{P}}$ of height at most two can be represented as $q_{\mathfrak{P}}$ where q is a prime ideal of S of height at most two. $(\Gamma_{\mathfrak{P}})_{q_{\mathfrak{P}}} = \Gamma_q$ and $(S_{\mathfrak{P}})_{q_{\mathfrak{P}}} = S_q$ so that $\text{gl.dim. } (\Gamma_{\mathfrak{P}})_{q_{\mathfrak{P}}} = \text{gl.dim. } (S_{\mathfrak{P}})_{q_{\mathfrak{P}}}$. Hence by our induction hypothesis $\text{Ext}_{\Gamma_{\mathfrak{P}}}^1(E_{\mathfrak{P}}, E_{\mathfrak{P}}) = 0$, i.e., $\text{Ext}_\Gamma^1(E, E) \otimes_S S_{\mathfrak{P}} = 0$. Thus no nonmaximal prime of S is an associated prime of $\text{Ext}_\Gamma^1(E, E)$, so if $\text{Ext}_\Gamma^1(E, E) \neq 0$, then $\text{Ass}_S(\text{Ext}_\Gamma^1(E, E)) = \{\mathfrak{M}\}$. But then $\text{codim}_S \text{Ext}_\Gamma^1(E, E) = 0$. By Proposition 2.1 B this is impossible, so $\text{Ext}_\Gamma^1(E, E) = 0$. ■

PROPOSITION 2.1 D (CF. [1, PROPOSITION 4.10]). *Suppose the S -algebra Γ is quasi-local and E is a Γ -module with $\text{p.d.}_\Gamma E = n$. If A is a nonzero Γ -module then $\text{Ext}_\Gamma^n(E, A) \neq 0$.*

Proof. Since $\text{p.d.}_\Gamma E = n$, $\text{Ext}_\Gamma^n(E, T) \neq 0$ for some simple Γ -module T [5, Corollary 1.5]. Since Γ is quasi-local, all simple Γ -modules are isomorphic. It follows that A/JA is a direct sum of copies of T , where $J = \text{Rad } \Gamma$. Ext commutes with direct sums, so $\text{Ext}_\Gamma^n(E, A/JA) \neq 0$. From the exact sequence

$$0 \rightarrow JA \rightarrow A \rightarrow A/JA \rightarrow 0$$

we obtain a long exact sequence

$$\cdots \rightarrow \text{Ext}_\Gamma^n(E, A) \rightarrow \text{Ext}_\Gamma^n(E, A/JA) \rightarrow \text{Ext}_\Gamma^{n+1}(E, JA) \rightarrow \cdots$$

$\text{Ext}_\Gamma^{n+1}(E, -) = 0$ since $\text{p.d.}_\Gamma E = n$. It follows that $\text{Ext}_\Gamma^n(E, A) \neq 0$. ■

We may now complete the proof of Lemma 2.1. Since $\text{gl.dim. } R = 3 = \text{codim}_R R$, $\text{codim}_R \text{Hom}_R(E^*, R) \geq 2$ by [1, Proposition 4.7] (X^* denotes $\text{Hom}_R(X, R)$). But $\text{Hom}_R(E^*, R) = E^{**}$ and by hypothesis $E = E^{**}$, so $\text{codim}_R E \geq 2$. Hence $\text{p.d.}_R E \leq 1$. Since Λ is R -free and $\text{gl.dim. } \Lambda = \text{gl.dim. } R$, every R -free Λ -module is Λ -projective [5, Proposition 3.5]. From this it follows that for any Λ -module A , $\text{p.d.}_\Lambda A = \text{p.d.}_R A$. Thus $\text{p.d.}_\Lambda E \leq 1$. But since $\text{End}_\Lambda(E)$ is R -free, $\text{Ext}_\Lambda^1(E, E) = 0$ by Proposition 2.1 C. (Since $\text{gl.dim. } \Lambda = \text{gl.dim. } R$, for any prime $\mathfrak{P}$ in R $\text{gl.dim. } \Lambda_{\mathfrak{P}} = \text{gl.dim. } R_{\mathfrak{P}}$ by [5, Corollary 3.4].) Now by Proposition 2.1 D $\text{p.d.}_\Lambda E$ cannot be 1, so it must be 0 and E is Λ -projective. ■

3. More practical sufficient conditions. We shall give some more practical sufficient conditions for the maximality of R -orders having finite global dimension.

