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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF HYPERBOLIC INEQUALITIES(1)

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Abstract. This paper discusses the asymptotic behavior of C^2 solutions $u=u(t, x_1, \ldots, x_\nu)$ of the inequality (1) $|Lu| \leqq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$, in domains in (t, x)-space which grow unbounded in x as $t \to \infty$. The operator L is a second order hyperbolic operator with variable coefficients. The main results establish the maximum rate of decay of nonzero solutions of (1). This rate depends on the asymptotic behavior of k_1, k_2, and the time derivatives of the coefficients of L.

1. **Introduction.** Let L be defined on C^2 functions $u=u(t, x_1, \ldots, x_\nu)$ by

$$Lu = Au - \partial^2 u/\partial t^2$$

where

$$A \equiv \sum_{ij=1}^{\nu} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}$$

is a symmetric uniformly elliptic operator. Thus we assume that the $a_{ij}=a_{ij}(t, x)$ are C^1 functions on $\mathscr{H} \equiv \{(t, x) \in \mathbf{R} \times \mathbf{R}^\nu : t \geqq 0\}$ with $a_{ij}=a_{ji}$ for $1 \leqq i, j \leqq \nu$. We also make the assumption

(A_0) There are positive constants m, M such that

$$m^2 \leqq \sum_{ij=1}^{\nu} a_{ij}(t, x)\xi_i \xi_j \leqq M^2$$

whenever $t \geqq 0$ and $\sum_{i=1}^{\nu} \xi_i^2 = 1$.

For $\varepsilon \geqq 0$ and $R \geqq 0$ let $D(\varepsilon, \infty, R)$ denote the set $\{(t, x) \in \mathbf{R} \times \mathbf{R}^\nu : \varepsilon \leqq t;\ |x| \leqq Mt+R\}$, and let $S(T, R)$ denote $\{(T, x) : |x| \leqq MT+R\}$. Suppose u is a solution of

$$|Lu| \leqq k_1(t, x)|u| + k_2(t, x)\|\nabla u\| \tag{1.1}$$

in some $D(\varepsilon, \infty, R)$. The decay rate of u is measured in terms of the energy integral

$$\mathscr{E}(u, T, R) \equiv \int_{S(T,R)} \{u^2 + \|\nabla u\|^2\}\, dx.$$

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As a special case of our main theorem we get the following results:

(I) Suppose that $|(a_{ij})_t| = O(t^{-1})$; $k_1(t, x) = O(t^{-2})$; and $k_2(t, x) = O(t^{-1})$. Then a nonzero solution of (1.1) in $D(\varepsilon, \infty, R)$ cannot decay so fast that, for every positive α, $\lim_{T\to\infty} T^\alpha \mathscr{E}(u, T, R) = 0$.

(II) Suppose that $|(a_{ij})_t| = O(1)$; $k_1(t, x) = O(1)$; $k_2(t, x) = O(1)$. Then a nonzero solution of (1.1) in $D(\varepsilon, \infty, R)$ cannot decay so fast that, for all $\alpha > 0$, $\lim_{T\to\infty} e^{\alpha T} \mathscr{E}(u, T, R) = 0$.

(III) Suppose there is a $\gamma > 1$ such that $|(a_{ij})_t| = O(t^{(\gamma-1)})$; $k_1(t, x) = O(t^{2(\gamma-1)})$; and $k_2(t, x) = O(t^{(\gamma-1)})$. Then a nonzero solution of (1.1) in $D(\varepsilon, \infty, R)$ cannot decay so fast that, for all $\alpha > 0$, $\lim_{T\to\infty} e^{\alpha T^\gamma} \mathscr{E}(u, T, R) = 0$.

The methods and immediate motivation for this work are derived from Protter's treatment [5], [6] of the asymptotic behavior of solutions of hyperbolic inequalities in interior domains. The crux of the method lies in finding appropriate families of weighted L_2 estimates for a C^2 function v and its gradient in terms of Lv. The estimates of this paper differ from those in [5], [6] by requiring either no boundary conditions or weaker boundary conditions. The changes in the derivation of the crucial estimates are suggested by techniques of Hörmander [1] for determining the sign of certain quadratic forms.

My decay rate results are comparable to those of Protter [5], [6] and Ogawa [4] for interior domains. Related problems about decay rates of solutions of hyperbolic equations have been studied by Morawetz [3], Strauss [7], and Littman [2] using different methods usually involving transform techniques. In particular, the partial differential inequality (1.1) arises from the uniqueness problem for the nonlinear equation

$$Lu = F(t, x, u, \nabla u)$$

provided F satisfies a Lipschitz condition of the form

$$|F(t, x, a, \boldsymbol{a}) - F(t, x, b, \boldsymbol{b})| \leqq k_1(t, x)|a-b| + k_2(t, x)\|\boldsymbol{a}-\boldsymbol{b}\|.$$

Let ν be a positive integer. We use the coordinates $(t, x_1, \ldots, x_\nu)$ in $\boldsymbol{R}\times\boldsymbol{R}^\nu$. The first coordinate t is the time coordinate; the rest make up the space variable $x = (x_1, \ldots, x_\nu)$. Let $r = (x_1^2 + \cdots + x_\nu^2)^{1/2}$.

In general we use subscripts to denote derivatives; thus

$$u_t = \partial u/\partial t; \qquad u_{x_i} = \partial u/\partial x_i \quad \text{for } i = 1, \ldots, \nu.$$

The gradient ∇u denotes the $(\nu+1)$-tuple $(u_t, u_{x_1}, \ldots, u_{x_\nu})$. However we take the Laplacian with respect to space coordinates only; $\Delta u = \sum_{i=1}^{\nu} u_{x_i x_i}$.

By a *domain* we mean the closure of an open set which has piecewise smooth boundary. Derivatives at boundary points of a domain are to be understood as the appropriate one-sided derivatives.

These results are excerpted from my Ph.D. thesis, University of California, Berkeley, 1970. I wish to thank my advisor, Professor M. H. Protter, for his assistance and encouragement.

2. **Basic lemmas.** We start with the formula for the integral of the expression $2\,Lv\lambda v_t$ over a bounded domain $D \subseteq \operatorname{Int}(\mathscr{H})$ where $\lambda \in C^1(D)$ and $v \in C^2(D)$. Let $n=(n_0, n_1, \ldots, n_\nu)$ be the outer unit normal along ∂D. Then integration by parts yields the formula [1], [7]

$$\begin{aligned} \int_D 2Lv\lambda v_t = & \int_D \Big\{\lambda_t\Big(v_t^2+\sum_{i,j=1}^{\nu} a_{ij}v_{x_i}v_{x_j}\Big)-2v_t\sum_{i,j=1}^{\nu} a_{ij}\lambda_{x_i}v_{x_j}\Big\} \\ & +\int_D \lambda \sum_{i,j=1}^{\nu} (a_{ij})_t v_{x_i}v_{x_j} \\ & -\int_{\partial D} \lambda\Big\{n_0\Big(v_t^2+\sum_{i,j=1}^{\nu} a_{ij}v_{x_i}v_{x_j}\Big)-2v_t\sum_{i,j=1}^{\nu} a_{ij}n_i v_{x_j}\Big\}. \end{aligned} \tag{2.1}$$

To exploit this formula we consider the region $\mathscr{H}$ as a Lorentz manifold with the Lorentzian metric

$$((b, c)) \equiv b_0c_0-\sum_{i,j=1}^{\nu} a_{ij}(t, x)b_ic_j. \tag{2.2}$$

For vector fields b and c on $\mathscr{H}$ we define a quadratic form $P_{b,c}$ at each point of $\mathscr{H}$ by

$$P_{b,c}(\xi) \equiv 2((b, \xi))((c, \xi))-((b, c))((\xi, \xi)). \tag{2.3}$$

Throughout we use h to denote the unit vector $h=(1, 0, \ldots, 0) \in \boldsymbol{R}^{\nu+1}$. Computing explicitly we find that

$$P_{h,c}(\xi) = c_0\Big\{\xi_0^2+\sum_{i,j=1}^{\nu} a_{ij}\xi_i\xi_j\Big\}-2\xi_0\sum_{i,j=1}^{\nu} a_{ij}c_i\xi_j.$$

Thus (2.1) can be rewritten more concisely as

$$\int_D 2Lv\lambda v_t = \int_D \Big\{P_{h,\nabla\lambda}(\nabla v)+\lambda\sum_{i,j=1}^{\nu} (a_{ij})_t v_{x_i}v_{x_j}\Big\}-\int_{\partial D} \lambda P_{h,n}(\nabla v). \tag{2.4}$$

We now describe conditions on D and λ under which the integrands on the right-hand side of (2.4) have definite sign.

