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SUMMABILITY IN AMENABLE SEMIGROUPS

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Abstract. A theory of summability is developed in amenable semigroups. We give necessary and (or) sufficient conditions for matrices to be almost regular, almost Schur, strongly regular, and almost strongly regular. In particular, when the amenable semigroup is the additive positive integers, our theorems yield those results of J. P. King, P. Schaefer and G. G. Lorentz for some of the matrices mentioned above.

1. **Introduction.** Let S be an infinite left amenable semigroup without any finite left ideals. Let C_∞ be the space of convergent functions, and F be the space of left almost convergent functions (see §2 for definitions). By an infinite matrix on S we shall mean a real-valued function on $S \times S$. If A is an infinite matrix on S, consider the following cases:

(1) Af (see §2 for the definition of Af) is convergent for every bounded real-valued function f on S (Schur matrices).

(2) Af is left almost convergent for every bounded real-valued function f on S (almost Schur matrices).

(3) Af is convergent to k whenever f is convergent to k (regular matrices).

(4) Af is left almost convergent to k whenever f is convergent to k (almost regular matrices).

(5) Af is convergent to k whenever f is left almost convergent to k (strongly regular matrices).

(6) Af is left almost convergent to k whenever f is left almost convergent to k (almost strongly regular matrices).

In the case when S is the semigroup of positive integers under addition, necessary and sufficient conditions for matrices satisfying (1), (3), (4) and (5) have been obtained by J. Schur in [13], O. Toeplitz in [14], J. P. King in [8] and G. G. Lorentz in [9], respectively. More recently, P. Schaefer in [12] gave *sufficient* conditions for matrices satisfying (6). It is the purpose of this paper to give necessary and (or)

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sufficient conditions for matrices satisfying (1) to (6) when S is any left amenable semigroup without any finite left ideals. The reason for this restriction is that only in such semigroups is almost convergence a generalization of convergence in the sense defined in §2. (See Theorem 3.1.)

It should be pointed out here that in the cases where we are not concerned with the space F, (as in (1) and (3)) the results do not depend on the fact that S is a left amenable semigroup. Consequently, the results of J. Schur and O. Toeplitz for Schur matrices and regular matrices can be carried over to any set S without too much difficulty.

One of the main results in this paper is the following theorem, which is a generalization of a result of G. G. Lorentz:

THEOREM. *Let S be a left cancellative left amenable semigroup without any finite left ideals. Let S be generated by $B\subset S$. Then the following conditions are both necessary and sufficient for an infinite matrix A on S to be strongly regular*:

(i) $\sup_s \sum_t |A(s, t)| < M$ *for some* $M > 0$.

(ii) $\lim_s \sum_t A(s, t) = 1$.

(iii) $\lim_s \sum_t |(A(s, t) - A(s, at)| = 0$ *for every* $a \in B$.

If S is extremely left amenable, not necessarily left cancellative, then condition (iii) *above can be replaced by*

(iv) $\lim_s \sum_{t\in S\sim aS} |A(s, t)| = 0$ *for every $a \in S$ such that $a \in Sa$.*

If S is the semigroup of additive positive integers, the above theorem yields the following theorem of G. G. Lorentz [9, p. 181, Theorem 8].

THEOREM. *Let N be the semigroup of additive positive integers. Then an infinite matrix A on N is strongly regular iff A is regular and* $\lim_n \sum_m |A(n, m) - A(n, m+1)| = 0$.

Our theorems seem to be new even when applied to the multiplicative semigroup of positive integers with $B = \{\text{prime positive integers}\}$.

2. **Definitions and notations.** Let S be a set. A function f on S with values in a linear topological space L is called unconditionally summable to g in L if $\lim_{\sigma\in\Sigma} \sum_{s\in\sigma} f(s) = g$, where Σ is the family of all finite subsets of S directed by inclusion. We shall denote this by $g = \sum_{s\in S} f(s)$ and say the sum $\sum_{s\in S} f(s)$ converges to g [2]. In particular, we may take L to be the reals. Then $\sum_{s\in S} f(s) = g$ if for every $\varepsilon > 0$ there is a finite subset σ_1 such that if $\sigma \supseteq \sigma_1$ then $|\sum_{s\in\sigma} f(s) - g| < \varepsilon$. It is well known that the above definition implies only countably many $f(s)$ are different from 0 [7, p. 19, Theorem 1].

Let S be a set and $S \cup \{\infty\}$ be the one-point compactification of S when S has the discrete topology. Let $m(S)$ be the linear space of all bounded real-valued functions on S with the sup norm, and let C_∞ be the closed linear subspace of all those f in $m(S)$ such that $\lim_{s\to\infty} f(s)$ exists. From now on, we shall write $\lim_s f(s)$ for $\lim_{s\to\infty} f(s)$, so that $\lim_s f(s) = k$ means that for every $\varepsilon > 0$ there is a finite $\sigma \subset S$ such that $|f(s) - k| < \varepsilon$ if $s \in S \sim \sigma$. If, in addition, S is a semigroup, then, for

