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TENSOR PRODUCTS OF POLYNOMIAL IDENTITY ALGEBRAS(1)

BY
ELIZABETH BERMAN(2)

Abstract. We investigate matrix algebras and tensor products of associative algebras over a commutative ring R with identity, such that the algebra satisfies a polynomial identity with coefficients in R. We call A a P. I. algebra over R if there exists a positive integer n and a polynomial f in n noncommuting variables with coefficients in R, not annihilating A, such that for all $a_1, \ldots, a_n$ in $A, f(a_1, \ldots, a_n)=0$. We call A a *P-algebra* if f is homogeneous with at least one coefficient of 1. We define the *docile identity*, a polynomial identity generalizing commutativity, in that if A satisfies a docile identity, then for all n, A_n, the set of n-by-n matrices over A, satisfies a standard identity. We similarly define the *unitary identity*, which generalizes anti-commutativity. Claudio Procesi and Lance Small recently proved that if A is a P. I. algebra over a field, then for all n, A_n satisfies some power of a standard identity. We generalize this result to P-algebras over commutative rings with identity. It follows that if A is a P-algebra, A satisfies a power of the docile identity.

It is well known that if A is a P. I. algebra over a field, then A also satisfies a homogeneous multilinear polynomial of the same degree with coefficients in the field [2, Chapter 10]. Clearly, the same is true if A is a P. I. algebra over a commutative ring with identity. If A is a P-algebra over a commutative ring with identity, then A satisfies a homogeneous multilinear polynomial with at least one coefficient of 1. Every P. I. algebra over a field is a P-algebra.

The following lemma generalizes the pigeonhole principle.

Lemma 1. *If k and q are positive integers ($q>1$), S a set with k elements, B a set with $k(q-1)+1$ elements, and g a function from B into S, then there exists i in S such that* $|\{g^{-1}(i)\}| \geqq q$.

Proof. Choose a subset A of B with $k(q-1)$ elements. Let g' be the restriction of g to A. Now if there exists i in S such that $|\{g'^{-1}(i)\}| \geqq q$, then we are done. Suppose that no such i exists. Then for all i in S, $|\{g'^{-1}(i)\}| \leqq q-1$. But

$$k(q-1) = |A| = \sum_{i=1}^{k} |\{g'^{-1}(i)\}|.$$

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Hence for all i, $|\{g'^{-1}(i)\}|=q-1$. Let b be the one remaining element of B and let $g(b)=i$. Then $|\{g^{-1}(i)\}|=q$.

DEFINITION 1. Let S be a fixed subset of the integers, 1 through n. An *S-permutation* of these n integers is any permutation of them sending S into S. In particular, a *checkered permutation* sends even integers into even integers.

DEFINITION 2. Let n be a positive integer and let S be a fixed subset of the integers, 1 through n. An *S-identity* is the following polynomial in n noncommuting variables, $x_1, \ldots, x_n$: $\sum_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$, where the sum is over all the S-permutations σ. If S is the entire set of integers, 1 through n, then we call the resulting polynomial identity the *unitary identity*, because all coefficients are 1.

Note that an algebra satisfies the unitary identity of degree 2 if and only if it is anticommutative. Thus the unitary identity generalizes anticommutativity.

DEFINITION 3. Let n be a positive integer. The *docile identity* is the following polynomial in n noncommuting variables, $x_1, \ldots, x_n$: $\sum_\sigma (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$, where the sum is over all the checkered permutations σ.

LEMMA 2. *If A is an algebra satisfying an S-identity of degree n, then A satisfies a unitary identity of degree n.*

Proof. Let S_n be the group of all permutations of the integers 1 through n. Let P be the set of all S-permutations of those integers. Clearly P is a subgroup of S_n. Let G be a set of distinct left coset representatives. Let $a_1, \ldots, a_n$ be elements of A, and evaluate g, the unitary identity of degree n:

$$\begin{aligned} g(a_1, \ldots, a_n) &= \sum_{\sigma \in S_n} a_{\sigma(1)} \cdots a_{\sigma(n)} \\ &= \sum_{\gamma \in G} \sum_{\pi \in P} a_{\gamma\pi(1)} \cdots a_{\gamma\pi(n)}. \end{aligned}$$

Let $\gamma \in G$. Let $b_1 = a_{\gamma(1)}, \ldots, b_n = a_{\gamma(n)}$. Then

$$\sum_{\pi \in P} a_{\gamma\pi(1)} \cdots a_{\gamma\pi(n)} = \sum_{\pi \in P} b_{\pi(1)} \cdots b_{\pi(n)} = 0.$$

Thus A satisfies g.