We first consider the boundary integrand, $\lambda P_{h,n}(\nabla v)$. Following Hörmander [1] we classify tangent vectors in terms of the Lorentz metric (2.2). Thus we say $b=(b_0, b_1, \ldots, b_\nu)$ is *timelike* at (t, x) if $((b, b))>0$; *spacelike* if $((b, b))<0$; and *characteristic* if $((b, b))=0$. Furthermore, we call b *positive* if $b_0>0$, and *negative* if $b_0<0$. Hörmander has proved that if b and c are any two positive timelike vectors with respect to a Lorentz metric $((\ ,\ ))$, then the form $P_{b,c}$ defined by (2.3) is positive definite. Since any positive characteristic vector is a limit of positive timelike vectors, it follows that $P_{h,n}(\xi) \geqq 0$ for all ξ provided that n is positive non-spacelike.

We call a domain D *convenient* if the outer unit normal n along ∂D is never spacelike, i.e. if $((n, n)) \geqq 0$ along ∂D. The boundary of a convenient domain D can be decomposed into two parts: the part $S^+(D)$ on which n is positive, and the part $S^-(D)$ on which n is negative. Since $P_{h,n}$ is linear in n, we have $P_{h,n}(\nabla v)$ nonnegative along $S^+(D)$ and nonpositive along $S^-(D)$. For a positive function λ we therefore have determined the sign of the integrand $\lambda P_{h,n}(\nabla v)$.

Next we turn to the integrand

$$\mathscr{I} = \left\{P_{h,\nabla\lambda}(\nabla v) + \lambda \sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j}\right\}$$

in (2.4) and confine our attention to functions λ of the form

$$\lambda = \lambda(\alpha) = e^{\alpha f(t)}$$

where f is a smooth function of positive t and α is a positive parameter. Three specific functions f to which our results apply are

(i) $f(t) = \ln(t)$,

(ii) $f(t) = t$,

(iii) $f(t) = t^\gamma$, for a constant $\gamma > 1$.

Let $\mathscr{F}$ denote the set containing these specific functions. Rather than compute separately for each $f \in \mathscr{F}$, we do a single computation appealing to several technical properties (P_i) which are easily verified for each $f \in \mathscr{F}$. The first of these properties is

$$(P_1) \qquad f \in C^3(\boldsymbol{R}^+); \qquad f_t > 0; \qquad \lim_{t\to\infty} f(t) = \infty.$$

If f satisfies (P_1) then each $\lambda(\alpha) = e^{\alpha f(t)}$ is a C^1 function such that

$$\lambda(\alpha) > 0; \qquad \lambda_t(\alpha) = \alpha f_t e^{\alpha f}; \qquad \nabla\lambda = \lambda_t h.$$

Then

$$P_{h,\nabla\lambda}(\nabla v) = \lambda_t P_{h,h}(\nabla v) = \alpha f_t \lambda(\alpha) P_{h,h}(\nabla v)$$

and

$$\mathscr{I} = \lambda(\alpha)\left\{\alpha f_t P_{h,h}(\nabla v) + \sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j}\right\}.$$

In order to be sure that $\mathscr{I}$ is a positive definite form in ∇v we make the following assumption about the growth of the time derivatives $(a_{ij})_t$:

(A_1) For each $\varepsilon > 0$ there is a $B(\varepsilon) > 0$ such that $|(a_{ij})_t| \leqq B(\varepsilon) f_t$ at all points (t, x) where $t \geqq \varepsilon$.

LEMMA 2.1. *Suppose L satisfies* (A_1) *with respect to an f satisfying* (P_1). *If $\frac{1}{2}\alpha m^2 \geqq \nu B(\varepsilon)$ and $v \in C^2(\mathscr{H})$, then at all (t, x) with $t \geqq \varepsilon$ we have*

$$(2.5) \qquad \left\{\alpha f_t P_{h,h}(\nabla v) + \sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j}\right\} \geqq \tfrac{1}{2}\alpha f_t P_{h,h}(\nabla v) \geqq 0.$$

Proof. From (2.3) and (A_0) it follows that

$$P_{h,h}(\nabla v) \equiv v_t^2 + \sum_{ij=1}^{\nu} a_{ij} v_{x_i} v_{x_j} \geqq m^2 \sum_{i=1}^{\nu} v_{x_i}^2.$$

Because of (P_1) we get for all positive α

$$\alpha f_t P_{h,h}(\nabla v) \geqq \tfrac{1}{2}\alpha f_t P_{h,h}(\nabla v)+\tfrac{1}{2}\alpha f_t m^2 \sum_{i=1}^{\nu} v_{x_i}^2.$$

On the other hand, whenever $t \geqq \varepsilon$ we have

$$\begin{aligned}\left|\sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j}\right| &\leqq \sum_{ij=1}^{\nu} |(a_{ij})_t|\,|v_{x_i}|\,|v_{x_j}| \\ &\leqq B(\varepsilon) f_t \left(\sum_{i=1}^{\nu} |v_{x_i}|\right)^2 \leqq \nu B(\varepsilon) f_t \sum_{i=1}^{\nu} v_{x_i}^2.\end{aligned}$$

So

$$\left\{\alpha f_t P_{h,h}(\nabla v)+\sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j}\right\} \geqq \tfrac{1}{2}\alpha f_t P_{h,h}(\nabla v)+\tfrac{1}{2}\alpha m^2 f_t \sum_{i=1}^{\nu} v_{x_i}^2 - \nu B(\varepsilon) f_t \sum_{i=1}^{\nu} v_{x_i}^2.$$

Thus if $\frac{1}{2}\alpha m^2 \geqq \nu B(\varepsilon)$, the inequality (2.5) follows.

Using this result we derive two important integral inequalities.