$f \in m(S)$, $a \in S$, $p_a(f) = f(a)$, and l_a $[r_a]$ is the left [right] translation operator on $m(S)$ defined by $l_a f(s) = f(as)$ $[r_a f(s) = f(sa)]$. The conjugate mapping of l_a will be denoted by L_a. If Co A denotes the convex hull of A then elements in Co $\{p_a : a \in S\}$ are called finite means. A linear functional φ on $m(S)$ is a left invariant mean (LIM) if $\varphi(f) \geqq 0$ for $f \geqq 0$, $\varphi(1) = 1$ and $\varphi(l_a f) = \varphi(f)$ for all $f \in m(S)$ and all $a \in S$, where 1 is the constant one function on S, and $f \geqq 0$ means $f(s) \geqq 0$ for all $s \in S$. We denote the set of all left invariant means by $Ml(S)$. If $Ml(S) \neq \varnothing$, where $\varnothing$ denotes the empty set, then the semigroup S is said to be left amenable (LA). If, in addition, φ is multiplicative, i.e. $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in m(S)$ then S is said to be extremely left amenable (ELA). Examples of left amenable semigroups are commutative semigroups, solvable groups and locally finite groups. For details and an excellent reference see [1]. Extremely left amenable semigroups are precisely those semigroups in which every two elements have a common right zero. For details and other interesting results see [3], [4], [5], and [11].

If S is LA, then a function $f \in m(S)$ is said to be left almost convergent to k if $\varphi(f) = \psi(f) = k$ for every $\varphi, \psi \in Ml(S)$. We shall denote the set of all almost convergent functions by F, and write f is lac to k to mean f is left almost convergent to k.

If $A = (A(s, t))$ is an infinite matrix on S and $f \in m(S)$, let Af be the function defined on S by $Af(s) = \sum_t A(s, t)f(t)$, whenever the sum on the right-hand side converges for each $s \in S$. We say f is F_A-summable to k iff $\lim_s \sum_t A(s, t)f(tb) = k$ uniformly in b, where $b \in S$. This generalizes the definition by G. G. Lorentz [9, p. 171].

3. Convergence and left almost convergence. We show, with our definition of convergence, that $C_\infty \subset F$ in a left amenable semigroup without any finite left ideals.

3.1. THEOREM. *Let S be a LA semigroup. Then f is lac to k whenever f is convergent to k iff S does not contain any finite left ideals.*

Proof. Suppose S does not contain any finite left ideals. We first show $\varphi(1_a) = 0$ for any LIM φ and any $a \in S$, where 1_A, here and elsewhere, denotes the characteristic function of A. We shall write $\varphi(A)$ for $\varphi(1_A)$. If $\varphi(a) > 0$ then since φ is left invariant, $\varphi(sa) \geqq \varphi(a) > 0$ for all $s \in S$. Since $\varphi(S) = 1$, Sa has to be a finite left ideal, which cannot be.

Suppose now $f \in C_\infty$ and $\lim_s f(s) = 0$. For $\varepsilon > 0$, let H be the finite subset of S for which $|f(s)| < \varepsilon$ whenever $s \in S \sim H$. Let $M = \max_{s \in H} |f(s)|$. Then $|f(s)| \leqq \sum_{a \in H} M 1_a + \varepsilon 1_{S \sim H}$. Hence if φ is any LIM then $|\varphi(f)| < \varepsilon$. And since ε is arbitrary, we see that $\varphi(f) = 0$. If now $\lim_s f(s) = k$, then, by considering $f - k$, we see that f is lac to k.

Conversely, suppose S has a finite left ideal A. Then using an argument employed in the proof of Theorem II-2 in [6] one can show that A contains a minimal left ideal G of S which is also a group. Define φ on $m(S)$ by $\varphi(f) = (1/N(G)) \sum_{a \in G} f(a)$, where $N(G)$ is the cardinality of $G, f \in m(S)$. It can easily be checked that φ is indeed a LIM. Clearly the function 1_G is convergent to 0, while 1_G is not lac to 0 since $\varphi(1_G)=1$.

In view of the above theorem, whenever we consider LA semigroups *we shall always assume the semigroup to be infinite and without any finite left ideals*, even though we might not explicitly mention so.

4. **Almost regular matrices.** We say an infinite matrix A is *almost regular* if Af lac to k whenever $\lim_s f(s)=k$.

We first prove the following useful lemma which will be used throughout this paper.

4.1. Lemma. *Let A be an infinite matrix on S. A necessary and sufficient condition for $Af\in m(S)$ whenever $f\in C_\infty$ is that there exists an $M>0$ such that*

$$\sup_s \sum_t |A(s,t)| < M.$$

Proof. If $\sup_s \sum_t |A(s,t)|<\infty$ then clearly for each $s\in S$ the sum $\sum_t A(s,t)f(t)$ exists and $\|Af\|<M\|f\|$ for each $f\in C_\infty$.

Assume that $Af\in m(S)$ whenever $f\in C_\infty$. Then for each s, $\sum_t A(s,t)$ exists and therefore $\sum_t |A(s,t)|<\infty$ [Kelley, *General topology*, Van Nostrand, 1955, p. 78]. Let $C_0=\{f\in C_\infty : \lim_s f(s)=0\}$. If we write $g_s(t)=A(s,t)$ then $g_s\in l_1(S)=C_0^*$. By the assumption, for each $f\in C_0$,

$$\sup_s |g_s(f)| = \sup_s |Af(s)| < \infty.$$

Hence by the Banach-Steinhaus theorem

$$\sup_s \|g_s\|_1 = \sup_s \sum_t |A(s,t)| < \infty,$$

where $\|\quad\|_1$ denotes the l_1-norm.