LEMMA 3. *If A is an algebra satisfying the docile identity of degree n, then A satisfies the standard identity of degree n.*

Proof. The proof is as above.

DEFINITION 4. Suppose that we have an ordered set of j elements of an algebra, $\{a_1, \ldots, a_j\}$. *To parenthesize them into clumps* is to insert adjacent pairs of parentheses such that each element lies in exactly one pair of parentheses. The elements within parentheses form a *clump*. The product of the elements in a clump in the stated order is the *value of the clump*.

THEOREM 1. *If A is an algebra satisfying an S-identity of degree m, then for all n, A_n satisfies f, the unitary identity of degree n^2m+1.*

Proof. If Z is the ring of the integers, then A_n is ring-isomorphic to $A \otimes_Z Z_n$. Z_n has n^2 matrix units. Let K be the set of simple tensors in $A \otimes_Z Z_n$ of form $a \otimes e_{ij}$, where a is an element of A and e_{ij} is a matrix unit in Z_n. Choose n^2m+1 elements of K in some order. We will show that f vanishes on them and hence, by Lemma 3, on A_n.

Let $p=n^2m+1$ and let C be the set of chosen elements, $u_1, \ldots, u_p$. Let B be the set of integers, 1 through p. Define a function f, B onto C: $f(t)=u_t$. Each u_t can be written as $a_t \otimes e_{ij}$ for unique e_{ij}. Let $h(u_t)=e_{ij}$. Let g be the composition function, $h \circ f$, sending B into S, the set of n^2 matrix units in Z_n. Now we apply Lemma 4, with $k=n^2$ and $q=m+1$. There exist at least $m+1$ elements of B on which g takes the same value in S. Choose $m+1$ such elements and call the common value of g, y. Write the corresponding simple tensors as $a_1 \otimes y, \ldots, a_{m+1} \otimes y$. Call the other elements, $b_1 \otimes z_1, \ldots, b_s \otimes z_s$. When we evaluate f on these elements, we get a sum of simple tensors, such as the following example, where we have parenthesized some elements into clumps:

$$b_1b_6(a_1b_2)(a_2b_3b_4)(a_3b_5)\cdots(a_m)a_{m+1}b_j \otimes z_1z_6(y\, z_2)(y\, z_3z_4)(y\, z_5)\cdots(y\,)yz_j.$$

Now we insert parentheses, as above, in each term of the sum, working from left to right in this way: start with the first a. Enclose it, along with all b's to the right, if any, up to the next a, in parentheses. Similarly enclose the next a with its b's, and so forth, up to the next-to-last a. The last a gets no parentheses. Do the same kind of parenthesizing to the corresponding y's and z's on the right.

We introduce a relation $*$ on the terms of the sum: we write $w * w'$ if the following three conditions hold:

1. The set of b's to the left of the first a in w and the corresponding set of b's to the left of the first a in w' are the same sets in the same order.

2. On the left sides the m clumps created by parenthesizing are the same, possibly in different order.

3. The last a's are the same and the b's following to the right are the same sets in the same order. For instance, the term above has the $*$ relation to

$$b_1b_6(a_3b_5)(a_1b_2)(a_m)\cdots(a_2b_3b_4)a_{m+1}b_j \otimes z_1z_6(y\, z_5)(y\, z_2)(y\,)\cdots(y\, z_3z_4)yz_j.$$

It is easy to see that $*$ is an equivalence relation and so induces a partition of the terms into disjoint equivalence classes. We will show that the sum of the terms in each class is 0.

Let C be an equivalence class. We will show that all the terms in C have right sides with the same value. If all terms in C have right sides 0, then the statement is certainly true. Suppose, then, that a term in C has nonzero right side.

Case 1. The right side has two adjacent y's. Then there exists i such that $y=e_{ii}$. Hence, in each clump on the right containing z's, the product of the z's in the clump must be e_{ii}. Thus all the right clumps have value e_{ii}.

Case 2. Every y before the last y has one or more z's to the right. Let $y=e_{ij}$. Then in each clump the product of the z's must be e_{ji}. So the value of each right clump is $e_{ij}e_{ji}=e_{ii}$.

In both cases all the right clumps are equal. Thus all terms in C have right sides with equal value; call it r. The sum of all the terms in C is then a simple tensor whose right side is r. But the left side of this simple tensor is the product of the following three factors:

1. The product of all the b's to the left of the first a.

2. The unitary identity of degree m evaluated on the values of the m clumps on the left.

3. The last a and all the b's to the right of it.

(The reader will modify this statement slightly in case there are no b's to the left of the first a or no b's to the right of the last a. A similar modification will be necessary in the proof of Theorem 2.)