LEMMA 2.2. *Suppose* (A_1) *holds for some f satisfying* (P_1). *Let D be a bounded convenient domain in which $t \geqq \varepsilon > 0$. If $v \in C^2(D)$ and $\frac{1}{2}\alpha m^2 \geqq \nu B(\varepsilon)$, then*

$$\int_D 2Lve^{\alpha f(t)} v_t \geqq -\int_{S^+(D)} e^{\alpha f(t)} P_{h,n}(\nabla v) \tag{2.6}$$

and

$$\tfrac{1}{2}\alpha \int_D f_t e^{\alpha f(t)} P_{h,h}(\nabla v) \leqq 2\int_D e^{\alpha f(t)} |Lvv_t| + \int_{S^+(D)} e^{\alpha f(t)} P_{h,n}(\nabla v). \tag{2.7}$$

Proof. For each function $\lambda(\alpha)=e^{\alpha f(t)}$, $\alpha>0$, we can apply (2.4) to obtain

$$\int_D 2Lv\lambda(\alpha) v_t = \int_D e^{\alpha f(t)}\left\{\alpha f_t P_{h,h}(\nabla v)+\sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j}\right\} - \int_{\partial D} e^{\alpha f(t)} P_{h,h}(\nabla v). \tag{2.8}$$

Since D is convenient, we have $P_{h,n}(\nabla v) \leqq 0$ on $S^-(D)$, and thus

$$-\int_{\partial D} e^{\alpha f} P_{h,n}(\nabla v) \geqq -\int_{S^+(D)} e^{\alpha f} P_{h,n}(\nabla v).$$

If $\frac{1}{2}\alpha m^2 \geqq \nu B(\varepsilon)$, then by the preceding lemma

$$\int_D e^{\alpha f}\left\{\alpha f_t P_{h,h}(\nabla v)+\sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j}\right\} \geqq \tfrac{1}{2}\alpha \int_D f_t e^{\alpha f} P_{h,h}(\nabla v) \geqq 0.$$

Thus, for large enough α we have

$$\int_D 2Lv\lambda(\alpha) v_t \geqq \tfrac{1}{2}\alpha \int_D f_t \lambda(\alpha) P_{h,h}(\nabla v) - \int_{S^+(D)} \lambda(\alpha) P_{h,n}(\nabla v).$$

The inequalities (2.6) and (2.7) now follow directly since $f_t\lambda(\alpha)P_{h,h}(\nabla v)$ is non-negative.

COROLLARY 2.3. *Let $m'=\min\{1, m^2\}$. Then for α sufficiently large*

$$\tfrac{1}{2}\alpha m' \int_D f_t\lambda(\alpha)\|\nabla v\|^2 \leqq 2\int_D \lambda(\alpha)|Lvv_t| + \int_{S^+(D)} \lambda(\alpha)P_{h,n}(\nabla v). \tag{2.9}$$

Proof. This follows from (2.7) since we have

$$P_{h,h}(\nabla v) = \left(v_t^2+\sum_{ij=1}^{\nu} a_{ij}v_{x_i}v_{x_j}\right) \geqq v_t^2+m^2\sum_{i=1}^{\nu} v_{x_i}^2 \geqq m'\left(v_t^2+\sum_{i=1}^{\nu} v_{x_i}^2\right).$$

3. **Basic a priori estimates.** Suppose v is a C^2 function. In this section we derive the crucial family of weighted L_2 estimates for v and ∇v in terms of Lv.

Let f satisfy (P_1). We assume that L satisfies (A_1) relative to this f. In the course of the derivation we will impose additional technical conditions (P_i) on f, conditions which are verified for all $f\in\mathscr{F}$.

First we apply a variant of Protter's method [5], [6] to estimate v on a bounded convenient domain D in which $t\geqq\varepsilon>0$. We employ the parametrized family of weight functions $\lambda=\lambda(\alpha)=e^{\alpha f(t)}$, $\alpha>0$.

For an $\alpha>0$, let z denote the auxiliary function $z=\lambda(\alpha)v$. Since $v=e^{-\alpha f}z$, computation shows that

$$\lambda(\alpha)Lv = Lz+2\alpha f_t z_t+\alpha(f_{tt}-\alpha f_t^2)z. \tag{3.1}$$

Applying the elementary inequality $(A+B+C)^2\geqq 2(A+C)B$, we find

$$\lambda(2\alpha)(Lv)^2 \geqq 2\{Lz+\alpha(f_{tt}-\alpha f_t^2)z\}\{2\alpha f_t z_t\}. \tag{3.2}$$

Expanding the right-hand side of (3.2) we obtain

$$\lambda(2\alpha)(Lv)^2 \geqq 2\alpha\{2Lzf_tz_t\}+2\alpha^2 f_t(f_{tt}-\alpha f_t^2)\,\partial(z^2)/\partial t. \tag{3.3}$$

Integrating (3.3) over D does not yield a useful estimate. Instead we must first multiply (3.3) through by the positive quantity $f_t^{-1}\lambda(\beta)$ where β is chosen large enough that $\tfrac{1}{2}\beta m^2\geqq \nu B(\varepsilon)$. This yields

$$\int_D f_t^{-1}\lambda(\beta+2\alpha)(Lv)^2 \geqq 2\alpha\int_D Lz\lambda(\beta)z_t+2\alpha^2\int_D \lambda(\beta)(f_{tt}-\alpha f_t^2)\frac{\partial}{\partial t}(z^2). \tag{3.4}$$

Now β was chosen so that Lemma 2.2 applies. Thus

$$\int_D Lz\lambda(\beta)z_t \geqq -\int_{S^+(D)} \lambda(\beta)P_{h,n}(\nabla z). \tag{3.5}$$

Integration by parts yields

$$\begin{aligned}\int_D \lambda(\beta)(f_{tt}-\alpha f_t^2)\frac{\partial}{\partial t}(z^2) &= \int_D \frac{\partial}{\partial t}\{z^2e^{\beta f}(f_{tt}-\alpha f_t^2)\}-\int_D z^2\frac{\partial}{\partial t}\{e^{\beta f}(f_{tt}-\alpha f_t^2)\}\\ &= \int_{\partial D} n_0z^2e^{\beta f(t)}(f_{tt}-\alpha f_t^2)\\ &\quad+\int_D z^2e^{\beta f}\{\beta f_t(\alpha f_t^2-f_{tt})+(2\alpha f_tf_{tt}-f_{ttt})\}.\end{aligned} \tag{3.6}$$

We add two more technical properties of f to improve (3.5):

(P_2) For all $t>0$, $\alpha>0$, $(f_{tt}-\alpha f_t^2)\geqq -(\alpha+1)f_t^2$. For each $\varepsilon>0$, there is an $\alpha_1(\varepsilon)>0$ such that $(f_{tt}-\alpha f_t^2)$ is negative whenever $t\geqq\varepsilon$ and $\alpha\geqq\alpha_1(\varepsilon)$.

(P_3) For each $\varepsilon>0$ and $\beta\geqq 3$, there is an $\alpha_2(\varepsilon,\beta)$ such that

$$\alpha\beta f_t^3+(2\alpha-\beta)f_tf_{tt}-f_{ttt} \geqq \alpha f_t^3$$

whenever $t\geqq\varepsilon$ and $\alpha\geqq\alpha_2(\varepsilon,\beta)$.

The property (P_2) simplifies the boundary integral in (3.6). We have assumed that $t\geqq\varepsilon>0$ in D and we know that $n_0>0$ on $S^+(D)$. Thus by (P_2) if $\alpha>\alpha_1(\varepsilon)$ we have

$$n_0z^2e^{\beta f}(f_{tt}-\alpha f_t^2) \geqq 0 \quad \text{on } S^-(D),$$

$$n_0z^2e^{\beta f}(f_{tt}-\alpha f_t^2) \geqq -(\alpha+1)n_0z^2e^{\beta f}f_t^2 \quad \text{on } S^+(D),$$

and therefore

$$\int_{\partial D} n_0z^2e^{\beta f}(f_{tt}-\alpha f_t^2) \geqq -(\alpha+1)\int_{S^+(D)} n_0z^2e^{\beta f}f_t^2.$$