4.2. Theorem. *Let S be an LA semigroup. Then a matrix A is almost regular iff the following conditions are satisfied*:

(4.2.1) $\sup_s \sum_t |A(s,t)|<M$ *for some* $M>0$.

(4.2.2) $A(s,t)$, *as a function of s, is lac to* 0 *for each* $t\in S$.

(4.2.3) $\sum_t A(s,t)$, *as a function of s, is lac to* 1.

Proof. Suppose A is almost regular. Then (4.2.1) follows from Lemma 4.1. Conditions (4.2.2) and (4.2.3) follow if we note that $A1_t(s)=A(s,t)$ and $A1(s)=\sum_t A(s,t)$.

Conversely, suppose (4.2.1), (4.2.2) and (4.2.3) hold. Then (4.2.1) together with 4.1 implies $Af\in m(S)$ for every $f\in C_\infty$. Using (4.2.2) and (4.2.3), Af is lac to $\lim_s f(s)$ whenever f is in the set $B=\{1, 1_t : t\in S\}$. The proof is then completed by noting that B is fundamental in C_∞, i.e. the uniform closure of the linear span of B is C_∞.

4.3. REMARK. (a) J. P. King was the first to consider almost regular matrices, and for the semigroup of additive positive integers, 4.2 yields King's Theorem 3.2 in [8].

(b) Let $(N, +)$ and $(N, \cdot)$ denote the semigroup of additive positive integers and the semigroup of multiplicative positive integers respectively. Define the matrix A by

$$\begin{aligned} A(m, n) &= \delta(n, 1) \quad \text{if } m \text{ is odd,} \\ &= \delta(m, n) \quad \text{if } m \text{ is even,} \end{aligned}$$

where $\delta(m, n) = 1$ iff $m = n$ and 0 otherwise. Then with $(N, +)$, A is not almost regular because the sequence $(1, 0, 1, 0, \ldots)$ which appears in the first column of the matrix is lac to $\frac{1}{2}$. However, with $(N, \cdot)$, the sequence $(1, 0, 1, 0, \ldots)$ restricted to the ideal $2N$ is the identically 0 sequence and hence, by Proposition 4.4 below, is lac to 0. We leave to the reader to check that the conditions in 4.2 are satisfied, so that *A is almost regular when the semigroup is $(N, \cdot)$*. We feel that this example together with those that will follow justify the study of summability in LA semigroups.

4.4. PROPOSITION. *Suppose S is an LA semigroup. Let $f \in m(S)$ and A be any right ideal of S. If $\pi f \in m(A)$ is the restriction of f to A then πf is lac to k iff f is lac to k.*

Proof. The map $\pi: m(S) \to m(A)$ is defined by $\pi f(t) = f(t)$ for $t \in A$, $f \in m(S)$. Suppose φ_α is a net of finite means on $m(S)$ such that $\lim_\alpha \|L_s\varphi_\alpha - \varphi_\alpha\| = 0$ for each $s \in S$. We may assume each φ_α has its support in A, otherwise we replace φ_α by $L_a\varphi_\alpha$ for a fixed $a \in A$. Let $\varphi_\alpha(f) = \sum_{i=1}^n \varphi_\alpha(t_i) f(t_i)$ and $(T\varphi_\alpha f)(s) = \sum_{i=1}^n \varphi_\alpha(t_i) f(t_i s)$. If f is lac to k then, for all $s \in A$,

$$\begin{aligned} |(T\varphi_\alpha \pi f)(s) - k| &= \left| \sum_{i=1}^n \varphi_\alpha(t_i) \pi f(t_i s) - k \right| \\ &= \left| \sum_{i=1}^n \varphi_\alpha(t_i) f(t_i s) - k \right| \\ &\leqq \|T\varphi_\alpha f - k1\| \to 0 \qquad [5, \text{p. } 71, \text{Theorem } 7(1)]. \end{aligned}$$

Hence πf lac to k [5, p. 71, Theorem 7(2)].

Similarly by taking φ_α to be a net of finite means on $m(A)$ such that

$$\lim_\alpha \|L_s\varphi_\alpha - \varphi_\alpha\| = 0 \quad \text{for all } s \in S$$

we can show f lac to k whenever πf lac to k.

5. Strongly regular matrices.

Let S be a semigroup. Let

$$H = \left\{ \sum_1^n f_i(g_i - l_{a_i} g_i) : a_i \in S, f_i, g_i \in m(S), n = 1, 2, \ldots \right\},$$

$$K = \left\{ \sum_1^n (f_i - l_{a_i} f_i) : a_i \in S, f_i \in m(S), n = 1, 2, \ldots \right\}.$$

We denote the uniform closure of H and K by Cl (H) and Cl (K) respectively. It is known that $F=C \oplus \mathrm{Cl}\,(K)$ for LA semigroups, where C is the set of constant functions; and $F=C \oplus \mathrm{Cl}\,(H)$ for ELA semigroups. See [5] for details.

