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A CLASS OF COMPLETE ORTHOGONAL SEQUENCES OF STEP FUNCTIONS(1)

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Abstract. A class of orthogonal sets of step functions is defined and each member is shown to be complete in $L_2(0, 1)$. Pointwise convergence theorems are obtained for the Fourier expansions relative to these sets. The classical Haar orthogonal set is shown to be a set of this class and the class itself is seen to be a subclass of the "generalized Haar systems" defined recently by Price.

1. **Introduction.** Convergence theorems for the Fourier expansions relative to the classical set of Haar functions have interestingly weak hypotheses. If $f \in L_1(0, 1)$ is the derivative of its indefinite integral at $x \in [0, 1]$, the Haar expansion converges to $f(x)$ at this point, and, if f is continuous on $[0, 1]$, the convergence is uniform on this interval.

In this paper we prove that each sequence of points which is dense in [0, 1] determines a complete orthonormal set of step functions whose associated Fourier expansion has convergence properties similar to the Haar expansion. In fact, the Haar set is a member of the class of sets defined in §2. This class, in turn, is seen to be a subclass of the class of "generalized Haar systems" defined by Price [3].

2. **Definition of the sequences $\{\theta_n\}$.** Suppose that $A=\{a_1, a_2, \ldots\}$ is a sequence of distinct points in (0, 1) which is dense in [0, 1] and let $\{g_n\}$, $n=0, 1, 2, \ldots$, be the set of unit step functions defined by

$$g_0(x) \equiv 1 \quad \text{on } [0, 1]$$

and, for $n \geqq 1$,

$$\begin{aligned} g_n(x) &= 0, \qquad x \in [0, a_n), \\ &= 1, \qquad x \in [a_n, 1]. \end{aligned}$$

Since no two of these functions have discontinuities at the same point, it is clear that the g_i are linearly independent on [0, 1]. Consequently, one can use the Gram-Schmidt process to construct an orthonormal sequence of functions $\{\theta_n(x)\}$

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such that each θ_n is a linear combination of the g_k, $k \leqq n$. Because of the triangular nature of this construction, each g_i can also be expressed as a linear combination of the θ_k, $k \leqq i$.

3. **Completeness of $\{\theta_n\}$.** To obtain convergence theorems for the Fourier expansions relative to the orthonormal sequences defined in §2 one needs a rather obvious property of the sequence A which is given in Lemma 1. In the statement of this lemma and throughout this paper the term "adjacent points" of a finite subset $A_N \subset A$ will be used to denote successive elements of this subset when the elements are arranged in order of magnitude; i.e. a_m and a_n are adjacent points of A_N if and only if there is no $a_k \in A_N$ such that $a_m < a_k < a_n$ or $a_n < a_k < a_m$.

LEMMA 1. *Let $\{a_1, a_2, \ldots\}$ be a sequence of distinct points of* $(0, 1)$ *which is dense in* $I = [0, 1]$. *Then for each $\delta > 0$ there is an integer N_δ such that for each $N > N_\delta$,*

(i) *any pair of adjacent points a_m and a_n in the subset $A_N = \{a_1, a_2, \ldots, a_N\}$ satisfy* $|a_m - a_n| < \delta$,

(ii) $d(x, A_N) < \delta$ *for all $x \in I$. ($d(x, A_N)$ is the distance from x to the set A_N defined in the usual manner.)*

THEOREM 1. *The orthonormal sequence of functions $\{\theta_n\}$ is complete in* $L_2(0, 1)$.

Proof. Suppose that $f \in L_2(0, 1)$ and $\int_0^1 f\theta_n\,dx = 0$ for all n. Then since each g_n is a linear combination of the θ_i, $i \leqq n$,

$$\int_0^1 fg_n\,dx = \int_{a_n}^1 f\,dx = 0 \quad \text{for all } n,$$

and since A is dense in [0, 1],

$$\int_x^1 f\,dx = 0 \quad \text{for all } x \in [0, 1];$$

i.e.

$$f \overset{\text{a.e.}}{=} 0 \quad \text{on } [0, 1].$$

4. **Pointwise convergence of the Fourier $\{\theta_n\}$ expansion.** Since the orthonormal sequence $\{\theta_n\}$ is complete in $L_2(0, 1)$, any function f in this space has the norm-convergent Fourier expansion

$$f(x) \sim \sum b_n\theta_n(x), \quad \text{where } b_n = \int_0^1 f\theta_n\,dx. \tag{1}$$