The property (P_3) has the consequence that

$$\int_D z^2\lambda(\beta)\{\alpha\beta f_t^3+(2\alpha-\beta)f_tf_{tt}-f_{ttt}\} \geqq \alpha\int_D z^2\lambda(\beta)f_t^3 = \alpha\int_D f_t^3\lambda(\beta+2\alpha)v^2$$

provided that $\beta\geqq 3$ and $\alpha\geqq\alpha_2(\varepsilon,\beta)$. Now if β is chosen so that both $\beta\geqq 3$ and $\frac{1}{2}\beta m^2\geqq\nu B(\varepsilon)$, then with the help of (P_2) and (P_3) we strengthen (3.6) to obtain the inequality

$$\text{(3.7)}\quad \int_D \lambda(\beta)(f_{tt}-\alpha f_t^2)\frac{\partial z^2}{\partial t} \geqq -(\alpha+1)\int_{S^+(D)} n_0z^2\lambda(\beta)f_t^2+\alpha\int_D f_t^3\lambda(\beta+2\alpha)v^2$$

for all $\alpha\geqq\max\{\alpha_1(\varepsilon),\alpha_2(\varepsilon,\beta)\}$. Combining (3.5) and (3.7) with (3.4) we obtain the estimate

$$\text{(3.8)}\quad \begin{aligned} 2\alpha^3\int_D f_t^3\lambda(\beta+2\alpha)v^2 &\leqq \int_D f_t^{-1}\lambda(\beta+2\alpha)(Lv)^2 \\ &\quad +2\alpha\int_{S^+(D)}\lambda(\beta)P_{h,n}(\nabla z)+2\alpha^2(\alpha+1)\int_{S^+(D)} n_0\lambda(\beta)f_t^2z^2 \end{aligned}$$

which is valid for all sufficiently large α.

With this indication of the purpose of (P_2) and (P_3) we can summarize the derivation of (3.8) in the following lemma:

LEMMA 3.1. *Suppose f satisfies (P_i) for $i=1, 2, 3$ and L satisfies (A_1) relative to f. Let D be a bounded convenient domain in which $t\geqq\varepsilon>0$. Let β be a constant such that $\beta\geqq 3$ and $\frac{1}{2}m^2\beta\geqq\nu B(\varepsilon)$. If $v\in C^2(D)$ then for all sufficiently large α*

$$\text{(3.9)}\quad 2\alpha^3\int_D f_t^3e^{(\beta+2\alpha)f}v^2 \leqq \int_D f_t^{-1}e^{(\beta+2\alpha)f}(Lv)^2+E_1$$

where, letting $z=e^{\alpha f}v$, E_1 *denotes the quantity*

$$E_1 = 2\alpha \int_{S^+(D)} e^{\beta f} P_{h,n}(\nabla z) + 2\alpha^2(\alpha+1) \int_{S^+(D)} n_0 f_t^2 e^{\beta f} z^2.$$

Proof. Since we have defined $\lambda(\alpha)=e^{\alpha f}$ for all $\alpha>0$, the inequality (3.9) is essentially a restatement of (3.8).

The next task is to derive a companion estimate for $\|\nabla v\|$ in terms of Lv, one with the same weight function multiplying $(Lv)^2$ as in (3.9). This is done using (2.8) from Corollary 2.3. Recall that $m'=\min\{1, m^2\}$.

LEMMA 3.2. *Suppose* f *satisfies* (P_i) *for* $i=1, 2, 3$ *and* L *satisfies* (A_1) *relative to* f. *Let* D *be a bounded convenient domain in which* $t \geqq \varepsilon > 0$. *Let* β *be a constant such that* $m'\beta > 2$ *and* $\frac{1}{2}m^2\beta \geqq \nu B(\varepsilon)$. *If* $v \in C^2(D)$, *then for all* $\alpha > 0$

$$m'\alpha \int_D f_t e^{(\beta+2\alpha)f} \|\nabla v\|^2 \leqq \int_D f_t^{-1} e^{(\beta+2\alpha)f} (Lv)^2 + E_2 \tag{3.10}$$

where E_2 *denotes the quantity* $\int_{S^+(D)} e^{(\beta+2\alpha)f} P_{h,n}(\nabla v)$.

Proof. If $\frac{1}{2}\delta m^2 \geqq \nu B(\varepsilon)$, then by Corollary 2.3 we have the inequality

$$\tfrac{1}{2}\delta m' \int_D f_t e^{\delta f} \|\nabla v\|^2 \leqq 2 \int_D e^{\delta f} |Lvv_t| + \int_{S^+(D)} e^{\delta f} P_{h,n}(\nabla v). \tag{2.8}$$

Since $f_t > 0$ everywhere we have

$$2|Lvv_t| \leqq f_t^{-1}(Lv)^2 + f_t v_t^2 \leqq f_t^{-1}(Lv)^2 + f_t \|\nabla v\|^2.$$

Thus for sufficiently large δ we get

$$(\tfrac{1}{2}\delta m' - 1) \int_D f_t e^{\delta f} \|\nabla v\|^2 \leqq \int_D f_t^{-1} e^{\delta f} (Lv)^2 + \int_{S^+(D)} e^{\delta f} P_{h,n}(\nabla v). \tag{3.11}$$

If we set $\delta = 2\alpha + \beta$, then (2.8) and (3.11) hold for all $\alpha > 0$. Furthermore $\frac{1}{2}\delta m' - 1 = \alpha m' + \frac{1}{2}\beta m' - 1 \geqq \alpha m'$, so (3.10) follows.

The boundary integral terms E_1 and E_2 are easily computed for the particular bounded convenient domains we employ. For any nonnegative ε and R we define the unbounded domain

$$D(\varepsilon, \infty, R) = \{(t, x) \in \boldsymbol{R} \times \boldsymbol{R}^\nu : \varepsilon \leqq t < \infty; r \leqq Mt + R\}.$$

If L is the wave operator, $L = \Delta - \partial^2/\partial t^2$, then $m = M = 1$ in (A_0) and $D(0, \infty, R)$ is the "domain of influence" of the sphere of radius R at time $t=0$. In general the lateral boundary, $r = Mt + R$, will not be a characteristic hypersurface for L. The outer normal along $r = Mt + R$ is given by

$$n = (1+M^2)^{1/2}(-M, x_1/r, \ldots, x_\nu/r)$$

and thus

$$((n, n)) = (1+M^2)\left\{(-M)^2 - \sum_{ij=1}^{\nu} a_{ij} \frac{x_i}{r} \frac{x_j}{r}\right\}.$$

Because of (A_0)

$$((n, n)) \geqq (1+M^2)\{M^2-M^2\} = 0$$

and therefore the lateral boundary of $D(\varepsilon, \infty, R)$ is never spacelike. The domains $D(0, \infty, R)$ are the smallest convenient conical regions containing the "domain of influence" of the initial sphere of radius R.

