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REGULARITY CONDITIONS IN NONNOETHERIAN RINGS[1]

BY
T. KABELE

Abstract. We show that properties of R-sequences and the Koszul complex which hold for noetherian local rings do not hold for nonnoetherian local rings. For example, we construct a local ring with finitely generated maximal ideal such that $\mathrm{hd}_R\, M<\infty$ but M is not generated by an R-sequence. In fact, every element of $M-M^2$ is a zero divisor. Generalizing a result of Dieudonné, we show that even in local (nonnoetherian) *integral domains* a permutation of an R-sequence is not necessarily an R-sequence.

Introduction. A fundamental theorem of local algebra states that for a *noetherian* local ring R with maximal ideal M the following are equivalent: (i) M is generated by a regular sequence (also called R-sequence or prime sequence), (ii) $\mathrm{hd}_R\, M<\infty$. If these conditions are satisfied, R is called *regular*. A natural question to ask is: Is the theorem true if we remove the noetherian hypothesis (but still assume M is finitely generated)? In this paper we answer this question in the negative (§3, Example 3) and investigate several alternate "regular conditions."

Once and for all, all rings are commutative with unit. We use the phrase "(R, M) is local" to mean R is any commutative ring with unique maximal ideal M.

Let R be a ring, $J=(x_1,\dots,x_n)$ an ideal of R. Let $X_1,\dots,X_n$ be indeterminates. Following Grothendieck, we say $\boldsymbol{x}=(x_i)_{1\leqq i\leqq n}$ is a *regular sequence* iff, for $1\leqq i\leqq n$, x_i is not a zero divisor of $R/\sum_{j=1}^{i-1} Rx_j$. We say $\boldsymbol{x}$ is a *quasi-regular* sequence iff the canonical surjection $\alpha\colon R/J[X_1,\dots,X_n]\twoheadrightarrow \sum J^i/J^{i+1}$ defined by $\alpha(X_i)=x_i+J^2$ is bijective [5, 15.1.7, p. 15]. We let $K(\boldsymbol{x}, R)$ or $R\langle X_i\rangle{:}\partial X_i=x_i$ denote the Koszul complex; that is, the exterior algebra generated by $\{X_1,\dots,X_n\}$ with boundary map $\partial X_i=x_i$. The homology groups of $K(\boldsymbol{x}, R)$ are denoted by $H_i(\boldsymbol{x}, R)$. The symbol hd_R means homological dimension.

We introduce the following new terminology:

Definition 1. A sequence $\boldsymbol{x}=(x_i)_{1\leqq i\leqq n}$ is called *Koszul-regular* (resp. *H_1-regular*) iff $H_i(\boldsymbol{x}, R)=0$ for all $i\geqq 1$ (resp. $H_1(\boldsymbol{x}, R)=0$).

Definition 2. Let (R, M) be local with M finitely generated. We say (a) M is a *regular ideal* (resp. (b) *Koszul-regular*, (b′) *H_1-regular*, (c) *quasi-regular*) iff M is

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generated by some regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence.

REMARK. Grothendieck [6, 16.9.2, 16.9.7, p. 46] and Berthelot [2] give similar definitions of *regular ideal* for finitely generated ideals in an arbitrary ringed space. They use their concepts to define *regular immersions.*

The concept of Koszul-regular sequence (or ideal) seems to be most suitable in the nonnoetherian case (cf. [2]). It, unlike regularity, is independent of order (see [4] and §3, Examples 4 and 5) and, unlike quasi-regularity, implies that the ideal generated by the sequence has finite homological dimension (see §3, Examples 1 and 2).

In §1 we show that for the maximal ideal M of a nonnoetherian local ring R we still have (a) $\Rightarrow$ (b) $\Rightarrow \mathrm{hd}_R M<\infty \Rightarrow$ (b′) $\Rightarrow$ (c) and (b), (b′), (c) are "independent of base." In §2 we show (c) is invariant under completion. In §3, using a class of local rings invented by Nagata, we show (Examples 1, 2, 3) that (a) $\Leftarrow$ (b) $\Leftarrow$ (b′) $\Leftarrow$ (c) do not hold and that (a) depends on both order and choice of minimal generating set (Example 4). In our last example (Example 5) we show that even in *local integral domains* regularity of a sequence of generators of the maximal ideal depends on order. This generalizes [4] and contradicts the converse part of Lemma 2 in [8, p. 42].

This paper was inspired by a problem posed by Gerson Levin [7]. The author wishes to express his deep appreciation to his thesis advisor Professor E. Matlis for suggesting this problem and for encouraging and inspirational advice.

1. **Basic facts.** Here we prove a generalization of a theorem of Eilenberg and, putting this together with some obvious generalizations of classical results, we conclude

THEOREM 1.1. *Let* (R, M) *be local with* M *finitely generated. Then, with the notation of Definition* 2,

(i) (a) $\Rightarrow$ (b) $\Rightarrow$ (b′) $\Rightarrow$ (c).

(ii) *M satisfies* (b) (*resp.* (b′), (c)) *iff every* (*some*) *minimal generating set of M is a Koszul-regular* (*resp. H_1-regular, quasi-regular*) *sequence.*

In fact, (i) follows from 1.4(i) and 1.6; while (ii) follows from 1.4(ii) and 1.5.