THEOREM 3.1. *Let R be regular local of dimension n . Assume that the R -order Λ is R -reflexive. Suppose that for some $x \in \mathfrak{M} - \mathfrak{M}^2$, $\Lambda_{(x)}$ is $R_{(x)}$ -separable and $\Lambda/x\Lambda$ is a maximal R/x -order in $\Lambda(x)/x\Lambda_{(x)}$.*

Then Λ is maximal.

Proof. It suffices, by [1, Theorem 1.5], to show that for every height one prime $\mathfrak{Q}$ in R , $\Lambda_{\mathfrak{Q}}$ is maximal.

If $\mathfrak{Q} = (x)$ we are done since $\Lambda_{(x)}$ is $R_{(x)}$ -separable and therefore maximal [2, Proposition 7.2]. Suppose $\mathfrak{Q} \neq (x)$. Let $\mathfrak{P}$ be a minimal associated prime of $xR + \mathfrak{Q}$. Then $\text{height } \mathfrak{P} \leq 1 + \text{height } \mathfrak{Q} = 2$. Since $\mathfrak{Q} \neq (x)$ and $\text{height } \mathfrak{Q} = 1$, $x \notin \mathfrak{Q}$. Therefore $\mathfrak{P}$ properly contains $\mathfrak{Q}$, and so $\text{height } \mathfrak{P} = 2$. $\mathfrak{P}/x$ is a height one prime in R/x . Since $\Lambda/x\Lambda$ is maximal, $(\Lambda/x\Lambda)_{\mathfrak{P}/x}$ is maximal over a discrete valuation ring and is therefore quasi-local. But $(\Lambda/x\Lambda)_{\mathfrak{P}/x} \simeq \Lambda_{\mathfrak{P}}/x\Lambda_{\mathfrak{P}}$, so $\Lambda_{\mathfrak{P}}/x\Lambda_{\mathfrak{P}}$ is hereditary and quasi-local. Also, since x is regular on $\Lambda_{\mathfrak{P}}$ and $\text{gl.dim. } \Lambda_{\mathfrak{P}}/x\Lambda_{\mathfrak{P}} < \infty$, by [5, Proposition 5.6] $\text{gl.dim. } \Lambda_{\mathfrak{P}} = 1 + \text{gl.dim. } \Lambda_{\mathfrak{P}}/x\Lambda_{\mathfrak{P}} = 2$, and so $\Lambda_{\mathfrak{P}}$ is maximal by [5, Theorem 5.4]. Hence $(\Lambda_{\mathfrak{P}})_{\mathfrak{Q}\mathfrak{P}}$ is maximal, i.e. $\Lambda_{\mathfrak{Q}}$ is maximal. ■

COROLLARY 3.2. *Let R be regular local and let Λ be a maximal R -order in the central simple K -algebra Σ . Let $\Lambda[[X]] = \Lambda \otimes_R R[[X]]$ and $\Sigma((X)) = \Sigma \otimes_K K((X))$, where $R[[X]]$ denotes the formal power series ring over R in one indeterminate and $K((X))$ is its quotient field. Then $\Lambda[[X]]$ is a maximal $R[[X]]$ -order in the central simple $K((X))$ -algebra $\Sigma((X))$.*

Proof. $\Sigma((X))$ is central separable (and hence central simple) over $K((X))$ by [2, Corollary 1.6]. To simplify notation, let $S = R[[X]]$ and $\Gamma = \Lambda[[X]]$. Λ is R -reflexive and S is R -free and hence R -flat, so Γ is S -reflexive. $S/XS \simeq R$ and $\Gamma/X\Gamma \simeq \Lambda$. $\text{gl.dim. } S = 1 + \text{gl.dim. } R$, so S is regular local. $\Gamma_{(x)}/X\Gamma_{(x)} \simeq \Lambda \otimes_R K = \Sigma$. So $\Gamma_{(x)}/X\Gamma_{(x)}$ is central separable over $S_{(x)}/XS_{(x)}$ and thus $\Gamma_{(x)}$ is central separable over $S_{(x)}$. By our hypothesis on Λ , $\Gamma/X\Gamma$ is a maximal S/XS -order in $\Gamma_{(x)}/X\Gamma_{(x)}$. The maximality of Γ now follows from Theorem 3.1. ■