The bounded convenient domains for which we specialize the estimates (3.10) and (3.11) are the "cork-shaped" domains

$$D(\varepsilon, T, R) = \{(t, x) : \varepsilon \leqq t \leqq T; r \leqq Mt+R\}$$

where $0<\varepsilon<T$ and $0<R$. The boundary $\partial D(\varepsilon, T, R)$ has three smooth parts:

$$\begin{aligned} S_{\text{lat}} &= \{(t, x) \in \partial D(\varepsilon, T, R) : r = Mt+R\}, \\ S(T, R) &= \{(t, x) \in \partial D(\varepsilon, T, R) : t = T\}, \\ S(\varepsilon, R) &= \{(t, x) \in \partial D(\varepsilon, T, R) : t = \varepsilon\}. \end{aligned}$$

On the lateral boundary S_{lat} we have seen that the outer unit normal n is negative nontimelike. On $S(\varepsilon, R)$ clearly $n=(-1, 0, \ldots, 0)$. And on $S(T, R)$, $n=(1, 0, \ldots, 0)$. Thus these $D(\varepsilon, T, R)$ are indeed convenient domains, and

$$\begin{aligned} S^+(D(\varepsilon, T, R)) &= S(T, R) \\ S^-(D(\varepsilon, T, R)) &= S_{\text{lat}} \cup S(\varepsilon, R). \end{aligned}$$

Suppose D is one of the $D(\varepsilon, T, R)$. The terms E_1 and E_2 of the Lemmas 3.1 and 3.2 can be estimated in terms of the energy integral

$$\mathscr{E}(v, T, R) = \int_{S(T,R)} \{v^2+\|\nabla v\|^2\}\,dx.$$

Since t is constant on $S(T, R)$ we have

$$E_1 = 2\alpha e^{\beta f(T)} \int_{S(T,R)} P_{h,h}(\nabla z)+2\alpha^2(\alpha+1)f_t^2(T)e^{\beta f(T)} \int_{S(T,R)} z^2$$

and

$$E_2 = e^{(\beta+2\alpha)f(T)} \int_{S(T,R)} P_{h,h}(\nabla v).$$

Let $M'=\max\{1, M^2\}$. So for any C^1 function w, it follows that

$$P_{h,h}(\nabla w) = w_t^2+\sum_{ij=1}^{\nu} a_{ij}w_{x_i}w_{x_j} \leqq w_t^2+M^2\sum_{i=1}^{\nu} w_{x_i}^2 \leqq M'\|\nabla w\|^2.$$

Since $z=e^{\alpha f}v$, we find that

$$\begin{aligned} \|\nabla z\|^2 &= (e^{\alpha f}v)_t^2+\sum_{i=1}^{\nu}(e^{\alpha f}v)_{x_i}^2 \\ &\leqq e^{2\alpha f}\left\{(\alpha f_t v+v_t)^2+\sum_{i=1}^{\nu} v_{x_i}^2\right\} \\ &\leqq 2e^{2\alpha f}\{\alpha^2 f_t^2 v^2+\|\nabla v\|^2\}. \end{aligned}$$

Thus

$$E_1 \leqq 2\alpha^2\{2M'\alpha+\alpha+1\}f_t^2(T)e^{(\beta+2\alpha)f(T)}\int_{S(T,R)} v^2$$
$$+4\alpha M'e^{(\beta+2\alpha)f(T)}\int_{S(T,R)} \|\nabla v\|^2$$

and

$$E_2 \leqq M'e^{(\beta+2\alpha)f(T)}\int_{S(T,R)} \|\nabla v\|^2.$$

At this point we introduce the following additional restriction on f:

(P_4) There is a constant $\mu \geqq 1$ such that $f_t^2(T) \leqq \mu e^{f(T)}$ whenever $T>1$.

This property is easily verified for $f \in \mathscr{F}$; indeed for $f(t)=\ln(t)$ or $f(t)=t$ we can take $\mu=1$, and if $f(t)=t^\gamma$ with $\gamma>1$ then $\mu=2\gamma^2$ will suffice.

If we now assume that f satisfies (P_i) for $i=1, 2, 3, 4$, then for sufficiently large T

$$f_t^2(T) \leqq \mu e^{f(T)}, \qquad 1 < e^{f(T)} \leqq \mu e^{f(T)},$$

and thus

$$E_1+E_2 \leqq 2\alpha^2\{2\alpha M'+\alpha+1\}\mu e^{(1+\beta+2\alpha)f(T)}\int_{S(T,R)} v^2$$
(3.12)
$$+(4\alpha+1)M'\mu e^{(1+\beta+2\alpha)f(T)}\int_{S(T,R)} \|\nabla v\|^2.$$

Letting

(3.13) $$p(\alpha) = 2\alpha^2\{2\alpha M'+\alpha+1\}\mu+(4\alpha+1)M'\mu$$

we get the bound

(3.14) $$E_1+E_2 \leqq p(\alpha)e^{(1+\beta+2\alpha)f(T)}\mathscr{E}(v, T, R)$$

for sufficiently large α and T.

The next result gives the crucial family of a priori inequalities from which our decay results follow.

THEOREM 3.3. *Suppose that f satisfies* (P_i) *for* $i=1, 2, 3, 4$ *and that L satisfies* (A_1) *relative to f. Let ε and R be positive constants. For each $T>\varepsilon$ let D_T denote $D(\varepsilon, T, R)$ and let S_T denote $S(T, R)$. Choose β so that $\beta \geqq 3$, $m^2\beta > 2\nu B(\varepsilon)$, and $m'\beta>2$. If $v \in C^2(D(\varepsilon, \infty, R))$ and if α and T are sufficiently large, then*

(3.15) $$2\alpha^3\int_{D_T} f_t^3 e^{(\beta+2\alpha)f}v^2+m'\alpha\int_{D_T} f_t e^{(\beta+2\alpha)f}\|\nabla v\|^2 \leqq 2\int_{D_T} f_t^{-1}e^{(\beta+2\alpha)f}(Lv)^2+E_3$$

where, using the $p(\alpha)$ defined in (3.13),

$$E_3 = p(\alpha)e^{(1+\beta+2\alpha)f(T)}\mathscr{E}(v, T, R).$$

Proof. Each D_T is a bounded convenient domain in which $t \geqq \varepsilon > 0$. The constant β is chosen to meet the conditions of Lemmas 3.1 and 3.2. Thus there is a constant $\boldsymbol{\alpha}$, depending on ε and β and R only, such that (3.9) and (3.10) hold when $\alpha \geqq \boldsymbol{\alpha}$ and $T > \varepsilon$. Adding (3.9) and (3.10) we get

$$2\alpha^3 \int_{D_T} f_t^3 e^{(\beta+2\alpha)f} v^2 + m'\alpha \int_{D_T} f_t e^{(\beta+2\alpha)f} \|\nabla v\|^2 \leqq 2 \int_{D_T} f_t^{-1} e^{(\beta+2\alpha)f} (Lv)^2 + E_1 + E_2.$$

But with the help of (P_1) and (P_2) we have seen that (3.14) holds for sufficiently large α and T. The inequality (3.15) now follows.

4. **Decay rate results.** Let f be a function with the properties (P_i) for $i = 1, 2, 3, 4$. In this section we study the inequality

$$(1.1) \qquad |Lu| \leqq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$$

under the following assumptions about the coefficients:

(A_0) There are positive constants m, M such that

$$m^2 \leqq \sum_{ij=1}^{\nu} a_{ij}(t, x)\xi_i\xi_j \leqq M^2$$

whenever $t \geqq 0$ and $\sum_{i=1}^{\nu} \xi_i^2 = 1$.

(A_1) For each $\varepsilon > 0$ there is a $B(\varepsilon) > 0$ such that $|(a_{ij})_t| \leqq B(\varepsilon) f_t$ whenever $t \geqq \varepsilon$.

(A_2) In each $D(\varepsilon, \infty, R)$ with $\varepsilon > 0$ and $R > 0$,

$$k_1(t, x) = O(f_t^2) \quad \text{and} \quad k_2(t, x) = O(f_t).$$

Notice that (A_2) does not control the asymptotic behavior of k_1 and k_2 uniformly in x, but only on each truncated cone $D(\varepsilon, \infty, R)$. Actually ($A_1$) could also be given as $|(a_{ij})_t| = O(f_t)$ in each $D(\varepsilon, \infty, R)$ at the cost of more complicated technical requirements in the preceding sections.