We will quite often have occasions to use the following: If S is left cancellative, $b \in S$, and $\sup_s \sum_t |A(s, t)| < M$, then

$$\text{(i)} \qquad \sum_{t\in bS} A(s,t) = \sum_{t\in S} A(s,bt),$$

$$(5.0.1)\qquad \begin{aligned} \text{(ii)} \quad \sum_{t\in S\sim bS} |A(s,t)| &= \sum_{t\in S} |A(s,t)| - \sum_{t\in bS} |A(s,t)| \\ &= \sum_{t\in S} (|A(s,t)| - |A(s,bt)|) \\ &\leqq \sum_{t\in S} |A(s,t)-A(s,bt)|. \end{aligned}$$

We say an infinite matrix A is *strongly regular* if $\lim_s Af(s)=k$ whenever f lac to k.

The following Theorem 5.1 contains one of the main results of this paper. When S is the semigroup of additive positive integers, Theorem 5.1 yields G. G. Lorentz's theorem in [9, p. 181].

5.1. THEOREM. *Let S be a left cancellative LA semigroup generated by $B \subset S$. The following conditions are necessary and sufficient for an infinite matrix A to be strongly regular*:

(5.1.1) $\sup_s \sum_t |A(s, t)| < M$ *for some* $M>0$.

(5.1.2) $\lim_s \sum_t A(s, t)=1$.

(5.1.3) $\lim_s \sum_t |A(s, t)-A(s, at)|=0$ *for each* $a \in B$.

Proof. Assume (5.1.1), (5.1.2) and (5.1.3). Then (5.1.1) implies that $A: m(S) \to m(S)$ is a bounded linear operator, and (5.1.2) says that $\lim_s A1(s)=1$. If K is as previously defined, then we note that K is the linear subspace of $m(S)$ spanned by $\{f-l_a f : a \in B, f \in m(S)\}$, and $h \in m(S)$ lac to k iff $h-k1 \in \mathrm{Cl}\,(K)$. Hence in order to prove that A is strongly regular, it is sufficient to prove that $A(\mathrm{Cl}\,(K)) \subset C_0$, where $C_0=\{f \in C_\infty : \lim_s f(s)=0\}$, or $A(f-l_a f) \in C_0$ for each $a \in B$ and $f \in m(S)$, because A is continuous and C_0 is closed in $m(S)$. By using (5.0.1), we have

$$\begin{aligned} |A(f-l_a f)(s)| &= \Big|\sum_t A(s,t)f(t) - \sum_t A(s,t)f(at)\Big| \\ &= \Big|\sum_{t\in S\sim aS} A(s,t)f(t) + \sum_{t\in S}(A(s,at)-A(s,t))f(at)\Big| \\ &\leqq 2\|f\| \sum_t |A(s,t)-A(s,at)|. \end{aligned}$$

Hence by (5.1.3), $A(f-l_a f) \in C_0$ whenever $f \in m(S)$ and $a \in B$.

Conversely, suppose A is strongly regular. Then (5.1.1) follows from Lemma 4.1 and (5.1.2) is clear. If (5.1.3) does not hold for some $a \in B$ then there is an $\varepsilon > 0$

such that $\sum_t |A(s, t)-A(s, at)| > 5\varepsilon$ for an infinite number of $s \in S$. Using this and the fact that

$$\lim_s [A(s, t)-A(s, at)] = 0$$

for each $t \in S$, we now choose an increasing sequence $\sigma(k)$ of finite subsets of S and an infinite subset $\{s_k\}$ of S as follows: For convenience, denote $A(s, t)-A(s, at)$ by $B(s, t)$. In general, for $k=1, 2, \ldots$, let $\sigma(2k-1)\subset\sigma(2k-2)$ (where $\sigma(0)=\varnothing$) and let $s_k \in S$ be such that

$$\sum_{t\in S} |B(s_k, t)| > 5\varepsilon \tag{5.1.4}$$

and

$$\sum_{t\in\sigma(2k-1)} |B(s_k, t)| < \varepsilon. \tag{5.1.5}$$

And since $\sum_{t\in S} |B(s_k, t)|$ exists there is a finite subset $\sigma(2k)\supset\sigma(2k-1)$ such that

$$\sum_{t\in S\sim\sigma(2k)} |B(s_k, t)| < \varepsilon. \tag{5.1.6}$$

Then from (5.1.4), (5.1.5) and (5.1.6) we have

$$\begin{aligned}\sum_{t\in\sigma(2k)\sim\sigma(2k-1)} |B(s_k, t)| &= \Big(\sum_{t\in S} - \sum_{t\in\sigma(2k-1)} - \sum_{t\in S\sim\sigma(2k)}\Big)|B(s_k, t)| \\ &> 5\varepsilon-\varepsilon-\varepsilon = 3\varepsilon.\end{aligned} \tag{5.1.7}$$

Now define $f \in m(S)$ by

$$\begin{aligned} f(t) &= \operatorname{sgn} B(s_k, t) \quad \text{if } t \in a(\sigma(2k)\sim\sigma(2k-1)),\\ &= 0 \qquad\qquad\qquad \text{otherwise.}\end{aligned}$$

Since S is left cancellative, f is well defined. Moreover, $\|f\| \leqq 1$ and l_af-f is lac to 0. But for $k=1, 2, \ldots$, it follows from (5.1.5), (5.1.6) and (5.1.7) that

$$\begin{aligned} |A(l_af-f)(s_k)| &= \Big|\sum_t A(s_k, t)f(at)-\sum_t A(s_k, t)f(t)\Big| \\ &= \Big|\sum_t [A(s_k, t)-A(s_k, at)]f(at)\Big| \\ &\geqq \Big(\sum_{t\in\sigma(2k)\sim\sigma(2k-1)} - \sum_{t\in\sigma(2k-1)} - \sum_{t\in S\sim\sigma(2k)}\Big)|B(s_k, t)| \\ &> 3\varepsilon-\varepsilon-\varepsilon = \varepsilon.\end{aligned}$$

But this cannot be since $A(l_af-f) \in C_0$. Thus (5.1.3) holds.