Now since A satisfies an S-identity of degree m, it satisfies the unitary identity of degree m, by Lemma 5. Thus the second factor above must be 0. Hence f vanishes on K and the theorem follows.

COROLLARY TO THEOREM 1. *If A is an algebra of characteristic* 2, *and A satisfies the standard identity of degree m, then for all n, A_n satisfies the standard identity of degree n^2m+1.*

Proof. If the characteristic is 2, then the unitary and standard identities are the same.

Let us investigate the problem of making A_n satisfy a standard identity. Evaluating a standard identity, we get some terms with coefficient $+1$ and some with coefficient -1, depending on whether the permutation is odd or even. To analyze the problem of the sign of the term, we will need the following lemma:

LEMMA 4. HYPOTHESIS: *We have an ordered set of n elements of some set. We form two adjacent clumps and exchange them. The first clump has i elements and the second clump has j elements.*

CONCLUSION. *The sign of the resulting permutation of the original n elements is* $(-1)^{ij}$.

Proof. Without loss of generality, we can call the original n elements, $\{1, \ldots, n\}$. Here is an example of an exchange of adjacent clumps:

$$\{1, (2, 3,)(4, 5, 6,)7\} \to \{1, (4, 5, 6,)(2, 3,)7\}.$$

In general, if either clump has an even number of elements, then there will be an even number of inversions resulting from the exchange. But if both clumps have an odd number of elements, then there will be an odd number of inversions.

Henceforth we will call a clump containing an odd number of elements an *odd clump*; with an even number of elements, an *even clump*.

COROLLARY TO LEMMA 4. *If we have n ordered elements of a set and parenthesize them to form j odd clumps, and permute these clumps with some clump permutation σ, then the resulting permutation of the original elements is even if σ is even and odd if σ is odd.*

NOTE. If A is an algebra, and $a_1, \ldots, a_n$ are elements of A, and σ is a permutation of the integers, 1 through n, then, evaluating the standard identity of degree n:

$$[a_1, \ldots, a_n] = \pm [a_{\sigma(1)}, \ldots, a_{\sigma(n)}].$$

Whether we get a " + "or " − " sign depends on whether σ is even or odd. With similar reasoning, we see that if σ is a checkered permutation of the integers 1 through n, and we evaluate h, the docile identity of degree n, then

$$h(a_1, \ldots, a_n) = \pm h(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

Also note that the docile identity has an interesting property, which it shares with the standard identity: If A satisfies the docile identity of degree n, and n is less than m, then A also satisfies the docile identity of degree m. Thus if A satisfies a docile identity, it satisfies a docile identity of even degree.

THEOREM 2. *If A is an algebra satisfying the docile identity of degree $2k$, then for all n, A_n satisfies the standard identity of degree $2k^2n^2+1$.*

Proof. This proof will be quite similar to that of Theorem 1. Note that A_n is ring-isomorphic to $A \otimes_Z Z_n$. It suffices to show that f, the standard identity of degree $2k^2n^2+1$, vanishes on all elements of form $a \otimes e_{ij}$. Choose $2k^2n^2+1$ elements of this form in some order. As in Theorem 1, at least $2k^2+1$ of the e_{ij}'s must be the same; call them y. Let the corresponding simple tensors be $a_1 \otimes y, \ldots, a_p \otimes y$, with $p=2k^2+1$. Call the remaining elements, $b_1 \otimes z_1, \ldots, b_s \otimes z_s$.

Evaluating f, we get a sum of simple tensors. As in Theorem 1, insert parentheses on the left side of each term: start with the first a and enclose it with all b's to the right, if any. Similarly parenthesize the next a with its b's, and so on, up to but not including the last a. Thus we get $2k^2$ clumps on the left. Similarly make $2k^2$ clumps on the right.

We have a sum of terms, each with a coefficient of 1 or -1, depending on whether the permutation is even or odd. Let us analyze the situation into two cases.

Consider first, subset B: those terms with $2k$ or more odd clumps. We use the standard identity to attack B. On the terms of B we introduce a new parenthesizing with brackets: start with the first odd clump on the left side. Enclose it in brackets, along with all even clumps to the right, if any, up to the next odd clump. Call the bracketed elements a *super-clump*. Similarly enclose the next odd clump with all even clumps to the right of it, and so on, making a total of $2k$ adjacent super-clumps. Each super-clump contains an odd number of elements.