The rate of decay of a solution u of (1.1) in some $D(\varepsilon, \infty, R)$ is described in terms of the energy integral

$$\mathscr{E}(u, T, R) = \int_{S(T,R)} \{u^2 + \|\nabla u\|^2\}.$$

Let $g(t)$ be a continuous function with $\lim_{t\to\infty} g(t) = 0$. We say that u decays faster than $g(t)$ in $D(\varepsilon, \infty, R)$ if

$$\lim_{T\to\infty} \left\{ \frac{\mathscr{E}(u, T, R)}{g(T)} \right\} = 0.$$

We say u does not decay as fast as $g(t)$ if the ratio of $\mathscr{E}(u, T, R)$ to $g(T)$ grows unbounded as $T \to \infty$.

THEOREM 4.1. *Let u be a C^2 solution of*

$$(1.1) \qquad |Lu| \leqq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$$

in some $D(\varepsilon, \infty, R)$. There is a positive constant ρ such that if u decays faster than $e^{-\rho f(T)}$ in $D(\varepsilon, \infty, R)$, then u must vanish identically in $D(\varepsilon, \infty, R)$.

Proof. For each $T>\varepsilon$ let D_T denote $D(\varepsilon, T, R)$. To show $u=0$ in $D(\varepsilon, \infty, R)$ it suffices to show that u vanishes in all D_T.

Since u is C^2 on $D(\varepsilon, \infty, R)$ we may apply Theorem 3.3. Thus there is a β depending on ε such that for all sufficiently large α and T

$$(3.15)\quad 2\alpha^3 \int_{D_T} f_t^3 e^{(\beta+2\alpha)f} u^2 + \alpha m' \int_{D_T} f_t e^{(\beta+2\alpha)f} \|\nabla u\|^2 \leqq 2 \int_{D_T} f_t^{-1} e^{(\beta+2\alpha)f} (Lu)^2 + E_3$$

where $E_3 = p(\alpha) e^{(1+2\alpha+\beta)f(T)} \mathscr{E}(u, T, R)$.

Because u is a solution of (1.1) we have

$$(Lu)^2 \leqq \{k_1|u| + k_2\|\nabla u\|\}^2 \leqq 2\{k_1^2 u^2 + k_2^2 \|\nabla u\|^2\}.$$

By the assumption (A_2) we have constants K_1 and K_2 such that

$$k_1(t, x) \leqq K_1 f_t^2(t); \qquad k_2(t, x) \leqq K_2 f_t(t)$$

at all point of $D(\varepsilon, \infty, R)$. Hence,

$$(Lu)^2 \leqq 2K_1^2 f_t^4 u^2 + 2K_2^2 f_t^2 \|\nabla u\|^2.$$

Combining this with (3.15) we obtain the inequality

$$\begin{aligned} &2\alpha^3 \int_{D_T} f_t^3 e^{(\beta+2\alpha)f} u^2 + \alpha m' \int_{D_T} f_t e^{(\beta+2\alpha)f} \|\nabla u\|^2 \\ (4.1)\qquad &\leqq 2 \int_{D_T} f_t^{-1} e^{(\beta+2\alpha)f} \{2K_1^2 f_t^4 u^2 + 2K_2^2 f_t^2 \|\nabla u\|^2\} + E_3 \\ &\leqq 4K_1^2 \int_{D_T} f_t^3 e^{(\beta+2\alpha)f} u^2 + 4K_2^2 \int_{D_T} f_t e^{(\beta+2\alpha)f} \|\nabla u\|^2 + E_3. \end{aligned}$$

Hence we have

$$(4.2)\quad (2\alpha^3 - 4K_1^2) \int_{D_T} f_t^3 e^{(\beta+2\alpha)f} u^2 \leqq (4K_2^2 - \alpha m') \int_{D_T} f_t e^{(\beta+2\alpha)f} \|\nabla u\|^2 + E_3.$$

The inequality (4.2) is true for all sufficiently large α and T. We now pick an α large enough that (4.2) holds and also so large that

$$(2\alpha^3 - 4K_1^2) \geqq 1; \qquad (4K_2^2 - \alpha m') \leqq 0.$$

For this α and all sufficiently large T we now have

$$(4.3)\qquad 0 \leqq \int_{D_T} f_t^3 e^{(\beta+2\alpha)f} u^2 \leqq E_3 = p(\alpha) e^{(1+\beta+2\alpha)f(T)} \mathscr{E}(u, T, R).$$

The integral over D_T in (4.3) is a nonnegative increasing function of T. Set $\rho = 1+\beta+2\alpha$. So, if

$$\lim_{T\to\infty} e^{\rho f(T)} \mathscr{E}(u, T, R) = 0$$

then it follows from (4.3) that

$$\int_{D_T} f_t^3 e^{(\beta+2\alpha)} u^2 = 0, \quad \text{for all } T > \varepsilon,$$

and therefore that $u=0$ in all D_T.

This theorem gives a bound on the rate of decay of a nonzero solution u in $D(\varepsilon, \infty, R)$: its energy $\mathscr{E}(u, T, R)$ cannot decay faster than $e^{-\rho f(T)}$.

COROLLARY 4.2. *Suppose u satisfies (1.1) in $\mathscr{H}=\{(t, x) : t \geqq 0\}$. If, for every $\varepsilon > 0$ and $R > 0$, u decays faster than all $e^{-\rho f(T)}$, $\rho > 0$, in $D(\varepsilon, \infty, R)$, then $u=0$ in $\mathscr{H}$.*

Proof. For each positive ε and R the theorem applies to show that u vanishes in $D(\varepsilon, \infty, R)$. Thus $u(t, x)=0$ wherever $t>0$. By continuity $u=0$ in $\mathscr{H}$.

Each of the functions $f \in \mathscr{F}$ satisfy all the properties (P_i), $1 \leqq i \leqq 4$. The interpretations of Theorem 4.1 for $f(t)=\ln(t)$, $f(t)=t$, and $f(t)=t^\gamma$ with $\gamma>1$ respectively yield the results (I), (II), and (III) given in the introduction.

Corollary 4.2 is the best result possible if we add one more condition on f, namely

(P_5) For all $\alpha > \nu$, $\lim_{T\to\infty} T^\nu e^{-\alpha f(T)}=0$.

This property is easily verified for the functions in $\mathscr{F}$.

Suppose now that f satisfies (P_5) as well as (P_i), $i=1, \ldots, 4$. For a fixed $\alpha > \nu$ set $w(t, x)=e^{-\alpha f(t)}$. Then by a straightforward computation

$$Lw = \alpha(f_{tt}-\alpha f_t^2)w.$$

Set $k_1(t, x)=\alpha(\alpha+1)f_t^2$ and $k_2(t, x)=0$. These k_i satisfy (A_2). Because of (P_2) we have

$$|Lw| = \alpha|f_{tt}-\alpha f_t^2|\,|w| \leqq k_1(t, x)|w|+k_2(t, x)\|\nabla w\|.$$

So this w is a nonzero solution of an inequality of the form (1.1). Computation shows that

$$\mathscr{E}(w, T, R) = \int_{S(T,R)} e^{-2\alpha f}(1+\alpha^2 f_t^2)\,dx.$$

Let C_ν be the measure of the unit ball in $\boldsymbol{R}^\nu$. Using (P_4) we find

$$\mathscr{E}(w, T, R) \leqq \mu(1+\alpha^2)e^{(1-2\alpha)f(T)}C_\nu(MT+R)^\nu.$$

By using (P_5) we see that

$$\lim_{T\to\infty} (1+\alpha^2)C_\nu e^{-\alpha f(T)}(MT+R)^\nu = 0$$

for any $R>0$ and thus that w decays faster than $e^{-(\alpha-1)f(T)}$ in each $D(\varepsilon, \infty, R)$. Thus for any particular α we can find a nonzero solution of (1.1) which decays faster than $e^{-(\alpha-1)f(T)}$. So no rate of decay slower than that of Corollary 4.2 is sufficient to insure that a solution vanishes.