REMARK. Using a result of Tate [12, Theorem 8, p. 27] and Northcott [10, p. 239], Levin [7] has shown $\mathrm{hd}_R M<\infty \Rightarrow$ (b′). Trivially, (b) $\Rightarrow \mathrm{hd}_R M<\infty$. If M is generated by two elements, Levin has shown (b) $\Leftrightarrow \mathrm{hd}_R M<\infty$.

1.2. Let R be any ring and $J=(x_1, \dots, x_n)$ an ideal of R. Let $X_1, \dots, X_n$ be indeterminates. The following are equivalent: (i) $x_i, \dots, x_n$ is a quasi-regular sequence; (ii) for every integer $s\geqq 0$ and for every form $\psi(X_1, \dots, X_n)$ of degree s, $\psi(x_1, \dots, x_n)\in J^{s+1}$ implies the coefficients of ψ are in J; (iii) for every form $\varphi(X_1, \dots, X_n)$ of arbitrary degree, $\varphi(x_1, \dots, x_n)=0$ implies the coefficients of φ are in J.

In fact, (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) are trivial, while (iii) $\Rightarrow$ (ii) follows by modifying a proof of Northcott [11, 4.4, pp. 67–68]. When $J=M$ and (R, M) is a local noetherian ring, quasi-regularity is equivalent to *analytic independence*.

The next remark follows from the definition of the Koszul complex. We call an $n\times n$ matrix $C=[c_{ij}]$ *skew-symmetric* iff $c_{ii}=0$ and $c_{ij}=-c_{ji}$.

1.3. Let R be a ring with $x_1, \ldots, x_n \in R$. Let E denote the Koszul complex $R\langle X_1, \ldots, X_n\rangle : \partial X_i = x_i$. Let $T=[x_1 \cdots x_n]$ and $A=[a_1 \cdots a_n]$, where $a_i \in R$, be $1\times n$ matrices. Then $\sum_{i=1}^n a_i x_i$ is a one-cycle (resp. one-boundary) of E iff $AT^t=0$ (resp. $A=TB$ for some $n\times n$ skew-symmetric matrix B). Thus $H_1(E)=0$ iff $AT^t=0$ implies $A=TB$.

1.4. Let R be a ring, $\boldsymbol{x}=(x_i)_{1\leqq i\leqq n}$ a sequence of elements of R, and $J\neq R$ the ideal generated by $\boldsymbol{x}$. Consider the following conditions: (a) $\boldsymbol{x}$ is regular, (b) $\boldsymbol{x}$ is Koszul-regular, (b′) $\boldsymbol{x}$ is H_1-regular, (c) $\boldsymbol{x}$ is quasi-regular. Then (i) (a) $\Rightarrow$ (b) $\Rightarrow$ (b′) and (a) $\Rightarrow$ (c); (ii) if $\boldsymbol{x}$ satisfies any of the above regularity conditions, then it minimally generates J.

In fact, (i) is [6, 19.5.1, p. 204] while (ii) follows from 1.2 and 1.3.

If (R, M) is local and J a finitely generated ideal, then [9, 5.1, p. 13] every minimal generating set of J has the same number of elements and mod JM forms a free base over R/M. Thus 1.5 carries over from the noetherian case.

1.5. Let (R, M) be local with J an ideal minimally generated by $\boldsymbol{x}=(x_i)_{1\leqq i\leqq n}$ and $\boldsymbol{y}=(y_i)_{1\leqq i\leqq n}$. Then (i) $\boldsymbol{x}$ is quasi-regular iff $\boldsymbol{y}$ is quasi-regular, (ii) the Koszul complexes $K(\boldsymbol{x}, R)$ and $K(\boldsymbol{y}, R)$ are isomorphic. Thus $\boldsymbol{x}$ is Koszul-regular (resp. H_1-regular) iff $\boldsymbol{y}$ is.

Part (ii) is proved by the same method used by Tate in [12, p. 23]. Part (i) follows from the commutative diagram (cf. [6, 16.9.3, p. 46]):

$$\begin{tikzcd} R/J[X_1,\ldots, X_n] \arrow[r, dashed, two heads, "\alpha_x"] \arrow[d, "\varphi"'] & \sum J^i/J^{i+1} \arrow[d, equal] \\ R/J[Y_1,\ldots, Y_n] \arrow[r, dashed, two heads, "\alpha_y"] & \sum J^i/J^{i+1} \end{tikzcd}$$

If $x_i=\sum_{j=1}^n a_{ij}y_j$, then the vertical isomorphism is defined by $\varphi(X_i)=\sum_{j=1}^n a_{ij}Y_j$ where $a_{ij}\in R$; X_i and Y_j are indeterminates. We set $\alpha_x(X_i)=x_i+J^2$ and $\alpha_y(Y_i)=y_i+J^2$.

The following theorem is a generalization of a theorem of Eilenberg [12, p. 26]. We prove it by using a method of indeterminates employed by Northcott [11, Theorem 3, p. 68] to prove that every system of parameters of a *noetherian* local ring forms an analytically independent set.