COROLLARY 3.3. *Suppose R is regular local of dimension three. Assume that Λ is quasi-local, $\text{gl.dim. } \Lambda < \infty$, and $\mathfrak{M} \not\subset N^2$, where $N = \text{Rad } \Lambda$. Then Λ is maximal.*

Proof. Λ is R -free (and hence R -reflexive) by [5, Corollary 2.17]. Since $\mathfrak{M} \not\subset N^2$ there is an $x \in \mathfrak{M} - N^2$ such that $\Lambda_{(x)}$ is $R_{(x)}$ -separable. (For a proof of this, see the proof of [5, Theorem 5.7].) Furthermore, $\Lambda/x\Lambda$ is quasi-local, and since $x \in \mathfrak{M} - N^2$ and $\text{gl.dim. } \Lambda < \infty$, $\text{gl.dim. } \Lambda/x\Lambda = \text{gl.dim. } \Lambda - 1$ [5, Theorem 6.9]. It follows from [5, Theorem 5.4] that $\Lambda/x\Lambda$ is maximal in $\Lambda_{(x)}/x\Lambda_{(x)}$. The preceding theorem now applies. ■

EXAMPLE 1. (a) Let R be regular local of dimension 2 and suppose X and Y generate the maximal ideal $\mathfrak{M}$. Define a quaternion algebra $\Gamma = R[1, \sigma, \tau, \sigma\tau]$ (i.e.,

Γ is R -free with generators $1, \sigma, \tau, \sigma\tau$ by setting $\sigma^2 = X, \tau^2 = Y$, and $\tau\sigma = -\sigma\tau$. Then $\text{Rad } \Gamma = \Gamma(\sigma, \tau)$ and $\Gamma/\text{Rad } \Gamma \approx R/\mathfrak{M}$. Thus Γ is local. Define

$$f: \Gamma \oplus \Gamma \rightarrow \Gamma(\sigma, \tau) \quad \text{by } f(\gamma_1, \gamma_2) = \gamma_1\sigma + \gamma_2\tau.$$

$$0 \longrightarrow \Gamma \cdot \langle \tau, \sigma \rangle \longrightarrow \Gamma \oplus \Gamma \xrightarrow{f} \Gamma(\sigma, \tau) \longrightarrow 0$$

is then a Γ -free resolution of $\text{Rad } \Gamma$. Thus $\text{p.d.}_\Gamma \text{Rad } \Gamma \leq 1$, $\text{gl.dim. } \Gamma = \text{p.d.}_\Gamma \Gamma/\text{Rad } \Gamma = 1 + \text{p.d.}_\Gamma \text{Rad } \Gamma \leq 2$. Since Γ is local, $\text{gl.dim. } \Gamma = \text{gl.dim. } R = 2$.

(b) Let R be regular local of dimension three and assume that 2 is a unit in R . Suppose $\mathfrak{M} = (X, Y, Z)$. Define a quaternion algebra $\Lambda = R[1, \alpha, \beta, \alpha\beta]$ by setting $\alpha^2 = X + Z^2, \beta^2 = Y$, and $\alpha\beta = -\beta\alpha$. Now $\Lambda/Z\Lambda$ is isomorphic to the algebra Γ in part (a). So $\text{gl.dim. } \Lambda/Z\Lambda = 2$ and $\Lambda/Z\Lambda$ is local. Thus Λ is local, and by [5, Proposition 5.6] $\text{gl.dim. } \Lambda = 3$. $N = \text{Rad } \Lambda = \Lambda(\alpha, \beta, Z)$. $Z \notin N^2$, so $\mathfrak{M} \not\subset N^2$. By Corollary 3.3 Λ is maximal.