5. **Decay rates outside a characteristic conoid.** The maximal decay rate established in §4 holds for solutions in other domains provided appropriate boundary conditions are added. As examples we consider in this section solutions outside a characteristic conoid, and in the next section solutions outside a reflecting obstacle.

Throughout this section we assume that f satisfies (P_i) for $1 \leqq i \leqq 5$ and that L satisfies (A_1) relative to f.

The inequality (1.1) can be considered as describing the time course of a disturbance u in x-space. The form

$$d\rho^2 = \sum_{ij=1}^{\nu} a_{ij}(t, x)\, dx_i\, dx_j$$

is a time varying metric in x-space. The characteristic conoid $\mathscr{C}$ at the origin is the set of points (t, x) such that t is the ρ-distance between x and 0 at time t. It can be verified that $\mathscr{C}$ is a characteristic hypersurface in $\boldsymbol{R}\times\boldsymbol{R}^\nu$. In the special case that L is the wave operator and $\nu=3$, the conoid $\mathscr{C}$ is just the forward light cone. Saying that (t, x) lies outside $\mathscr{C}$ means that the ρ-distance between x and 0 at time t is not less than t.

Notice that (A_0) assures us that $\mathscr{C}$ lies in the region $\{(t, x) : r \leqq Mt\}$. Thus for $\varepsilon>0$ and $R>0$ it follows that the sets

$$P(\varepsilon, \infty, R) \equiv \{(t, x) \in D(\varepsilon, \infty, R) : (t, x) \text{ lies outside } \mathscr{C}\}$$

are unbounded domains. All the bounded domains

$$P(\varepsilon, T, R) \equiv \{(t, x) \in P(\varepsilon, \infty, R) : t \leqq T\}$$

are convenient since the boundary $\partial P(\varepsilon, T, R)$ is composed of smooth parts along the characteristic hypersurface $\mathscr{C}$, and along the nontimelike hypersurfaces $t=T$, $t=\varepsilon$, and $r=Mt+R$.

The first step in adapting Theorem 4.1 to the case of solutions outside $\mathscr{C}$ is to adapt Theorem 3.3.

THEOREM 5.1. *For fixed positive ε and R let P_T denote $P(\varepsilon, T, R)$ for each $T>\varepsilon$. Let S_T denote $\{(t, x) \in \partial P_T : t=T\}$. Suppose v is a C^2 function outside $\mathscr{C}$ which vanishes along $\mathscr{C}$. There is a β such that for all sufficiently large α and T*

$$(5.1)\quad 2\alpha^3 \int_{P_T} f_t^3 e^{(\beta+2\alpha)f} v^2 + \alpha m' \int_{P_T} f_t e^{(\beta+2\alpha)f} \|\nabla v\|^2 \leqq \int_{P_T} f_t^{-1} e^{(\beta+2\alpha)f} (Lv)^2 + E_4$$

where, for the polynomial p of (3.13), *E_4 denotes the quantity*

$$E_4 = p(\alpha) e^{(1+\beta+2\alpha)f(T)} \int_{S_T} \{v^2 + \|\nabla v\|^2\}\, dx.$$

Proof. Choose β so large that the Lemmas 3.1 and 3.2 hold for all P_T with $T>\varepsilon$. This choice of β is independent of v. Then under the hypotheses of Lemma 3.1 we get the estimates

$$(3.9)\qquad 2\alpha^3 \int_{P_T} f_t^3 e^{(\beta+2\alpha)f} v^2 \leqq \int_{P_T} f_t^{-1} e^{(\beta+2\alpha)f} (Lv)^2 + E_1(T)$$

where, in terms of $z=e^{\beta f}v$, $E_1(T)$ denotes the quantity

$$E_1(T) = 2\int_{S^+(P_T)} e^{\beta f} P_{h,n}(\nabla z) + 2\alpha^2(\alpha+1) \int_{S^+(P_T)} n_0 f_t^{2} e^{\beta f} z^2.$$

The S^+ part of ∂P_T is composed of S_T and the portion of ∂P_T along $\mathscr{C}$. Since v vanishes along $\mathscr{C}$ so does z. Thus we know that ∇z is a scalar multiple, ζn, of the outer unit normal along $\mathscr{C} \cap \partial P_T$. Since $\mathscr{C}$ is characteristic we have $((n, n))=0$ on $\mathscr{C}$. Thus along $\mathscr{C}$

$$P_{h,n}(\nabla z) = \zeta^2 P_{h,n}(n) = \zeta^2\{2((h, n))((n, n))-((h, n))((n, n))\} = 0.$$

So both integrands in $E_1(T)$ vanish off S_T, and $E_1(T)$ takes the form

$$E_1(T) = 2\alpha \int_{S_T} e^{\beta f} P_{h,n}(\nabla z)+2\alpha^2(\alpha+1) \int_{S_T} n_0 f_t^2 e^{\beta f} z^2.$$

Similarly, under the hypotheses for Lemma 3.2 we find that

$$\alpha m' \int_{P_T} f_t e^{(\beta+2\alpha)f} \|\nabla v\|^2 \leqq \int_{P_T} f_t^{-1} e^{(\beta+2\alpha)f}(Lv)^2+E_2(T)$$

where

$$E_2(T) = \int_{S_T} e^{(\beta+2\alpha)f} P_{h,n}(\nabla v).$$

Now since $t=T$ along S_T, we can repeat the argument of §3 to obtain the estimate

$$E_1(T)+E_2(T) \leqq p(\alpha)e^{(1+\beta+2\alpha)f(T)} \int_{S_T} \{v^2+\|\nabla v\|^2\}$$

where $p(\alpha)$ is defined by (3.13).

With (5.1) established, we can mimic the proof of Theorem 4.1 exactly for solutions of (1.1) outside $\mathscr{C}$ which vanish on $\mathscr{C}$.

THEOREM 5.2. *Suppose k_1 and k_2 satisfy the conditions*

$$(\mathrm{A}_2) \qquad k_1(t, x) = O(f_t^2); \qquad k_2(t, x) = O(f_t),$$

in some $P(\varepsilon, \infty, R)$ with $\varepsilon>0$, $R>0$. Let u be a C^2 solution of

$$(1.1) \qquad |Lu| \leqq k_1(t, x)|u|+k_2(t, x)\|\nabla u\|$$

in $P(\varepsilon, \infty, R)$ which vanishes along $\mathscr{C}$. Then there is a positive ρ such that if

$$\lim_{T\to\infty} e^{\rho f(T)} \int_{S_T} \{u^2+\|\nabla u\|^2\} = 0,$$

then u vanishes identically in $P(\varepsilon, \infty, R)$.

Proof. For a fixed large β and all large enough α and T we have the estimate (5.1). Because of (1.1) and (A_2) we find that

$$(Lu)^2 \leqq 2K_1^2 f_t^4 u^2+2K_2^2 f_t^2\|\nabla u\|^2$$

in $P(\varepsilon, \infty, R)$. Combining this with (5.1) we obtain

$$(5.2) \quad (2\alpha^3-4K_1^2) \int_{P_T} f_t^3 e^{(\beta+2\alpha)f} u^2 \leqq (4K_2^2-\alpha m') \int_{P_T} f_t e^{(\beta+2\alpha)f}\|\nabla u\|^2+E_3(T)$$

for all large enough α and T. Now fix a value of α so large that (5.2) holds and that $(2\alpha^3-4K_1^2)\geqq 1$ and $(4K_2^2-\alpha m')\leqq 0$. With this fixed α and sufficiently large T we now obtain

$$0 \leqq \int_{P_T} f_t^3 e^{(\beta+2\alpha)f}u^2 \leqq p(\alpha)e^{(1+2\alpha+\beta)f(T)}\int_{S_T}\{u^2+\|\nabla u\|^2\}.$$

Picking $\rho=1+2\alpha+\beta$, the theorem follows.