THEOREM 1.6. *Let (R, M) be a local ring with J an ideal generated by $\boldsymbol{x}=(x_i)_{1\leqq i\leqq n}$. Then $\boldsymbol{x}$ is H_1-regular implies $\boldsymbol{x}$ is quasi-regular.*

We need some preliminary lemmas.

LEMMA 1.7. *Same hypothesis on R, J, $\boldsymbol{x}$ as in* 1.6. *If $\boldsymbol{x}$ is H_1-regular, then $cx_1^s \in (x_2, \ldots, x_n)$ implies $c \in (x_2, \ldots, x_n)$, where $c \in R$ and s is a positive integer.*

Proof. Suppose $cx_1^s \in (x_2, \ldots, x_n)$. Then $cx_1^s = \sum_{i=2}^n a_i x_i$ for some $a_i \in R$, and so $cx_1^{s-1}X_1 - \sum_{i=2}^n a_i X_i$ is a one-cycle of the Koszul complex $R\langle X_1, \ldots, X_n\rangle : \partial X_i = x_i$. By 1.3 $cx_1^{s-1} \in (x_2, \ldots, x_n)$. By induction we prove the lemma.

LEMMA 1.8. *Let R be any ring with $Z_1, \ldots, Z_d$ indeterminates and let $R^* = R[Z_1, \ldots, Z_d]$. If I is an ideal of R, let IR^* denote the ideal of R^* generated by I. Then* (i) *an element f in R^* is in IR^* iff all coefficients of f are in I*; (ii) *if P is a prime ideal of R, then PR^* is a prime ideal of R^**; (iii) *if $f \in R^*$ is a zero divisor, then there exists $c \neq 0$ in R such that $cf = 0$.*

Statements (i) and (ii) are proved by Northcott [11, Lemma 2, Proposition 2, p. 66, p. 81]. Northcott proves (iii) for noetherian rings but Nagata [9, 6.13, p. 17] gives a proof in the general case.

Proof of Theorem 1.6. To prove $\boldsymbol{x}$ is a quasi-regular sequence, it suffices to verify condition (iii) of 1.2. Let $\varphi(\boldsymbol{x}) = \varphi(x_1, \ldots, x_n) = 0$, where φ is a form of degree s; we have to show all coefficients of φ are in J. If c is the coefficient of x_1^s in φ, then from $\varphi(\boldsymbol{x}) = 0$ we get $cx_1^s \in (x_2, \ldots, x_n)$. By Lemma 1.7 $c \in (x_2, \ldots, x_n) \subset J$.

Following Northcott, we extend this result to all coefficients of φ by introducing n^2 indeterminates Z_{ij} $(1 \leq i \leq n, 1 \leq j \leq n)$. We put $R^* = R[Z_{ij}]$. By Lemma 1.8 $M^* = R^*M$ is a prime ideal. Since R is local, 1.8(iii) implies that no element of $R^* - M^*$ is a zero divisor in R^*; so we can form the ordinary ring of quotients Q' of R^* with respect to M^*. Q' is local with maximal ideal $M' = Q'M^* = Q'M$.

As in Northcott det $|Z_{ij}|$ is a unit in Q', so by Cramer's rule we can define $u_1, \ldots, u_n \in Q'$ such that $x_i = \sum_{j=1}^n Z_{ij} u_j$. Since $H_1(\boldsymbol{x}, R) = 0$ and since R^* is a free R-module, we see, using remark 1.3, that $H_1(\boldsymbol{x}, R^*) = 0$ and $H_1(\boldsymbol{x}, Q') = 0$. By a direct calculation, or by 1.4(ii), $\boldsymbol{x}$ *minimally* generates $J' = JQ'$. Since $|Z_{ij}|$ is an invertible matrix, $\boldsymbol{u} = (u_i)_{1 \leq i \leq n}$ minimally generates J'; and thus 1.5 shows $H_1(\boldsymbol{u}, Q') = 0$. As in Northcott, the first part of the proof shows that the coefficients of $\varphi(Z_{11}, Z_{12}, \ldots, Z_{n1})$ are in J. This proves 1.6.

2. **The completion.** In this section we show that quasi-regularity of a minimal generating set is preserved when we take the completion.

Let (R, M) be a local ring. Taking $\{M^n\}_{n=1}^\infty$ to be a base of neighborhoods of 0, we define a topology on R, called the M-adic topology. If $\bigcap_{n=1}^\infty M^n = \{0\}$, then $\{0\}$ is closed in the M-adic topology, and R is Hausdorff. We can then form the M-adic completion of R. The following is a result of Cohen [3, Theorem 2, Theorem 3, pp. 59–61].

2.1. Suppose (R, M) is a local ring with M finitely generated and $\bigcap_{n=1}^\infty M^n = 0$. Let R^* be the M-adic completion. Then R^* is a *noetherian* local ring with maximal ideal MR^*. Further, $M^iR^*/M^{i+1}R^* = M^i/M^{i+1}$ for every $i \geq 0$.

Note. Since $MR^*/M^2R^*=M/M^2$, every minimal generating set for M is also a minimal generating set MR^*.

Using 2.1 and a result of Nagata [9, 18.3, p. 59], we have the following interesting corollary (cf. [9, 31.8]):

2.2. Same hypothesis as 2.1. The following are equivalent: (1) $\otimes_R R^*$ is exact, (2) $AR^* \cap R=A$ for every ideal A of R, i.e. every ideal is closed, (3) R is noetherian.