5.2. Remark. If A is a strongly regular matrix we cannot hope that (5.1.3) hold in general as the following example shows: Let S be the set of positive integers with multiplication $*$ defined by $i * j=k$, where k is the smallest odd integer greater than or equal to $i \vee j=\max(i, j)$. It is clear that $*$ is commutative. We now show that $*$ is associative. Let $i, j, k \in S$.

(a) If $i\leqq j$ and j is odd, then, for all k, $(i * j) * k=(j * k)=i * (j * k)$.

(b) If $i\leqq j$ and j is even, then, for all k, $(i * j) * k=(j+1 * k)=(j * k)=i * (j * k)$.

Moreover for every $i, j \in S$ then either $i \vee j$ or $(i \vee j)+1$ is a right zero for i and j. Hence S is an ELA semigroup. Let now A be a matrix defined on S by

(i) $A(m, n)=0$ whenever n is even, or $n<2m-1$.

(ii) $A(m, 2n-1) \geqq A(m, 2n+1)>0$ whenever $2n-1 \geqq 2m-1$.

(iii) $\sum_n A(m, n)=1$ for each m.

Then A does not satisfy (5.1.3) since, for example, $\lim_m \sum_n |A(m, n)-A(m, 3 * n)| =1$. However, A is strongly regular as Theorem 5.4 below shows. We leave the details for the reader to check.

5.3. Remark. If S is a cancellative LA semigroup without any finite left ideals, then C_∞ is a proper subset of F, since otherwise the identity matrix would have to satisfy (5.1.3). Then there exist finite subsets $\sigma_1, \sigma_2, a, b \in S$, $a \neq b$, such that $at=t$ for $t \in S \sim \sigma_1$ and $bt=t$ for $t \in S \sim \sigma_2$. Hence if $t \in S \sim (\sigma_1 \cup \sigma_2)$ then $at=bt=t$. Since S is right cancellative, $a=b$, which cannot be.

5.4. Theorem. *If S is an ELA semigroup then the following conditions are both necessary and sufficient for an infinite matrix A on S to be strongly regular*:

(5.4.1) $\sup_s \sum_t |A(s, t)| < M$ *for some* $M>0$.

(5.4.2) $\lim_s \sum_t A(s, t)=1$.

(5.4.3) $\lim_s \sum_{t \in S \sim aS} |A(s, t)|=0$ *for every* $a \in S$ *such that* $a \in Sa$.

Proof. Since the proof is similar to that of Theorem 5.1 we shall only give the following essential steps: We estimate $Af(s)$ when $f=g-l_b g$, $g \in m(S)$, $g \neq 0$, and $b \in S$. Let $a \in S$ be such that $ba=a$. For $\varepsilon>0$, let H_0 be a finite subset such that if $s \notin H_0$ then $\sum_{t \in S \sim aS} |A(s, t)| < \varepsilon/2\|g\|$. Then for $s \notin H_0$, we have

$$
\begin{aligned}
|Af(s)| &= \Big|\sum_{t \in S} A(s, t)g(t) - \sum_{t \in S} A(s, t)g(bt)\Big| \\
&\leqq \Big|\sum_{t \in aS} A(s, t)g(t) - \sum_{t \in aS} A(s, t)g(bt)\Big| \\
&\qquad + 2\|g\| \sum_{t \in S \sim aS} |A(s, t)| \\
&< \varepsilon.
\end{aligned}
$$

Now it can easily be shown that A is strongly regular whenever (5.4.1), (5.4.2) and (5.4.3) hold.

Conversely, if A is strongly regular then (5.4.1) and (5.4.2) hold. If (5.4.3) does not hold there is an $\varepsilon>0$ and an $a \in S$ such that $\sum_{t \in S \sim aS} |A(s, t)| > 5\varepsilon$ for an infinite number of $s \in S$. Using this together with the fact that $\lim_s A(s, t)=0$ for each $t \in S$, we can choose, as in the proof of 5.1, an increasing sequence $\sigma(k)$ of finite subsets of $S \sim aS$ and an infinite subset $\{s_k\}$ of S so that the following conditions hold:

(5.4.4) $\sum_{t \in \sigma(2k-1)} |A(s_k, t)| < \varepsilon$.

(5.4.5) $\sum_{t \in S \sim aS \sim \sigma(2k)} |A(s_k, t)| < \varepsilon$.

(5.4.6) $\sum_{t \in \sigma(2k) \sim \sigma(2k-1)} |A(s_k, t) > 3\varepsilon$.