6. **Decay in exterior domains.** Much attention has been given to the asymptotic behavior of solutions of the wave equation outside a reflecting obstacle. In this section we consider solutions of

$$(1.1) \qquad |Lu| \leqq k_1(t,x)|u|+k_2(t,x)\|\nabla u\|$$

in an exterior domain. We assume that L satisfies (A_0) and (A_1) with respect to some function f satisfying (P_i) for $1\leqq i\leqq 5$.

Let $\mathcal{O}$ be a bounded domain in x-space with smooth boundary and connected complement. Let $R_0=\max\{r(x) : x\in\mathcal{O}\}$. We consider solutions outside $\mathcal{O}$, in other words, solutions in the region $\mathcal{R}$ where

$$\mathcal{R} = \{(t,x) : t \geqq 0; x\notin \operatorname{Int}(\mathcal{O})\}.$$

We further restrict our attention to solutions in the class

$$\mathcal{U} = \{u\in C^2(\mathcal{R}) : u(t,x) = 0 \text{ if } x\in\partial\mathcal{O}\}.$$

The technique employed to establish bounds on the decay rate in this situation is a slight modification of that used in §§3 and 4.

The standard domains in the a priori estimates are the sets

$$\begin{aligned} Q(\varepsilon,T,R) &= \mathcal{R}\cap D(\varepsilon,T,R) \\ &= \{(t,x) : \varepsilon \leqq t \leqq T; r \leqq Mt+R; x\notin \operatorname{Int}(\mathcal{O})\}\end{aligned}$$

for $\varepsilon>0$, $R>R_0$, and $\varepsilon<T<\infty$. The boundary $\partial Q(\varepsilon,T,R)$ is composed of three pieces:

$$\begin{aligned} S^+ &= \{(t,x)\in\partial Q(\varepsilon,T,R) : t = T\},\\ S^- &= \{(t,x)\in\partial Q(\varepsilon,T,R) : t = \varepsilon \text{ or } r = Mt+R\},\\ S^t &= \{(t,x)\in\partial Q(\varepsilon,T,R) : x\in\partial\mathcal{O}\}.\end{aligned}$$

On $S^+\cup S^-$ the outer unit normal n satisfies $((n,n))\geqq 0$. But on S^t n is spacelike; indeed the time component n_0 is zero. So the domains $Q(\varepsilon,T,R)$ fail to be convenient. However the formulas of Lemma 2.2 are still valid if we insist that $v\in\mathcal{U}$.

LEMMA 6.1. *Let Q denote one of the domains $Q(\varepsilon,T,R)$. If $v\in\mathcal{U}$, and $\frac{1}{2}\alpha m^2 \geqq \nu B(\varepsilon)$, then*

$$(6.1) \qquad \int_Q 2Lve^{\alpha f}v_t \geqq -\int_{S^+(Q)} e^{\alpha f}P_{h,n}(\nabla v)$$

and

$$\tfrac{1}{2}\alpha m' \int_Q f_t e^{\alpha f}\|\nabla v\|^2 \leqq 2\int_Q e^{\alpha f}|v_t L v| + \int_{S^+(Q)} e^{\alpha f} P_{h,n}(\nabla v). \tag{6.2}$$

Proof. We can apply (2.4) with $\lambda = e^{\alpha f}$ to obtain

$$\int_Q 2Lve^{\alpha f}v_t = \int_Q e^{\alpha f}\left\{\alpha f_t P_{h,h}(\nabla v) + \sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j}\right\} - \int_{\partial Q} e^{\alpha f} P_{h,n}(\nabla v).$$

Because of Lemma 2.1 and the definition of m', we get

$$\int_Q 2Lve^{\alpha f}v_t \geqq \tfrac{1}{2}\alpha m' \int_Q f_t e^{\alpha f}\|\nabla v\|^2 - \int_{\partial Q} e^{\alpha f} P_{h,n}(\nabla v).$$

On $S^-(Q)$ we know that $P_{h,n}(\nabla v) \leqq 0$. On S^t, we have $n_0 = 0$ and $v_t = 0$. Hence, on S^t,

$$P_{h,n}(\nabla v) = n_0 P_{h,h}(\nabla v) - 2v_t \sum_{ij=1}^{\nu} a_{ij} n_i v_{x_j} = 0.$$

Therefore

$$\int_Q 2Lve^{\alpha f}v_t \geqq \tfrac{1}{2}\alpha m' \int_Q f_t e^{\alpha f}\|\nabla v\|^2 - \int_{S^+(Q)} e^{\alpha f} P_{h,n}(\nabla v).$$

The inequalities (6.1) and (6.2) now follow directly.

Using this modified version of Lemma 2.2, the method of §3 adapts to establish the following estimates.

THEOREM 6.2. *For fixed $\varepsilon > 0$, $R > R_0$, let Q_T denote $Q(\varepsilon, T, R)$ for each $T > \varepsilon$. There is a β such that if $v \in \mathscr{U}$ and if α and T are sufficiently large then*

$$3\alpha^3 \int_{Q_T} f_t^3 e^{(\beta+2\alpha)f} v^2 + \alpha m' \int_{Q_T} f_t e^{(\beta+2\alpha)f}\|\nabla v\|^2 \leqq 2\int_{Q_T} f_t^{-1} e^{(\beta+2\alpha)f}(Lv)^2 + E_5 \tag{6.3}$$

where, for p defined in (3.13), E_5 *denotes the quantity*

$$E_5 = p(\alpha)e^{(1+\beta+2\alpha)f(T)} \int_{S^+(Q_T)} \{v^2 + \|\nabla v\|^2\}.$$

Letting $Q(\varepsilon, \infty, R)$ denote the unbounded domain $\mathscr{R} \cap D(\varepsilon, \infty, R)$ we make the following assumption about the coefficients k_i in (1.1):

(A_2) $k_1(t, x) = O(f_t^2)$; $k_2(t, x) = O(f_t)$ uniformly in each $Q(\varepsilon, \infty, R)$.

Suppose $u \in \mathscr{U}$ is a solution of (1.1) in some $Q(\varepsilon, \infty, R)$. Then following the method of proof of Theorem 4.1, we find an α so large that

$$0 \leqq \int_{Q_T} f_t^3 e^{(\beta+2\alpha)f} u^2 \leqq p(\alpha)e^{(1+2\alpha+\beta)f(T)} \int_{S^+(Q_T)} \{u^2 + \|\nabla u\|^2\} \tag{6.4}$$

for all sufficiently large T.

THEOREM 6.3. *Suppose* $u \in \mathscr{U}$ *is a solution of* (1.1) *in* $Q(\varepsilon, \infty, R)$. *Let* S_T *denote* $S^+(Q_T) = \{(T, x) : x \notin \operatorname{Int}(\mathscr{O}); r \leqq Mt + R\}$. *There is a* $\rho > 0$ *such that if*

$$\lim_{T \to \infty} e^{\rho f(T)} \int_{S_T} \{u^2 + \|\nabla u\|^2\} = 0 \tag{6.5}$$

then u *vanishes identically in* $Q(\varepsilon, \infty, R)$.

Proof. Set $\rho = 1 + 2\alpha + \beta$ for the α and β of (6.4). Since the integral over Q_T in (6.4) is an increasing nonnegative function of T, then (6.5) implies that u must vanish in all Q_T, and thus in $Q(\varepsilon, \infty, R)$.

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