THEOREM 2.3. *Let (R, M) be a local ring with M finitely generated. Let (S, N) be the completion of $R/\bigcap_{n=1}^{\infty} M^n$, and let $\varphi: R \to S$ be the canonical map. Then a minimal generating set $x_1, \ldots, x_n$ of M is a quasi-regular sequence in R if and only if $\varphi(x_1), \ldots, \varphi(x_n)$ is a quasi-regular sequence in S.*

Proof. We have induced maps

$$\varphi_1: R/M \to S/N,$$

$$\varphi_2: \sum_{i=0}^{\infty} M^i/M^{i+1} \to \sum_{i=0}^{\infty} N^i/N^{i+1},$$

$$\varphi_3: R/M[T_1, \ldots, T_n] \to S/N[T_1, \ldots, T_n],$$

where T_i are indeterminates and φ_3 is the obvious extension of φ_1; we set $\varphi_3(T_i)=T_i$. By 2.1 the maps φ_1 and φ_2 are bijective. The map φ_3 is also bijective. Thus, the theorem follows from the following commutative diagram:

$$\begin{tikzcd} R/M[T_1,\ldots,T_n] \arrow[r, dashed, two heads, "\alpha_R"] \arrow[d, dashed, "\varphi_3"'] & \sum M^i/M^{i+1} \arrow[d, dashed, "\varphi_2"] \\ S/N[T_1,\ldots,T_n] \arrow[r, dashed, two heads, "\alpha_S"] & \sum N^i/N^{i+1} \end{tikzcd}$$

COROLLARY 2.4. *Same hypothesis as* 2.3. *We have* (a) *$x_1, \ldots, x_n$ is a regular* (*resp. Koszul-regular, H_1-regular*) *sequence in R implies $\varphi(x_1), \ldots, \varphi(x_n)$ is a regular* (*resp. Koszul-regular, H_1-regular*) *sequence in S*; (b) $\mathrm{hd}_R\, M<\infty$ *implies* $\mathrm{hd}_S\, N<\infty$. *However, the converse implications are false* (*the counterexample is Example* 1 *of* §3).

Proof. Use Theorem 1.1, and note that the conditions of 1.1 are equivalent in the noetherian case.

3. **Counterexamples.** We now construct counterexamples to the converse implications of Theorem 1.1, using a class of local rings invented by Nagata [9, E 3.1, p. 206]. We first need some facts.

PROPOSITION 3.1 (NAGATA). *Let K be a field of characteristic $p\neq 0$ and $x_1,\ldots, x_n$ be indeterminates. Set $R^*=K[[x_1,\ldots, x_n]]$, $R=K^p[[x_1,\ldots, x_n]][K]$. Then R^* and R are both regular noetherian local rings with maximal ideals generated by $\{x_1, \ldots, x_n\}$, and R^* is the completion of R. Moreover,* (1) *an element h of R^* is in R iff the coefficients of h generate a finite extension of K^p,* (2) *if* $[K:K^2]=\infty$, *then*

$R^* \neq R$, (3) R^* is integral over R; in fact, $h \in R^*$ implies $h^p \in R$, (4) $AR^* \cap R = A$ for every ideal A of R.

Let K be a field of characteristic p. Recall from [13, Volume I, p. 129] that a subset $B \subset K$ is said to be a *p-independent* set iff for every finite subset $\{b_1, \ldots, b_n\} \subset B$ the monomials $b_1^{\nu_1} \cdots b_n^{\nu_n}$ (where $0 \leqq \nu_i \leqq p$) are linearly independent over K^p. The set B is called a p-base iff it is p-independent and $K^p[B] = K$. By Zorn's lemma we can prove that every field K contains a p-base.

Nagata states, without proof, the following result in [9, line 14, p. 209]. To prove it one could use the above definition of p-independent set and condition (1) of Proposition 3.1. We shall use generalizations of the result in constructing our counterexamples.

LEMMA 3.2. *Let R, R^* be as in 3.1 with $n=2$. Write x, y in place of x_1, x_2; and let $B = \{b_i, c_i\}_{i=1}^{\infty} \subset K$ be an infinite set of p-independent elements. Set $E = \sum_{i=1}^{\infty} b_i x^i$ and $F = \sum_{i=1}^{\infty} c_i y^i$. Then 1, E, F, EF are linearly independent over R.*

The following fact follows from a corollary of the "lying over theorem" (see [13, Volume I, Remark 2, p. 259]).

3.3. If (R^*, M^*) is a local ring and if R^* is integral over a ring R, then $(R, M^* \cap R)$ is a local ring.

We now list some examples of nonnoetherian local rings in which the maximal ideal is finitely generated by a regular sequence, i.e., the maximal ideal satisfies condition (a) of Theorem 1.1 and therefore also (b), (b′), and (c). These are (i) the ring B of germs of C^∞-functions of a real variable x in the neighborhood of 0, see [4]; (ii) the ring A of [4]; (iii) the integral domain T'' in [9, E 4.1, p. 207]; (iv) Example 4; (v) Example 5.

EXAMPLE 1. A local domain (T, N) for which N is quasi-regular but not H_1-regular. (By 1.1 and remark after $\mathrm{hd}_T\, N = \infty$.)