For example, in the following expression we have two odd clumps. Inserting brackets, we get two adjacent odd super-clumps:

$$b_1(a_1b_2)[(a_3b_4b_5)(a_6b_7)][(a_8)(a_9b_{10})]a_{11}b_{11}.$$

Now introduce a relation $*$ on the terms of B: write $w * w'$ if the following three conditions hold in w and w':

1. The elements to the left of the first super-clump are the same elements in the same order.

2. The $2k$ super-clumps are the same, but in any order.

3. The elements to the right of the last super-clump are the same elements in the same order.

The relation $*$ is an equivalence relation on B and induces a partition of B into disjoint equivalence classes. We will show that the sum of the signed terms in each. class is 0.

Let C be an equivalence class. All the terms in C have right sides equal in value. When we permute odd super-clumps, the resulting permutation of the original elements is just what we would like: odd if the super-clump permutation is odd, and even if the super-clump permutation is even. Thus the sum of the signed terms in C is a simple tensor whose left side is the product of the following three factors or its negative:

1. The product of all elements left of the first super-clump, in the order given.

2. The standard identity of degree $2k$ evaluated on the values of the $2k$ super-clumps, in some order.

3. The product of all elements right of the last super-clump, in the order given.

Now A satisfies the docile identity of degree $2k$, so by Lemma 6, A also satisfies the standard identity of degree $2k$. Thus the second factor is 0. Hence the sum of the signed terms in the class C is 0, and we dispose of B.

Now we must consider the remaining subset D of terms: those with $2k-1$ or fewer odd clumps. We will use the docile identity to attack D. A term in D has on the left $2k^2$ clumps, each containing an a and possibly some b's to the right of the a. We will call the b's in each clump the *b sub-clump* of that clump. An even clump consists of an a followed by an odd number of b's. We want to show that there are k adjacent even clumps in the term. Section off the $2k^2$ clumps into $2k$ adjacent sections, each containing k clumps. Now if each section contains one or more odd clumps, then there would be at least $2k$ odd clumps in the term, a contradiction. Hence at least one section consists entirely of even clumps. Thus there are k adjacent even clumps somewhere in the term. In each term of D we can find the first set of k adjacent even clumps; call this set T. (T depends on the particular term.)

We introduce an equivalence relation on the terms of D. Write $w \approx w'$ if the following three conditions hold:

1. The elements to the left of T are the same in the same order.

2. The k adjacent even clumps in T have the same a's and the same b sub-clumps.

We permit any order of the a's and b sub-clumps, so long as they alternate, starting with an a.

3. The elements to the right of T are the same in the same order.

For example, if we want three adjacent even clumps, terms with the following left sides have the $\approx$ relation:

$$(a_1)(a_2b_2)(a_3b_4b_5b_6)(a_4b_7)a_5b_8$$

and

$$(a_1)(a_3b_7)(a_2b_4b_5b_6)(a_4b_2)a_5b_8.$$

Now $\approx$ is an equivalence relation on D and induces a partition into disjoint equivalence classes. Let C be one class. All the terms in C have right sides equal in value. The reason is that if at least one term in C has nonzero right side, and $y=e_{ij}$ then each b sub-clump must have value e_{ji}. In the k adjacent even clumps, the b sub-clumps are all odd. Of course any single a forms an odd sub-clump. Thus the sum of all the signed terms in C is a simple tensor whose left side is the product of the following three factors or its negative:

1. The product of the elements to the left of T.

2. The docile identity of degree $2k$, evaluated on the k a's and k b sub-clumps of T in some order, starting with an a and alternating a's and b sub-clumps.

3. The product of the elements to the right of T.

Since A satisfies the docile identity of degree $2k$, the second factor is 0. Thus we dispose of D. Hence A_n satisfies f.

To demonstrate the necessity of the condition that A satisfy a docile identity, we prove the following:

THEOREM 3. *If A is an algebra and A_2 satisfies a standard identity of degree k, then A satisfies a docile identity of degree k.*

Proof. Choose k elements of A in some order, $a_1, \ldots, a_k$. Suppose that k is even. Consider

$$[a_1 \otimes e_{12}, a_2 \otimes e_{21}, a_3 \otimes e_{12}, \ldots, a_k \otimes e_{21}].$$

The value is a sum of two simple tensors, one with right side e_{11}; the other, e_{22}. Let P be the set of all checkered permutations of the integers, 1 through k. Then the simple tensor with e_{11} on the right side has left side

$$\sum_{\pi \in P} (-1)^{\pi} a_{\pi(1)} \cdots a_{\pi(n)}.$$

Since A_2 satisfies the standard identity of degree k, this sum must be 0. Hence A satisfies the docile identity of degree k. There is a similar argument if k is odd.