In this section we obtain sufficient conditions for the convergence of this expansion in the pointwise sense to $f(x)$. The main result may be stated as follows:

THEOREM 2. *The Fourier-θ_n expansion of a function $f \in L_1(0, 1)$ converges to $f(x)$ at each point $x \in [0, 1]$ at which f is the derivative of the indefinite integral F of f. This holds, in particular,* (a) *almost everywhere,* (b) *at every point of continuity of f.*

Proof. Fix $N>0$ and consider the partial sum S_N of f. Let $0<a_{i_1}<a_{i_2}<\cdots<a_{i_N}<1$ be the points $0, a_1, a_2, \ldots, a_N, 1$ arranged in increasing order. Since each θ_i, $i\leqq N$, is a linear combination of the g_i, $i\leqq N$, S_N is a step function whose intervals of constancy are $[0, a_{i_1})$, $[a_{i_1}, a_{i_2}), \ldots, [a_{i_N}, 1]$. We first show that for x in any such interval I, S_N is equal to the average of f over I; i.e.

$$S_N(x,f) = \frac{1}{|I|}\int_I f\,dt \quad \text{if } x\in I.$$

Suppose first that $f\in L_2(0, 1)$ and let $K_0, K_1, \ldots, K_N$ denote the characteristic functions of the intervals $[0, a_{i_1}), \ldots, [a_{i_N}, 1]$. Then

$$S_N = \sum_0^N b_n\theta_n = \sum_0^N B_nK_n,$$

where the B's are constants. Clearly B_i is the value S_N takes in the interval associated with K_i. Now if T_N is any linear combination of the θ_k, $k\leqq N$, it is well known that $\int_0^1 (f-T_N)^2\,dx$ assumes its minimum value when T_N is the partial sum S_N. Thus the B's must have values which minimize the integral $\int_0^1 (f-\sum_0^N B_nK_n)^2\,dx$ and, equating the partial derivatives with respect to the B's to 0, we obtain, for each $m=0, 1, 2, \ldots, N$,

$$\int_0^1 fK_m\,dx = B_m\int_0^1 K_m^2\,dx.$$

If f is merely in $L_1(0, 1)$, there is a sequence $\{f_k\}$ of functions in $L_2(0, 1)$ such that $\|f-f_k\|_1\to 0$. It is immediate that each Fourier coefficient of f is the limit of the corresponding coefficient of f_k and we have, for $x\in I$,

$$S_N(x,f) = \lim_k S_N(x,f_k) = \lim_k \frac{1}{|I|}\int_I f_k\,dx = \frac{1}{|I|}\int_I f\,dx.$$

Now if x_0 is in the interval $I=[a, b)$ where a and b are adjacent points of the subset A_N,

$$\begin{aligned}|f(x_0)-S_N(x_0,f)| &= \left|f(x_0)-\frac{1}{|I|}\int_I f\,dx\right|\\ &= \left|\frac{x_0-a}{|I|}\left[f(x_0)-\frac{1}{x_0-a}\int_a^{x_0} f\,dx\right]+\frac{b-x_0}{|I|}\left[f(x_0)-\frac{1}{b-x_0}\int_{x_0}^b f\,dx\right]\right|.\end{aligned}$$

Since by Lemma 1 the length of I can be made arbitrarily small by taking N sufficiently large, the last equation implies that the Fourier expansion of f converges to $f(x_0)$ at any point $x_0\in[0, 1]$ at which f is the derivative of its indefinite integral.

The expression for S_N obtained in the preceding proof and a theorem from [4, p. 32] can be used to establish the following corollaries to Theorem 2.

Corollary 1. *If* $f\in L_p(0, 1)$, *then* $\|f-S_N\|_p\to 0$ $(1<p<\infty)$.

COROLLARY 2. *If* $f \in L_p(0, 1)$, *then* $\|\text{Sup}_N S_N\| \leqq A_p \|f\|_p$ $(1 < p \leqq \infty)$ *where* A_p *is constant for each* p.

THEOREM 3. *The Fourier-*θ_n *expansion of a function* f *which is continuous on* $[0, 1]$ *converges uniformly to* $f(x)$ *on* $[0, 1]$.