We can choose the sets $\sigma(k)$ to be subsets of $S\sim aS$ since $S\sim aS$ is infinite (otherwise $\sum_{t\in S\sim aS} |A(s,t)|$ would be a finite sum of convergent functions) and the sum $\sum_{t\in S\sim aS} |A(s,t)|$ is finite. Define

$$\begin{aligned} f(t) &= \operatorname{sgn} A(s_k, t) \quad \text{if } t \in \sigma(2k) \sim \sigma(2k-1), \\ &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Now, observe that $\|f\| \leqq 1$, $l_a f=0$, the support of f is contained in $S\sim aS$, and that $f-l_a f$ is lac to 0. Using (5.4.4), (5.4.5) and (5.4.6) we can show (see the proof of 5.1) for all k, $|A(f-l_a f)(s_k)| > \varepsilon$, which cannot be.

5.5. Remark. (a) In the proof of the necessity in 5.4, we did not use the fact that a is a right zero of some element in S, so that, in any LA semigroup, (5.4.1), (5.4.2) and (5.4.3) are necessary conditions whenever A is strongly regular.

(b) We note that we actually proved more in the proofs of 5.1 and 5.4, namely that if g lac to k then $A(r_b g)$ converges to k uniformly in b. In his proof, for the semigroup of additive positive integers, G. G. Lorentz made the same observation (it should be pointed out that our proof differs in many ways from his). We now use this observation in the following theorem.

5.6. Theorem. *Let S be a left cancellative LA [ELA, not necessarily left cancellative] semigroup and A be an infinite matrix on S satisfying the conditions of Theorem* 5.1 [*Theorem* 5.4]. *Then f is F_A-summable to k iff f is lac to k.*

Proof. If f is lac to k then, as known $r_t f$ is lac to k for every $t \in S$. This can easily be seen from the fact that the left translation operator commutes with the right translation operator. By Remark 5.5(b),

$$\lim_s A(r_t f)(s) = \lim_s \sum_{t'} A(s,t')f(t't) = k$$

uniformly in t, i.e. f is F_A-summable.

The converse follows easily from Corollary 5.8 to the following theorem, proved first for the semigroup of additive positive integers by P. Schaefer [13, p. 51].

5.7. Theorem. *Let S be an LA semigroup. If A is almost regular and f is F_A-summable to k then f is lac to k.*

Proof. We basically adapt Schaefer's proof to the general semigroup case. Suppose f is F_A-summable to k. Then $\lim_s \sum_t A(s,t)f(tb)=k$ uniformly in b. Let g be a function of s and b be defined by $g(s,b)=\sum_t A(s,t)f(tb)$. Then $g(s,b) = k+h(s,b)$, where h is a function of s and b such that h, as a function of s, is convergent to 0 uniformly in $b \in S$. Now for each finite subset σ of S, define g_σ as a function of s and b by $g_\sigma(s,b)=\sum_{t\in\sigma} A(s,t)f(tb)$. Then g_σ converges uniformly to g for each fixed $s\in S$ since $\|g_\sigma - g\| = \sup_b |g_\sigma(b)-g(b)| \leqq \sum_{t\notin\sigma} |A(s,t)|\ \|f\|$ and this can be made as small as we please.

If now φ is any LIM then for each fixed $s \in S$,

$$\varphi(g) = \varphi\left(\lim_\sigma g_\sigma\right) = \lim_\sigma \sum_{t\in\sigma} A(s,t)\varphi(l_t f)$$
$$= \sum_{t\in S} A(s,t)\varphi(f) = \varphi(k+h) = k+\varphi(h).$$

Thus $\varphi(f)\sum_t A(s,t) = k+\varphi(h)$. Since h, as a function of s, converges to 0 uniformly in $b \in S$, for every $\varepsilon>0$ there is a finite subset H such that if $s \notin H$ then $|\varphi(h)|<\varepsilon$, i.e. $\varphi(h)$, as a function of s, is convergent to 0. If now ψ is any LIM then

$$\psi\left[\varphi(f)\sum_t A(s,t)\right] = \varphi(f)\psi\left[\sum_t A(s,t)\right] = \psi(k)+\psi(\varphi(h)). \tag{5.7.1}$$

Since A is almost regular, $\psi[\sum_t A(s,t)]=1$ and $\psi(\varphi(h))=0$. Therefore we see from (5.7.1) that $\varphi(f)=\psi(k)=k$, i.e. f is lac to k.

The following corollary, which is due to G. G. Lorentz for the additive positive integers [9, p. 171], is an immediate consequence of 5.7 since every regular matrix is almost regular.

5.8. Corollary. *Let S be an LA semigroup. If A is regular and f is F_A-summable to k then f is lac to k.*

6. **Almost Schur matrices.** We say an infinite matrix A is *almost Schur* if Af is lac for each $f \in m(S)$.

6.1. Theorem. *Let S be an LA semigroup. Let A be an infinite matrix on S satisfying the following conditions:*

(6.1.1) $\sup_s \sum_t |A(s,t)| < M$ *for some* $M>0$.

(6.1.2) *The sum* $\sum_t |A(s,t)|$ *converges uniformly in* s.

(6.1.3) $A(s,t)$, *as a function of* s, *is lac to* α_t *for each* $t \in S$.