If we call an algebra satisfying a docile identity a *docile algebra*, then we get the following theorem:

THEOREM 4. *If A is a docile algebra, then for all n, A_n is a docile algebra.*

Proof. Let A satisfy the docile identity of degree $2k$. It is well known that A_{2n} is isomorphic to $(A_n)_2$. By Theorem 2, A_{2n} satisfies the standard identity of degree $8k^2n^2+1$. By Theorem 3, A_n satisfies the docile identity of degree $8k^2n^2+1$.

DEFINITION 5. Let A be an algebra over a commutative ring R with identity. Suppose that there exists an index set I and a set J of nonzero elements $\{e_{ij}\}$ of A, for all i, j in I, such that if $i, j, k, p \in I$, then $e_{ij}e_{kq}=\delta_{jk}e_{ip}$. Then we call J a set of *matrix units* in A. If every element of A can be written as a linear combination of elements of J, then we say that A is *spanned by matrix units.*

Note that we do not assume that J is finite.

DEFINITION 7. Let A be an algebra over a commutative ring with identity. Let J be a set of matrix units in it. Let n and m be positive integers. An *n-subset* of J is a set of n elements of J of the form

$$\{e_{i(1),i(2)}, e_{i(2),i(3)}, \ldots, e_{i(n),i(n+1)}\}$$

where $j\neq k$ implies $i(j)\neq i(k)$. An *a-m row* in J is an ordered set of form

$$\{a_1, b_1, a_2, b_2, \ldots, a_m\}$$

where all the a's lie in J, and each b is either in J or is the scalar 1.

LEMMA 5. HYPOTHESIS: *A is an algebra and J is a set of matrix units in it. n is a positive integer.* $m=m_n=n+\sum_{k=1}^{n}k^2$.

CONCLUSION: *Given an a-m row, at least one of the following holds*:

1. $\prod_i a_ib_i=0$.
2. *At least two of the a's are identical.*
3. *There exists q such that we can take the first q elements of the row and parenthesize them into n clumps, such that the values of the clumps form an n-subset.*

Proof. We induct on n. Let $n=1$. Then $m=2$. A 1-subset is a set consisting of one element, $\{e_{ij}\}$, with $i\neq j$. Suppose that we have an a-2 row, $\{a_1, b_1, a_2\}$, with a_1 and a_2 in J and b either in J or the scalar 1. Suppose that $a_1\neq a_2$ and $a_1b_1a_2\neq 0$.

Case 1. $a_1=e_{ij}$, with $i\neq j$. Then $\{(a_1), b_1, a_2\}$ is the desired parenthesizing.

Case 2. $a_1=e_{ii}$. We divide this case into subcases.

Case 2a. $b_1=1$. Then $a_2=e_{ij}$ with $i\neq j$, because the a's are distinct. Then $\{(a_1, b_1, a_2)\}$ is the desired parenthesizing.

Case 2b. $b_1=e_{ij}$ with $i\neq j$. Then $\{(a_1, b_1,) a_2\}$ is the desired parenthesizing.

Case 2c. $b_1=e_{ii}$. Then $a_2=e_{ij}$ with $i\neq j$. Then $\{(a_1, b_1, a_2)\}$ is the desired parenthesizing.

We have established the theorem for $n=1$. Suppose that it is true for n; consider $n+1$. Let $m=m_n=n+\sum_{k=1}^{n}k^2$. By induction hypothesis, given an a-m row, either the product is 0, or at least two a's are identical, or we can parenthesize the first q elements into n clumps whose values form an n-subset. Let $s=m+(n+1)^2+1$. Suppose that we have an a-s row, $\{a_1, b_1, \ldots, a_s\}$, with all a's distinct and with nonzero product. The first $2m-1$ elements form an a-m row, so there exists

$q \leqq 2m-1$ such that we can parenthesize the first q elements into clumps, whose values form an n-subset:

$$\{e_{i(1),i(2)}, \ldots, e_{i(n),i(n+1)}\}.$$

Let B be the set of $n+1$ indices represented in these subscripts.