Proof. If $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ when $x_1, x_2 \in [0, 1]$ and $|x_1 - x_2| < \delta$. By Lemma 1 an integer N_δ exists such that if $N > N_\delta$, the set A_N determines a partition of $[0, 1]$ into subintervals of length less than δ. Thus if x_0 is in any subinterval $I = [a, b]$ of this partition we have

$$\begin{aligned} |f(x_0) - S_N(x_0, f)| &= \left| f(x_0) - \frac{1}{|I|} \int_I f(x)\, dx \right| \\ &= |f(x_0) - f(\xi)| \quad \text{where } \xi \in [a, b] \\ &< \varepsilon. \end{aligned}$$

5. **Behavior of the Fourier-θ_n expansion at a point of discontinuity of f.** Suppose f has an isolated finite discontinuity at a point a_i of the set A. Since the step function g_i has a unit jump at a_i, a function G which is continuous at a_i can be constructed by adding a constant multiple of g_i to f. By Theorem 2, the Fourier expansion of G converges to $G(a_i)$ and since the θ_n expansion of g_i is a finite sum, it follows that the Fourier expansion of f must converge at a_i. In fact, one can readily prove the following

THEOREM 4. *If* $f \in L_2(0, 1)$ *has a finite discontinuity at a point* a_i *of the sequence* A (*which determines* $\{\theta_n\}$) *the Fourier-*θ_n *expansion of* f *converges to* $f(a_i^+)$ *at this point.*

The fact that the expansion converges to $f(a_i^+)$ rather than $\frac{1}{2}[f(a_i^+) + f(a_i^-)]$ or some other value between $f(a_i^+)$ and $f(a_i^-)$ is not significant since this value is determined solely by the definition of $g_i(a_i)$.

No general statement can be made concerning the convergence of (1) at a point of discontinuity of f that is not in A. To show this we consider two expansions corresponding to different A sequences (hence to different θ_n sets) of a simple step function with a discontinuity at a point $c \in (0, 1)$ which is not in A. Let

$$\begin{aligned} g_c(x) &= 0, \qquad x \in [0, c), \\ &= 1, \qquad x \in [c, 1], \end{aligned}$$

where $c \notin A$. Suppose that N is an integer sufficiently large for the subset A_N to contain points in both $(0, c)$ and $(c, 1)$ and let S_N denote the partial sum of the Fourier expansion of g_c. If $a_l(N)$ and $a_r(N)$ are the points of A_N adjacent to c on the left and right, respectively, we find

$$S_N(c, g_c) = \frac{1}{a_r - a_l} \int_{a_l}^{a_r} g_c\, dx = \frac{a_r(N) - c}{a_r(N) - a_l(N)}. \tag{2}$$

Equation (2) suggests that the convergence of the Fourier expansion at the point c depends on the sequence A. To see that this is actually the case, construct a sequence A, by choosing $a_1=c/2$ and $a_2=(c+1)/2$, the respective midpoints of $(0, c)$ and $(c, 1)$. These two points along with c determine four successive subintervals of $(0, 1)$; let their midpoints (from left to right) be a_3, a_4, a_5, a_6. Continue this subdivision process (with 2^n new midpoints at the nth stage) to obtain A. In (2), $a_r(N)-c$ is the distance from c to the closest point of A_N on the right and has the form $(1-c)/2^k$ for some integer k. On the other hand, the distance from c to the closest point of A_N on the left will either be $c/2^k$ or $c/2^{k+1}$ depending on N. In the first case (2) gives $S_N=1-c$, and in the second $S_N=2(1-c)/(2-c)$. Since these expressions are independent of N, we see that the sequence $\{S_n(c, g_c)\}$ consists of two distinct constant subsequences and cannot converge.

As a second example we construct the sequence A as follows: $a_1=c/2$, $a_2=(1+c)/2$, $a_3=c/3$, $a_4=2c/3$, $a_5=(1+2c)/3$, $a_6=(2+c)/3, \ldots$; i.e. the elements of A are the distinct points which divide the intervals $(0, c)$ and $(c, 1)$ into two equal parts, three equal parts, four equal parts, etc. For this sequence, the distance from c to the closest point of A_N on the right has the form $(1-c)/k(N)$ where $k(N)$ is an integer that depends on N; and the distance from c to $a_l(N)$ is either $c/k(N)$ or $c/[k(N)+1]$. In the first case (2) gives $S_N=1-c$ and in the second

$$S_N = \frac{1-c}{1-c/(k(N)+1)}.$$

Since $k(N)$ approaches infinity with N, the limit of the subsequence given by the second equation is also $1-c$. Thus $\lim S_n(c)$ exists for this sequence A; i.e. the Fourier expansion of g_c determined by this particular sequence converges at c to $1-c$.