Let K be a field of characteristic 2 with $[K : K^2] = \infty$. Let $R^* = K[[x, y]]$ and let $R = K^2[[x, y]][K]$ where x, y are indeterminates. Let $\{b_i\}_{i=1}^{\infty} \subset K$ be an infinite set of p-independent elements. Set $E_n = \sum_{i=n}^{\infty} (xy)^i b_i$, $e_n = E_n / y^n$, $f_n = E_n / x^n$. We will show that

$$T = R[e_1, f_1, \ldots, e_n, f_n, \ldots] \text{ is the required integral domain.}$$

We have $e_n = y e_{n+1} + b_n x^n$, $f_n = x f_{n+1} + b_n y^n$, $x^n f_n = y^n e_n$, and $e_n f_n, e_n^2, f_n^2 \in R$. Thus, every $\alpha \in T$ is an R-linear combination of $\{1, y^m e_{n+m}, x^m f_{n+m}\}$, for some convenient n (depending on α) and for all m. Therefore, we can write

$$\alpha = \alpha_0 + \alpha_1 y^m e_{n+m} + \alpha_2 x^m f_{n+m} \tag{1}$$

where $\alpha_i \in R$. Hence,

$$[(x, y)R^*]^m \cap T = [(x, y)T]^m.$$

(In fact, $\alpha \in [(x, y)R^*]^m \cap T$ implies $\alpha_0 \in [(x, y)R^*]^m \cap R = [(x, y)R]^m$, which implies $\alpha \in [(x, y)T]^m$.) This proves T, with the $N=(x, y)T$ adic topology, is a subspace of R^*. T is dense in R^*, because R is dense in R^* and $R^* \supset T \supset R$. Thus, R^* is the completion of T.

By 3.3 T is local, with maximal ideal $(x, y)R^* \cap T = (x, y)T$.

The sequence (x, y) is quasi-regular in T, because it is quasi-regular in the completion of T, which is R^* (see 2.3).

Claim $e_1/x \notin T$. If, conversely, $e_1/x \in T$, then as in (1) above we can write, for all large n, $e_1/x = \alpha_0 + \alpha_1 e_n + \alpha_2 f_n$, where $\alpha_i \in R$. But $e_1 = y^{n-1} e_n + rx$ where $r = \sum_{i=1}^{n-1} (xy)^{i-1} b_i \in R$. Therefore,

$$(y^{n-1}e_n)/x + r = \alpha_0 + \alpha_1 e_n + \alpha_2 f_n.$$

Multiplying by x^n, using $x^n f_n = y^n e_n$, and applying linear independence of $\{1, e_n\}$ over R, we get $(xy)^{n-1} = \alpha_1 x^n + \alpha_2 y^n$. This contradicts unique factorization in R. Thus $e_1/x \notin T$. Similarly $f_1/y \notin T$.

Let E be the Koszul complex $E = T\langle X, Y\rangle : \partial X = x, \partial Y = y$. The above claim proves $H_1(E) \neq 0$. In fact, $Ye_1 - Xf_1$ is a nonbounding cycle.

Example 2. A local ring (S, N) for which N is H_1-regular but not Koszul-regular. (By remark after Theorem 1.1, $\mathrm{hd}_S\, N = \infty$.)

Let K be a field of characteristic 2 with $[K:K^2] = \infty$. Let $R^* = K[[x, y, z]]$, $R = K^2[[x, y, z]][K]$; and let $\{b_i\}_{i=1}^{\infty} \subset K$ be an infinite set of p-independent elements. Set

$$E_n = \sum_{i=n}^{\infty} (xy)^i b_i/(xy)^n, \qquad e_n = zE_n,$$

and let $T = R[e_1, \ldots, e_n, \ldots]$. Let $\mathfrak{A}$ be the *R-module* generated by $\{z, x^n e_n, y^n e_n, ze_n\}_{n=1}^{\infty}$. Note that $\mathfrak{A}$ is also a *T-ideal.* Then

$S = T/\mathfrak{A}$ *is the required example.*

As with the previous example, we can show T is a local ring with maximal ideal $(x, y, z)T$. Therefore, S is a local ring with maximal ideal $(\bar{x}, \bar{y})S$ (where $^-$ denotes residue class modulo $\mathfrak{A}$).

Since $e_n = xye_{n+1} + b_n z$, and $e_n^2 \in R$, we have that every $\alpha \in T$ is an R-linear combination of $\{1, e_n\}$ for every large n. Also every $\beta \in \mathfrak{A}$ is an R-linear combination of $\{z, x^n e_n, y^n e_n, ze_n\}$, for every large n.