Now there is exactly one way to arrange an n-subset so as to get a nonzero product: the order above. So the last clump must have value, $e_{i(n),i(n+1)}$. Call the next element x. (It could be an a or b.) From x through a_{s-1}, we have at least $(n+1)^2$ a's. Each a is of form e_{ij}. If one of these a's has an index element not in B appearing as a subscript, then we are done. Otherwise, each a from x through a_{s-1} has both subscripts in B. But there are exactly $(n+1)^2$ ordered pairs that one can choose from the elements of B. Furthermore, all the a's are distinct. So each ordered pair actually appears among these a's. Let $a_{s-1}=e_{ij}$, with i and j in B.

Case 1. $b_{s-1}=1$. Then $a_s=e_{jk}$ for some k not in B. Then the elements from x through a_s form a clump with value $e_{i(n+1),k}$ as desired.

Case 2. $b_{s-1}=e_{jk}$, with k not in B. Then the elements from x through b_{s-1} form the desired clump.

Case 3. $b_{s-1}=e_{jk}$ with k in B. Then $a_s=e_{kp}$ for some p not in B. Then the elements from x through a_s form the desired clump.

In any case, when we combine the new clump with the previous n clumps insured by the induction hypothesis, we get $n+1$ clumps whose values form an $(n+1)$-subset. Since $s=m_n+(n+1)^2+1$, $s=m_{n+1}$ and the theorem follows.

THEOREM 5. HYPOTHESIS: *A is a P. I. algebra over a commutative ring R with identity, satisfying a polynomial identity of degree n. A is spanned by a set of matrix units, J. q is a positive integer. $m=n+\sum_{k=1}^{n} k^2$.*

CONCLUSION: *A_q satisfies the standard identity of degree $q^2(m-1)+1$.*

Proof. A satisfies a homogeneous multilinear polynomial identity of degree n; call it h. Let

$$h(x_1, \ldots, x_n) = r'x_1 \cdots x_n + \sum r_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

where the sum is over all permutations σ of n elements except the identity permutation, the r's are scalars in R, and r' does not annihilate A.

Suppose that there exists a matrix unit e_{ij} in J such that $e_{ij}r'=0$. Let e_{kp} be any matrix unit. Then $e_{kp}r'=(e_{ki}e_{ij}e_{jp})r'=0$. Thus r' annihilates all the matrix units and hence r' annihilates A, a contradiction. Thus r' annihilates no matrix unit.

We claim that J cannot contain any n-subsets. For suppose that it does contain one:

$$\{e_{i(1),i(2)}, \ldots, e_{i(n),i(n+1)}\}.$$

Then

$$h(e_{i(1),i(2)}, \ldots, e_{i(n),i(n+1)}) = r'e_{i(1),i(n+1)} \neq 0.$$

Now by the previous lemma, given an a-m row, either the product is 0, or two a's are identical, or we can extract an n-subset. But the third possibility is outlawed in A, so the first two must prevail.

It is easy to see that A_q and $A \otimes_R R_q$ are isomorphic algebras. We will consider the latter. Let f be the standard identity of degree $q^2(m-1)+1$. Let K be the set of simple tensors of form $a \otimes e_{ij}$, where a is in J and e_{ij} is a matrix unit in R_q. Choose $q^2(m-1)+1$ elements of K in some order. At least m of the e_{ij}'s must be equal. Choose m equal e_{ij}'s and call them all y. Call the associated simple tensors $a_1 \otimes y, \ldots, a_m \otimes y$. Call the other elements $b_1 \otimes z_1, \ldots, b_s \otimes z_s$. Then, evaluating f, we get a sum of simple tensors. In each the left side is a product of the m a's and the s b's in some order. Now if each left side is 0, then certainly the sum of these simple tensors is 0. Suppose that at least one left side is different from 0. Then there exists $i \neq j$ such that $a_i = a_j$. So we are actually evaluating f on a set including two identical elements, $a_i \otimes y$ and $a_j \otimes y$. The standard identity always vanishes on such a set. Thus f vanishes on K and hence on A_q.

Up to now, we have been considering $A \otimes_R R_n$, where A is a P. I. algebra over R. More general is the problem: If A and B are P. I. algebras over R, what about $A \otimes_R B$? We proceed to this problem, proving first the following lemma.