6. **The structure of** $\{\theta_n\}$. The orthonormal sequence $\{\theta_n\}$ defined in §2 is obtained by applying the Gram-Schmidt orthogonalization process to the linearly independent sequence $\{g_i\}$. Since this process gives θ_n as a linear combination of the g_i, $i\leqq n$, it is clear that θ_n is a step function which is constant on the subintervals $[0, a_{i_1})$, $[a_{i_1}, a_{i_2}), \ldots, [a_{i_n}, 1]$ of $[0, 1]$ determined by the successive points of A_n. We shall now see that a precise expression for θ_n in terms of the points of the subset A_n can be obtained by induction.

Let $\theta_0=g_0$ and assume that θ_{n-1}, $n\geqq 1$, has been determined. Suppose a_n falls in the interval (a, b) whose endpoints are successive points of the partition of $[0, 1]$ determined by A_{n-1}, and let f_n be given by

$$\begin{aligned} f_n(x) &= 1/(a_n-a), & x &\in [a, a_n), \\ &= -1/(b-a_n), & x &\in [a_n, b), \\ &= 0, & &\text{otherwise.} \end{aligned}$$

Since a and b are either in A_{n-1} or are endpoints of [0, 1], it is obvious that f_n is a linear combination of the g_i, $i \leqq n$. Furthermore if $k<n$, θ_k is constant on (a, b) so

$$(\theta_k, f_n) = \int_0^1 \theta_k f_n \, dx = c \int_a^b f_n \, dx = 0$$

and we see that f_n is orthogonal to each θ_k, $k<n$. It follows, of course, that the function θ_n given by the Gram-Schmidt process must be $\pm f_n/\|f_n\|$.

7. **Examples and remarks.** (A) The set of rationals in (0, 1) is countable and different enumerations of this set lead to an infinity of sequences of the type A described in §2. For example, one can take $a_1=1/2$, $a_2=1/3$, $a_3=2/3$, $a_4=1/4$, $a_5=3/4, \ldots$ (where the irreducible fractions with denominator 2, 3, 4, ... are used successively in blocks) and the corresponding $\{g_i\}$ will be the set of all unit step functions with jumps at the rational points of (0, 1). The orthonormal set $\{\theta_n\}$, in this case, consists of step function with discontinuities at the rationals.

(B) If p is any prime, the set of all numbers of the form k/p^m where m and k are integers, $k<p^m$ and $k \not\equiv 0 \pmod p$, is countable since it is a subset of the rationals and is obviously dense in I. Any such set with an appropriate specific enumeration could therefore be used for the sequence A. In fact, if A is a particular sequence of this type with $p=2$, the corresponding g_i can be orthonormalized to obtain the familiar Haar functions $\{\chi_j^n\}$. Thus if $a_1=1/2$, $a_2=1/4$, $a_3=3/4$, $a_4=1/8$, $a_5=3/8$, $\ldots, a_n=(2n+1-2^k)/2^k, \ldots$, where k is the smallest integer such that $2^k>n$, the corresponding θ_i are the classical Haar functions.

Price [3] has defined a class of orthonormal sets of step functions of a more general type. Price's definition is not sufficiently restrictive to imply the completeness of his sets. It is readily verified that the class of sequences $\{\theta_n\}$ defined in this paper is a subclass of Price's and that our requirement that A be dense in [0, 1] (which insures completeness) is the essential difference.

Franklin [1] constructed a complete orthonormal sequence of linear functions related to the Haar functions. It can be readily generalized. Let $\{g_n\}$ be defined as in §2 and construct a sequence $\{h_n\}$ on [0, 1] with

$$h_0(x) \equiv 1 \quad \text{and} \quad h_n(x) = \int_0^x g_{n-1}(t) \, dt, \qquad n \geqq 1.$$

The sequence $\{h_n\}$ is linearly independent on [0, 1] and the Gram-Schmidt process yields a complete orthonormal sequence of continuous polygonal functions.

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