Then Af is lac to $\sum_t \alpha_t f(t)$ for each $f \in m(S)$.

Proof. Let Σ be the family of all finite subsets of S directed by inclusion. Let $f \in m(S)$. For each $\sigma \in \Sigma$ define g_σ by $g_\sigma(s) = \sum_{t\in\sigma} A(s,t)f(t)$. Then clearly g_σ is lac to $\sum_{t\in\sigma}\alpha_t f(t)$ by (6.1.3). Now (6.1.1) implies $Af \in m(S)$. And using (6.1.2), one can readily show that Af is the uniform limit of g_σ. Hence Af is lac and if φ is any LIM then

$$\varphi(Af) = \varphi\left(\lim_\sigma g_\sigma\right) = \lim_\sigma \varphi(g_\sigma) = \lim_\sigma \sum_{t\in\sigma}\alpha_t f(t) = \sum_t \alpha_t f(t).$$

6.2. Corollary. *If A is an almost regular matrix then A cannot be an almost Schur matrix.*

Proof. If A is an almost regular matrix then $A(s,t)$ is lac to 0 and $\sum_t A(s,t)$ is lac to 1. If A is also an almost Schur matrix then Af is lac to 0 by the theorem. In particular, $\sum_t A(s,t)$ is lac to 0, which cannot be.

6.3. REMARK. It is easy to see that if A is an almost Schur matrix then (6.1.1) and (6.1.3) are necessary. However, (6.1.2) is not necessary as the following example shows: Let S be the semigroup of ordinals less than the first uncountable ordinal Ω with the usual addition of order types. Then S is a noncommutative, left cancellative, ELA semigroup ([5], p. 73). Define A on S by

$$\begin{aligned} A(s, t) &= \delta(s, t) \quad \text{if } 1 \leqq s < \omega, t \in S, \\ &= 0 \qquad\quad \text{otherwise,} \end{aligned}$$

where ω is the first countable infinite ordinal. Then for any $f \in m(S)$, it is easy to see that $Af(s)=0$ for $s \in \alpha+S$ for any $\alpha>\omega$. By 4.4 Af is lac to 0. But clearly (6.1.2) is not satisfied.

6.4. EXAMPLE. Let $S=\{(m, n) : m=1, 2, \ldots, n=1, 2, \ldots\}$. Define the operation $*$ on S by

(a) $(m_1, n_1) * (m_2, n_2)=(m_1+m_2, n_1+n_2)$ if $m_1 \neq 1$ and $m_2 \neq 1$.

(b) $(m_1, n_1) * (1, n_2)=(1, n_2) * (m_1, n_1)=(1, n_2)$ if $m_1 \neq 1$.

(c) $(1, n_1) * (1, n_2)=(1, n_1 \vee n_2)$, where $n_1 \vee n_2=\max(n_1, n_2)$.

That S is an ELA semigroup actually follows from the following general construction: Let $S=S_1 \cup S_2$, where S_1 is any semigroup and S_2 is any ELA semigroup. For $a, b \in S$, define the product $a * b$ to be the product of a and b in S_i, $i=1, 2$, if both $a, b \in S_i$. If $a \in S_1$, $b \in S_2$ then $a * b=b * a=b$.

Now for each k and each l fixed, define

$$\begin{aligned} f(m, n) &= (\tfrac{1}{2})^{k+l} \quad \text{if } m = 1, \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

Then $g(m, n)=l_{(1,1)}f(m, n)=f[(1, 1) * (m, n)]=(\frac{1}{2})^{k+l}$ for all m and n. Then $g(m, n)-f(m, n)$ is lac to 0. Define the matrix A on S by $A(m, n; k, l)=g(m, n)-f(m, n)$. Then $A(m, n; k, l)$, as a function of (m, n), is lac to 0 for each k and l; and

$$\sum_{k,l} |A(m, n; k, l)| = \sum_{k,l} (\tfrac{1}{2})^{k+l}$$

converges uniformly in (m, n) to 1. By 6.1 A is an almost Schur matrix.

7. Almost strongly regular matrices.

We say an infinite matrix A is *almost strongly regular* if Af lac to k whenever f lac to k.

7.1. THEOREM. *Let S be a left cancellative LA semigroup generated by $B \subset S$. Let A be an infinite matrix on S such that the following conditions hold*:

(7.1.1) $\sup_s \sum_t |A(s, t)| < M$ *for some* $M>0$.

(7.1.2) $\sum_t A(s, t)$, *as a function of* s, *lac to* 1.

(7.1.3) $\sum_t |A(s, t)-A(s, at)|$, *as a function of* s, *lac to* 0 *for every* $a \in B$.

Then A is almost strongly regular.