LEMMA 6. HYPOTHESIS: *A is an algebra and J is a set of matrix units in it. p and n are positive integers, $p \geqq 2$. $r = r_n = n + (p-1)\sum_{k=1}^{n} k^2$.*

CONCLUSION: *Given r matrix units, $a_1, \ldots, a_r$, at least one of the following holds*:

1. $\prod_{i=1}^{r} a_i = 0$.
2. *At least p of the a's are identical.*
3. *There exists $t \leqq r$ such that we can parenthesize the first t a's into clumps whose values form an n-subset.*

Proof. We induct on n. If $n=1$, then $r = r_1 = 1 + p - 1 = p$. Suppose that the product is not 0 and the p a's are not identical. If $a_1 = e_{ij}$, $i \neq j$, then $\{(a_1)a_2, \ldots, a_r\}$ is the desired parenthesizing. If $a = e_{ii}$, then there exists $t < p$ such that $a_t \neq e_{ii}$. Choose smallest such t. Then $a_t = e_{ij}$, $i \neq j$. Then $\{(a_1, \ldots, a_t) \ldots, a_r\}$ is as desired.

Suppose that the theorem is true for n; consider $n+1$. Let $r = r_n$. Let $s = r + 1 + (p-1)(n+1)^2$. Suppose that we have s matrix units, $a_1, \ldots, a_s$, and neither statement (1) nor statement (2) holds. Then there exists $t \leqq r$ such that we can parenthesize the first t elements to form an n-subset:

$$\{e_{i(1),i(2)}, \ldots, e_{i(n),i(n+1)}\}.$$

Let B be the set of $n+1$ indices represented in these subscripts. There are $(n+1)^2$ ordered pairs of these indices. The last clump must have value $e_{i(n),i(n+1)}$. Consider the next $(p-1)(n+1)^2+1$ a's. If they all have subscripts in B, then by Lemma 4, at least p must be identical, a contradiction. So at least one has a subscript not in B, and if we take this element and parenthesize it with all preceding unclumped elements, we get a new clump whose value is $e_{i(n+1),i(n+2)}$, where $i(n+2)$ does not appear in B. Thus we have extracted an $(n+1)$-subset. Since $s=r_{n+1}$, the lemma is proven.

The following theorem has a proof similar to preceding proofs:

THEOREM 6. HYPOTHESIS: *A and B are algebras over a commutative ring R with identity. B is a P. I. algebra spanned by matrix units.*

CONCLUSION: *If A is also a P. I. algebra spanned by matrix units, then* $A \otimes_R B$ *satisfies a standard identity. If A satisfies an S-identity, then* $A \otimes_R B$ *satisfies a unitary identity. If A satisfies a docile identity, then* $A \otimes_R B$ *satisfies a standard identity.*

DEFINITION 8. If A is a P-algebra satisfying a homogeneous polynomial f with at least one coefficient of 1, of degree d, then we call d the *degree* of A.

THEOREM 7. *If A is a semiprime P-algebra of degree d over a commutative ring R with identity, and* $m=[d/2]^2$, *then there exists a commutative algebra B, a direct product of fields, such that A is isomorphic to a subalgebra of* B_m.

Proof. The proof is as in Jacobson [2, pp. 226–227]. A is a subalgebra of a direct product of primitive P-algebras, which can be considered P. I. algebras over fields.

Procesi and Small showed that if A is a P. I. algebra over a field, then for all n, A_n satisfies some power of a standard identity [3]. We prove next this theorem for P-algebras over rings. Their proof depends on a theorem by Amitsur: if A is a P. I. algebra over a field then A satisfies a power of the standard identity. Our proof is also an easy proof of Amitsur's theorem, generalizing it to algebras over rings as well as over fields.

LEMMA 7. *If A and B are algebras over a commutative ring with identity, and g is an algebra homomorphism, A into B, with kernel K, and g′ is the corresponding function,* A_n *into* B_n, *then g′ is an algebra homomorphism with kernel* K_n.

THEOREM 8. *If A is a P-algebra of degree d over a commutative ring R with identity, then for all n,* A_n *satisfies a power of the standard identity of degree* $2n[d/2]^2$.

Proof. Let A satisfy f, a homogeneous multilinear polynomial of degree d, coefficient in R, and at least one coefficient of 1. Let B be the free algebra over R, freely generated by $e_1, e_2, \ldots$. Let D be the algebra-ideal of B generated by all elements of B of form $f(b_1, \ldots, b_m)$, where, for all i, $b_i \in B$. Let $Q=B/D$. Let N be the nil radical of Q. Then Q/N is semiprime and satisfies f. Let $p=2n[d/2]^2$. Then the matrix algebra $(Q/N)_n$ satisfies the standard identity of degree p. Relabel all the e's in B, calling them $\{e_{ij}^k\}$, where $1 \leqq i, j \leqq n$ and $1 \leqq k$. Let $e_{ij}^{k\prime}=e_{ij}^k+D \in Q$. Let $e_{ij}^{k\prime\prime}=e_{ij}^{k\prime}+N \in Q/N$. Let (e_{ij}^k) be the n-by-n matrix in B_n whose entry in row i, column j is e_{ij}^k. Then in $(Q/N)_n$ we have