Let E be the Koszul complex $E = S\langle X, Y\rangle : \partial X = \bar{x}, \partial Y = \bar{y}$. Recall [1, p. 626] $H_2(E)$ equals the S-annihilator of $\{\bar{x}, \bar{y}\}$ (i.e., the set of all s in S such that $s\bar{x} = 0 = s\bar{y}$). Since $\bar{x}\bar{e}_1 = 0 = \bar{y}\bar{e}_1$, to prove $H_2(E) \neq 0$ we need only show $\bar{e}_1 \neq 0$. If, conversely, $e_1 \in \mathfrak{A}$, then we can write, for some n,

$$e_1 = \beta_0 z + \beta_1 x^n e_n + \beta_2 y^n e_n + \beta_3 z e_n$$

where $\beta_i \in R$. Since $e_1=(xy)^{n-1}e_n+zr$ (for some $r \in R$), we get from linear independence of $\{1, e_n\}$ over R that

$$(xy)^{n-1} = \beta_1 x^n+\beta_2 y^n+\beta_3 z.$$

This contradicts unique factorization in the regular noetherian local ring R/zR. Hence, $e_1 \notin \mathfrak{A}$.

We claim $H_1(E)=0$. Assume $\alpha, \beta \in T$ and $\alpha x+\beta y \in \mathfrak{A}$. We must find $\gamma \in T$ such that $\alpha \equiv -\gamma y$ and $\beta \equiv \gamma x \pmod{\mathfrak{A}}$. We can find an integer n and elements $\alpha_i, \beta_i, \gamma_i \in R$ such that

$$\begin{aligned} \alpha x+\beta y &= \gamma_0 z+\gamma_1 x^n e_n+\gamma_2 y^n e_n+\gamma_3 z e_n, \\ \alpha &= \alpha_0+\alpha_1 e_n, \\ \beta &= \beta_0+\beta_1 e_n. \end{aligned}$$

By linear independence of $\{1, e_n\}$ over R, we have

$$\begin{aligned} \alpha_0 x+\beta_0 y &= \gamma_0 z, \\ \alpha_1 x+\beta_1 y &= \gamma_1 x^n+\gamma_2 y^n+\gamma_3 z. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_0 x+\beta_0 y &\equiv 0, \\ (\alpha_1-\gamma_1 x^{n-1})x+(\beta_1-\gamma_2 y^{n-1})y &\equiv 0 \pmod{zR}. \end{aligned}$$

Since R/zR is a regular noetherian local ring, we have $0=H_1(R/zR\langle x+zR, y+zR\rangle)$. Therefore, for some $c, d \in R$,

$$\begin{aligned} \alpha_0 &\equiv -cy, & \beta_0 &\equiv cx, \\ \alpha_1-\gamma_1 x^{n-1} &\equiv -dy, & \beta_1-\gamma_2 y^{n-1} &\equiv dx \pmod{zR}. \end{aligned}$$

Now $e_n \equiv xye_{n+1}$ and $y^{n+1}e_{n+1} \equiv 0 \equiv x^{n+1}e_{n+1} \pmod{\mathfrak{A}}$. Therefore, $\alpha_1 e_n \equiv -\delta y$ and $\beta_1 e_n \equiv \delta x \pmod{\mathfrak{A}}$, where $\delta=[-\gamma_1 x^n+\gamma_2 y^n+dxy]e_{n+1}$. Then

$$\begin{aligned} \alpha &= \alpha_0+\alpha_1 e_n \equiv (\delta+c)(-y), \\ \beta &= \beta_0+\beta_1 e_n \equiv (\delta+c)x \mod (\mathfrak{A}). \end{aligned}$$

REMARK. [1, Proposition 2.6, p. 632]. If R is a *noetherian* ring and $\boldsymbol{x}=(x_i)_{1\le i\le n}$ a sequence of elements of R, then $H_1(\boldsymbol{x}, R)=0$ implies $H_j(\boldsymbol{x}, R)=0$ for all $j \geq 1$.

EXAMPLE 3. A local ring (S, N) for which N is Koszul-regular (and thus $\mathrm{hd}_S N<\infty$) but not regular. In fact, $N-N^2$ consists of zero divisors.

Before defining the example we need some essential facts.

Fact #1. There exists a field K of characteristic $p=2$, such that $[K:K^2]=2^{\aleph_0}$ $=\operatorname{card} K$.

In fact, let K be the integers mod 2 with $c=2^{\aleph_0}$ indeterminates adjoined. These indeterminates are linearly independent over K^2.

Fact #2. If K is as above and $B\subset K$ is a p-base, then card $B=c$.

We now define Example 3. Let K be a field as in Fact #1, and let $B\subset K$ be a p-base of K. Let x, y, z be indeterminates, let $R^*=K[[x, y, z]]$, $R=K^2[[x, y, z]][K]$, $R_0=K^2[[x, y]][K]$. Let $M=(x, y, z)R$ and $M_0=(x, y)R_0$ be the maximal ideals of R and R_0 respectively.

Divide B into $c=2^{\aleph_0}$ pairwise disjoint subsets, each of which is denumerable. Associate to each $v\in M_0-M_0^2$ one of these subsets; call it $\{b(v, i)\}_{i=1}^{\infty}$. This is possible since card $M_0-M_0^2\leqq$ card $R^*=$ card $K^{\omega\times\omega\times\omega}=c^{\aleph_0}=c=$ card B, where $\omega=\{0, 1, 2, \ldots\}$ is the set of nonnegative integers.