Proof. Assume (7.1.1), (7.1.2) and (7.1.3). Then (7.1.1) implies $A: m(S) \to m(S)$ is a bounded linear operator, and (7.1.2) says A1 lac to 1. By the same reasoning as in the proof of 5.1, in order to prove A is almost strongly regular, it suffices to show $A(f-l_af) \in \mathrm{Cl}(K)$ for each $a \in B$ and $f \in m(S)$. Let, then, φ_α be a net of finite means converging in norm to left invariance, i.e. $\lim_\alpha \|L_s\varphi_\alpha-\varphi_\alpha\|=0$ for all $s\in S$ [1, p. 524, Theorem 1]. Let $\varphi_\alpha(f)=\sum_{i=1}^n \varphi_\alpha(t_i)f(t_i)$. By using (5.0.1), we have

$$\begin{aligned}
&\left|\sum_{i=1}^n \varphi_\alpha(t_i)[Af-A(l_af)](t_is)\right| \\
&\quad = \left|\sum_{i=1}^n \varphi_\alpha(t_i)\left[\sum_t A(t_is,t)f(t)-\sum_t A(t_is,t)f(at)\right]\right| \\
&\quad = \left|\sum_{i=1}^n \varphi_\alpha(t_i)\left[\sum_{t\in S}(A(t_is,at)-A(t_is,t))f(at)+\sum_{t\in S\sim aS} A(t_is,t)f(t)\right]\right| \\
&\quad \leqq 2\|f\|\sum_{i=1}^n \varphi_\alpha(t_i)\sum_t |A(t_is,t)-A(t_is,at)|.
\end{aligned}$$

Hence by (7.1.3) and [5, p. 71, Theorem 7], $A(f-l_af)\in \mathrm{Cl}\,(K)$ whenever $a\in B$ and $f\in m(S)$.

7.2. Remark. (a) If S is the additive positive integers, Theorem 7.1 yields P. Schaefer's Theorem 2 [12, p. 52]. Our proof is entirely different from his. His proof does not seem to carry over to the general case.

(b) It is clear that if A is an almost strongly regular matrix then (7.1.1) and (7.1.2) are both necessary conditions. However, (7.1.3) does not always hold, since the identity matrix A is almost strongly regular but, for the additive positive integers, $\lim_m \sum_2 |A(m,n)-A(m,n+1)|=2$. When S is ELA (not necessarily left cancellative) we have the following stronger result.

7.3. Theorem. *Let S be ELA, and A be an infinite matrix on S satisfying the following conditions*:

(7.3.1) $\sup_s \sum_t |A(s,t)|<M$ *for some* $M>0$.

(7.3.2) $\sum_t A(s,t)$, *as a function of s, is lac to* 1.

(7.3.3) $\sum_{t\in S\sim aS} |A(s,t)|$, *as a function of s, is lac to* 0 *for every $a\in S$ such that $a\in Sa$.*

Then Af is lac to k whenever f is lac to k.

Proof. Let f lac to k. By [5, p. 72, Theorem 8] for $\varepsilon>0$ there is a $b\in S$ such that if $t\in bS$ then $|f(t)-k|<\varepsilon$. Let $a\in S$ be such that $ba=a$. Then if $t\in aS\subset bS$, $|f(t)-k|<\varepsilon$. By (7.3.2) and (7.3.3) let $c,d\in S$ be such that if $s\in cS$ then $|\sum_t A(s,t)-1|<\varepsilon$, and if $s\in dS$ then $\sum_{t\in S\sim aS}|A(s,t)|<\varepsilon$. Now $Af\in m(S)$ by (7.3.1) and if $s\in cS\cap dS\neq\varnothing$ (since S is ELA), then

$$\begin{aligned}
|Af(s)-k| &\leqq \left|\sum_t A(s,t)(f(t)-k)\right|+\left|\sum_t A(s,t)k-k\right| \\
&\leqq \sum_{t\in S\sim aS}|A(s,t)|\,|f(t)-k|+\sum_{t\in aS}|A(s,t)|\,|f(t)-k|+|k|\left|\sum_t A(s,t)-1\right| \\
&< \varepsilon(\|f\|+|k|)+\varepsilon M+|k|\varepsilon \\
&< (2|k|+\|f\|+M)\varepsilon.
\end{aligned}$$

By [5, p. 72 Theorem 8], Af is lac to k.

7.4. Remark. If A is nonnegative, i.e. $A(s,t)\geqq 0$ for all $s,t\in S$, then (7.3.1), (7.3.2) and (7.3.3) are necessary also. For $\sum_{t\in S\sim aS} |A(s,t)| = \sum_{t\in S\sim aS} A(s,t) = A(1_{S\sim aS})(s)$. Since $1_{S\sim aS}$ is lac to 0 it follows that (7.3.3) holds.

7.5. Example. Let S be the semigroup described in 5.2. Let A be defined for each m, n, by

(i) $A(2m-1, 2n-1) \geqq A(2m-1, 2n+1) > 0$ whenever $2n-1 \geqq 2m-1$ and 0 otherwise.

(ii) $A(2m, n) = 1$ only if $n=1$ and 0 otherwise.

(iii) $A(m, n) = 0$ whenever n is even.

(iv) $\sum_n A(m, n) = 1$ for each m.

By 7.3 A is almost strongly regular. However, if we replace the operation $*$ by the ordinary addition, then A is not almost strongly regular since the sequence $f = (1, 0, 1, 0, \ldots)$ is lac to $\frac{1}{2}$ while Af is the sequence $(1, 1, 1, \ldots)$, which is lac to 1.

7.6. Remark. The referee has raised the following interesting question: Does each left cancellative LA semigroup admit a matrix of the types that we have considered in this paper? We do not have complete answers to this question, but only give some examples of these matrices for some particular semigroups.

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