Λ is rather strange, though. It itself is local, and since it is maximal, Λ_q is quasi-local for every height one prime q of R . However, $\mathfrak{P} = (X, Y)R$ is a height two prime, such that $\Lambda_{\mathfrak{P}}$ is *not* quasi-local. It is easily seen that $\Lambda_{\mathfrak{P}}(\alpha - Z, X, \beta)$ and $\Lambda_{\mathfrak{P}}(\alpha + Z, X, \beta)$ are maximal two-sided ideals in $\Lambda_{\mathfrak{P}}$; indeed $\Lambda_{\mathfrak{P}}$ modulo either ideal is isomorphic to the field $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$. The two are distinct since $\alpha + Z - (\alpha - Z) = 2Z$ is a unit in $R_{\mathfrak{P}}$.

COROLLARY 3.4. *Let R be regular local of dimension three and Λ a quasi-local central R -algebra. Suppose that, for some $x \in \mathfrak{M} - \mathfrak{M}^2$, $\text{gl.dim. } \Lambda/x\Lambda < \infty$ and the center of $\Lambda/x\Lambda$ is R/x . Then $\text{gl.dim. } \Lambda = 3$ and Λ is a maximal order in the central simple K -algebra $\Lambda \otimes_R K$.*

Proof. By Proposition 1.1, $\Lambda/x\Lambda \otimes_{R/x} R_{(x)}/xR_{(x)} = \Lambda_{(x)}/x\Lambda_{(x)}$ is central simple (or equivalently, central separable) over $R_{(x)}/xR_{(x)}$. Hence $\Lambda_{(x)}$ is $R_{(x)}$ -separable [2, Theorem 4.7].

$\Lambda/x\Lambda$ is quasi-local, has finite global dimension and $\text{gl.dim. } R/x = 2$, so by [5, Theorem 5.4] $\Lambda/x\Lambda$ is a maximal R/x -order in $\Lambda_{(x)}/x\Lambda_{(x)}$, and $\text{gl.dim. } \Lambda/x\Lambda = 2$. By [5, Proposition 5.6] $\text{gl.dim. } \Lambda = \text{gl.dim. } \Lambda/x\Lambda + 1 = 3$. It follows from Proposition 1.1 that $\Lambda \otimes_R K$ is a central simple K -algebra. Since Λ is quasi-local and $\text{gl.dim. } \Lambda < \infty$, Λ is R -free. Hence by Theorem 3.1, Λ is maximal in $\Lambda \otimes_R K$. ■

We conclude with an example for which Corollary 3.4, but not Corollary 3.3, is applicable.

EXAMPLE 2. Let R be regular local of dimension three, $\mathfrak{M} = (X, Y, Z)$, and characteristic $(R/\mathfrak{M}) \neq 2$. Let $\Lambda = R[1, \alpha, \beta, \alpha\beta]$ where $\alpha^2 = X, \beta^2 = Y$, and $\beta\alpha = -\alpha\beta + Z$.

Then $N = \text{Rad } \Lambda = \Lambda(\alpha, \beta)$. Notice that $Z \in N^2$, so that $\mathfrak{M} \subset N^2$. Thus Corollary 3.3 does not apply. However, consider $\Lambda/Z\Lambda$. It is $R/Z[1, \bar{\alpha}, \bar{\beta}, \bar{\alpha}\bar{\beta}]$ where $\bar{\alpha}^2 = X, \bar{\beta}^2 = Y$, and $\bar{\beta}\bar{\alpha} = -\bar{\alpha}\bar{\beta}$. This is the ring Γ in part (a) of Example 1. Hence $\Lambda/Z\Lambda$ is

local and $\text{gl.dim. } \Lambda/Z\Lambda = 2$. Since characteristic $(R/\mathfrak{M}) \neq 2$, 2 is a unit in R/Z and therefore the center of $\Lambda/Z\Lambda$ is R/Z . From Corollary 3.4 it follows that Λ is maximal.

REFERENCES

1. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. **97** (1960), 1–24. MR **22** #8034.
2. ———, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409. MR **22** #12130.
3. M. Deuring, *Algebren*, Springer, Berlin, 1935.
4. M. Harada, *Hereditary orders*, Trans. Amer. Math. Soc. **107** (1963), 273–290. MR **27** #1474.
5. M. Ramras, *Maximal orders over regular local rings of dimension two*, Trans. Amer. Math. Soc. **142** (1969), 457–479. MR **39** #6878.

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