Define, for every integer $n\geqq 1$,

$$G(x, n) = \sum_{i=n}^{\infty} y^i b(x, i)/y^n, \qquad E(u, n) = \sum_{i=n}^{\infty} x^i b(u, i)/x^n,$$

where u ranges over all elements satisfying

$$(*) \qquad u\in M_0-M_0^2 \quad\text{and}\quad u\notin xR_0.$$

Note that $E(u, n)^2$ and $G(x, n)^2\in R_0$. Define $e(u, n)=zE(u, n)$, $g(x, n)=zG(x, n)$. Set

$$\begin{aligned} C_n &= \{e(u, n), g(x, n) : u \text{ satisfies } (*)\},\\ D_n &= \{\text{finite products of two or more distinct elements of } C_n\},\\ F_n &= \{ue(u, n), ze(u, n), xg(x, n), zg(x, n) : u \text{ satisfies } (*)\}.\end{aligned}$$

Let $C=\bigcup_{n=1}^{\infty} C_n$, let $T=R[C]$, let $\mathfrak{A}$ be the *R-module* generated by

$$\{z\}\cup\bigcup_{n=1}^{\infty} F_n\cup\bigcup_{n=1}^{\infty} D_n.$$

Note that $\mathfrak{A}$ is also a *T-ideal.* Then

$$S = T/\mathfrak{A} \textit{ is the required example.}$$

By definition $e(u,n)=xe(u,u+1)+b(u,n)z, g(x,n)=yg(x,n+1)+b(x,n)z,$ and $E(u,n)^2$, $G(x, n)^2\in zR$. As in previous examples we can prove that T is local with maximal ideal $(x, y, z)T$. Thus, S is local with maximal ideal $N=(\bar{x}, \bar{y})S$, where $^-$ denotes residue class mod $\mathfrak{A}$. Also for every $\alpha\in T$ (resp. $\beta\in\mathfrak{A}$) and for all large n, α (resp. β) is an R-linear combination of $\{1\}\cup C_n\cup D_n$ (resp. $\{z\}\cup F_n\cup D_n$). The proof of 3.2 generalizes to show $\{1\}\cup C_n\cup D_n$ is R-linearly independent for every n.

Using the above facts, we can show $e(u, 1)$ and $g(x, 1)\notin\mathfrak{A}$, while $e(u, 1)\alpha$ and $g(x, 1)\alpha\in\mathfrak{A}$ for every $\alpha\in T\cdot C$ (where $T\cdot C$ is the T-ideal generated by C). Since $z\in\mathfrak{A}$, every element of N is of the form $\bar{u}+\bar{\alpha}$ where $u\in M_0$ and $\alpha\in T\cdot C$. But $\bar{\alpha}\in\bigcap N^i$ since, for example, $e(u, n)\equiv x^m e(u, n+m)$ (mod $\mathfrak{A}$). Thus, $\bar{u}+\bar{\alpha}\in N-N^2$ implies $u\in M_0-M_0^2$. If $u\in xR_0$, then $g(x, 1)(u+\alpha)\in\mathfrak{A}$. If $u\notin xR_0$, then u satisfies (*) and $e(u, 1)(u+\alpha)\in\mathfrak{A}$. Thus, every element of $N-N^2$ is a zero divisor.

Let $E=S\langle X, Y\rangle : \partial X=\bar{x}, \partial Y=\bar{y}$ be the Koszul complex. Claim $H_2(E)=0$. Suppose $\alpha \in T$ and $\alpha x, \alpha y \in \mathfrak{A}$. We must show $\alpha \in \mathfrak{A}$. By the above we can write, for large n,

$$\alpha = \alpha_0+\alpha_1 g(x, n)+\textstyle\sum_u \alpha_2(u)e(u, n)+\text{terms involving } D_n,$$

$$x\alpha = \beta_0 z+[\beta_1 x+\beta_2 z]g(x, n)+\textstyle\sum_u [\beta_3(u)u+\beta_4(u)z]e(u, n)+\text{terms involving } D_n,$$

$$y\alpha = \gamma_0 z+[\gamma_1 x+\gamma_2 z]g(x, n)+\textstyle\sum_u [\gamma_3(u)u+\gamma_4(u)z]e(u, n)+\text{terms involving } D_n,$$

where $\alpha_0, \alpha_1, \alpha_2(u)$, etc., $\in R$ and where u ranges over a finite set of elements L satisfying (*). Since $D_n \subset \mathfrak{A}$, we need only show $\alpha_0, \alpha_1 g(x, n), \alpha_2(u)e(u, n) \in \mathfrak{A}$ for all $u \in L$. Applying linear independence of $\{1\} \cup C_n \cup D_n$ over R, we get

$$x\alpha_0 = \beta_0 z, \qquad y\alpha_0 = \gamma_0 z,$$
$$x\alpha_1 = \beta_1 x+\beta_2 z, \qquad y\alpha_1 = \gamma_1 x+\gamma_2 z,$$
$$x\alpha_2(u) = \beta_3(u)u+\beta_4(u)z, \qquad y\alpha_2(u) = \gamma_3(u)u+\gamma_4(u)z.$$

Now zR, $(u, z)R$, and $(x, z)R$ are prime ideals of R. (By [11, p. 72] if $u_1, \dots, u_n$ generate the maximal ideal of a regular noetherian local ring, then $(u_1, \dots, u_i)$ is a prime ideal for $i=1, \dots, n$.) Hence, $x\alpha_0 \in zR$, $y\alpha_1 \in (x, z)R$, $x\alpha_2(u) \in (u, z)R$ imply $\alpha_0 \in zR$, $\alpha_1 \in (x, z)R$, $\alpha_2(u) \in (u, z)R$. Since $z, xg(x, n), ue(u, n) \in \mathfrak{A}$, we have that α_0, $\alpha_1 g(x, n)$, and $\alpha_2(u)e(u, n) \in \mathfrak{A}$. This proves $\alpha \in \mathfrak{A}$.