$$[(e_{ij}^{1\prime\prime}), \ldots, (e_{ij}^{p\prime\prime})] = 0.$$

Therefore in Q_n:

$$[(e_{ij}^{1\prime}), \ldots, (e_{ij}^{p\prime})] = w \in N_n.$$

Let the matrix w be (w_{ij}), with $w_{ij} \in N$.

Now it is well known that every nil P-algebra is locally nilpotent [3, p. 232]. (The proof there for algebras over fields also applies to P-algebras over a commutative ring with identity.) Since N is a nil P-algebra, the finite set $\{w_{ij}\}$ generates a nilpotent subalgebra W of N. There exists a positive integer t such that $W^t=0$. Hence in Q_n, $w^t=0$. Thus in B_n,

$$[(e_{ij}^1), \ldots, (e_{ij}^p)]^t = (d_{ij}) \in D_n.$$

Now let $\{a_{ij}^k\}$ be any n^2p elements of A, $1 \leqq i, j \leqq n$; $1 \leqq k \leqq p$. Let $g(e_{ij}^k)=a_{ij}^k$ and extend g to get a homomorphism, B into A, inducing a homomorphism g', B_n into A_n. Then

$$[(a_{ij}^1), \ldots, (a_{ij}^p)]^t = g'((d_{ij})) = 0.$$

Procesi and Small showed that if A and B are P. I. algebras over a field and A is semiprime, then $A \otimes B$ is a P. I. algebra [3]. We use this to prove

THEOREM 9. *If A and B are P. I. algebras over a field, and A has nilpotent upper nilradical N, then $A \otimes B$ is a P. I. algebra.*

Proof. A/N is semiprime; hence $A/N \otimes B$ satisfies a polynomial $f(x_1, \ldots, x_n)$. There exists t such that $N^t=0$. Let $w_{11}, \ldots, w_{nt}$ be nt elements of $A \otimes B$. Let g be the obvious homomorphism of $A \otimes B$ onto $A/N \otimes B$. $\ker g = A \otimes N$. Let $g(w)=w'$. For all j, $f(w'_{1j}, \ldots, w'_{nj})=0$. Thus $f(w_{1j}, \ldots, w_{nj})=w_j \in \ker g$. Hence $\prod_j w_j=0$. Thus $A \otimes B$ satisfies the polynomial $\prod_j f(x_{1j}, \ldots, x_{nj})$.

THEOREM 10. *If A is a P-algebra over a commutative ring R with identity, then A satisfies a power of the docile identity.*

Proof. Let A_2 satisfy $[x_1, \ldots, x_n]^t$. Suppose that n is even. Consider $A \otimes_R R_2$, isomorphic to A_2. Choose n elements of A: $a_1, \ldots, a_n$. Consider

$$[a_1 \otimes e_{12}, a_2 \otimes e_{21}, a_3 \otimes e_{12}, \ldots, a_n \otimes e_{21}]^t.$$

Let P be the set of all checkered permutations of the integers 1 through n. Then the value of the expression within brackets is

$$\sum_{\pi \in P} (-1)^\pi a_{\pi(1)} \cdots a_{\pi(n)} \otimes e_{11} + a \otimes e_{22},$$

where a is some element of A.

Raising the above expression to the tth power, we see that

$$\left(\sum_{\pi \in P} (-1)^\pi a_{\pi(1)} \cdots a_{\pi(n)}\right)^t = 0.$$

Hence A satisfies the tth power of the docile identity of degree n. There is a similar proof for n odd.

THEOREM 11. *Suppose that A and B are P. I. algebras over a field F and A is semiprime. Then if B satisfies a docile identity, $A \otimes B$ satisfies a standard identity. If B satisfies an S-identity, then $A \otimes B$ satisfies a unitary identity.*

Proof. A is a subalgebra of $C \otimes F_m$, where C is a commutative algebra over F. Thus $A \otimes B$ is a subalgebra of $(A \otimes C)_m$. $A \otimes C$ satisfies any homogeneous multilinear identity satisfied by A. Use Theorems 1 and 2.

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UNIVERSITY OF MISSOURI,
KANSAS CITY, MISSOURI 64110