The proof that $H_1(E)=0$ is similar to the proof of a similar statement in Example 2. For instance, let $u=rx+sy \in M_0-M_0^2$, where $r, s \in R_0$ and $s \neq 0$. Let $e_n=e(u, n)$. Then $0 \equiv ue_n \pmod{\mathfrak{A}}$ implies $\bar{r}\bar{e}_n X+\bar{s}\bar{e}_n Y$ is a one-cycle of E. This cycle is a one-boundary since $0 \equiv ue_{n+1}$ and $e_n \equiv xe_{n+1} \pmod{\mathfrak{A}}$ imply

$$re_n \equiv re_{n+1}x \equiv (-se_{n+1})y, \qquad se_n \equiv (se_{n+1})x \pmod{\mathfrak{A}}.$$

Remark 1. Not every element of N^2 is a zero divisor in S. In fact, the prime element x^2+y^3 of R gives rise to a nonzero divisor when reduced mod $\mathfrak{A}$. However, by modifying construction of S we can make every element of N a nonzero divisor.

Remark 2. Suppose R is a *noetherian* local ring with maximal ideal M. If $M-M^2$ consists of zero divisors, then M does also and the annihilator of M is not zero [1, p. 653].

Example 4. A local ring (S, N) (with N finitely generated) in which regularity depends on the choice of the minimal generating set.

We will construct a local ring (S, N) where N is generated by two elements $\bar{x}, \bar{y}$ such that both $\bar{x}$ and $\bar{y}$ are zero divisors (and therefore neither $(\bar{x}, \bar{y})$ nor $(\bar{y}, \bar{x})$ are regular sequences) but $(\bar{x}+\bar{y}, \bar{x})$ is a regular sequence.

Let K be a field of characteristic 2 with $[K:K^2]=\infty$. Let $R^*=K[[x, y, z]]$, $R=K^2[[x, y, z]][K]$, and $\{b_i, c_i\}_{i=1}^{\infty} \subset K$ be an infinite set of p-independent elements. Define

$$E_n = \sum_{i=n}^{\infty} b_i x^i/x^n, \qquad F_n = \sum_{i=n}^{\infty} c_i y^i/y^n, \qquad e_n = zE_n, \qquad f_n = zF_n.$$

Set $T=R[e_1, f_1, \ldots, e_n, f_n, \ldots]$. Let $\mathfrak{A}$ be the R-module (or, equivalently, T-ideal) generated by $\{z, ye_n, ze_n, xf_n, zf_n, e_nf_n\}_{n=1}^{\infty}$. Then

$$S = T/\mathfrak{A} \text{ is the required example.}$$

The proof is much like the preceding ones. Note that for all n, the set $\{1, e_n, f_n, e_nf_n\}$ is linearly independent over R. The elements x, y are zero divisors since e_1y and $f_1x \in \mathfrak{A}$. We are done with Example 4.

Now let T be an arbitrary integral domain with x, y elements of T. We easily see $H_1(x, y; T)=0$ iff (x, y) is a regular sequence. The next example shows this to be false for sequences of three or more elements.

EXAMPLE 5. A local *integral domain* (T, N) in which regularity of the minimal generating set depends on order.

Let K, R^*, R, $\{b_i\}$ be as in Example 2. Set $E_n=\sum_{i=n}^{\infty} b_iz^i/z^n$, $e_n=yE_n, f_n=xE_n$. Then

$$T = R[e_1, f_1, \ldots, e_n, f_n, \ldots] \text{ is the required example.}$$

We have $yf_n=xe_n$, $e_n=ze_{n+1}+b_ny$, $f_n=zf_{n+1}+b_nx$ and e_nf_n, e_n^2, $f_n^2 \in R$. As with the previous examples, for every $\alpha \in T$ there is an n (depending on α) such that for all m we can write

$$\alpha = \alpha_0+\alpha_1z^me_{n+m}+\alpha_2z^mf_{n+m},$$

where $\alpha_i \in R$. We conclude $[(x, y, z)T^*]^m \cap T=[(x, y, z)T]^m$ for all m, $zR^* \cap T=zT$, $(x, z)R^* \cap T=(x, z)T$, $(y, z)R^* \cap T=(y, z)T$.

The first equation shows us that the maximal ideal of T is $(x, y, z)T$. The last three say that zT, $(xz)T$, and $(y, z)T$ are prime ideals of T. Since (0) is also a prime ideal (z, x, y), (z, y, x), (y, z, x), and (x, z, y) are all regular sequences.

However, neither (y, x, z) nor (x, y, z) are regular sequences since $e_1 \in (yT, xT)$ but $e_1 \notin yT$, and $f_1 \in (xT, yT)$ but $f_1 \notin xT$.

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NORTHWESTERN UNIVERSITY,
EVANSTON, ILLINOIS 60201