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A RANK THEOREM FOR COHERENT ANALYTIC SHEAVES

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Abstract. Let S be an analytic subvariety in C^n and $\mathscr{F}$ a coherent analytic sheaf on C^n, such that $\mathscr{F}$ is locally free on $C^n - S$ and $\Gamma(U, \mathscr{F}) = \Gamma(U-S, \mathscr{F})$ for every open set $U \subset C^n$. It is shown that $\mathscr{F}$ is locally free everywhere, if codh $\mathscr{F} \geqq n-1$ and dim $S + \text{rank } \mathscr{F} \leqq n-2$.

In this note the following question is examined. Let S be an analytic subvariety in C^n and V be a vector bundle on $C^n - S$. Is it true that V is globally trivial if dim $S + \text{rank } V \leqq n-2$? If we use results on the extension of coherent analytic sheaves, we may reformulate the question as follows. Let $\mathscr{F}$ be a coherent analytic sheaf on C^n and assume that

(a) $\mathscr{F} | C^n - S$ is locally free,

(b) for any open subset $U \subset C^n$, $\Gamma(U, \mathscr{F}) \simeq \Gamma(U-S, \mathscr{F})$,

(c) dim $S + \text{rank } \mathscr{F} \leqq n-2$.

Is then $\mathscr{F}$ locally free everywhere? In the case dim $S = 0$ an affirmative answer of this question was conjectured by Wolfgang Barth. We prove here that this conjecture is right for general S if we make the additional assumption codh $\mathscr{F}_x \geqq n-1$ for every $x \in S$.

It turned out that the above question is only local in nature and in addition purely algebriac, see §1. It is related to a conjecture on the number of generators of analytic modules. In §2 we use a result of [3] and [4] to establish a canonical resolution of a certain ideal sheaf which describes the singularities of the sheaf in question, and derive from this result the theorem.

1. Let X be a domain in C^n, $\mathscr{F}$ a coherent analytic sheaf on X and $S \subset X$ an analytic subvariety. We denote by $\mathcal{O}$ the sheaf of holomorphic functions on X. By $\mathscr{R}^i_S \mathscr{F}$ we denote the analytic sheaf on X defined by the presheaf $U \rightsquigarrow H^i(U-S, \mathscr{F})$. In general the $\mathscr{R}^i_S \mathscr{F}$ are no longer coherent, see [7]. The following result however was proved in [5] and [6].

1.1. $\mathscr{F} = \mathscr{R}^0_S \mathscr{F}$ and $\mathscr{R}^i_S \mathscr{F} = 0$, for $1 \leqq i \leqq q$, if and only if dim $S \cap S_{k+q+2}(\mathscr{F}) \leqq k$ for all k, where $S_m(\mathscr{F})$ denotes the analytic subvariety of all $x \in X$ for which

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codh $\mathscr{F}_x \leqq m$. For every $x \in X$ we define rank $\mathscr{F}_x = \dim_{\mathscr{M}_x} \mathscr{M}_x \otimes_{\mathscr{O}_x} \mathscr{F}_x$ where $\mathscr{M}_x$ is the field of meromorphic function germs at x. Since $\mathscr{F}$ is coherent and X connected rank $\mathscr{F}_x$ does not depend on x and is denoted simply by rank $\mathscr{F}$.

1.2. The following three statements for a subvariety germ S_0 at $0 \in \boldsymbol{C}^n$ are equivalent:

(1) Let $\mathscr{F}$ be a coherent analytic sheaf in a neighborhood X of 0 in $\boldsymbol{C}^n$, where S_0 has a representative S. If $\mathscr{F}|X-S$ is locally free, $\mathscr{F} = \mathscr{R}_S^0 \mathscr{F}$, and $\dim S + \operatorname{rank} \mathscr{F} \leqq n-2$, then $\mathscr{F}_0$ is free.

(2) Let $\mathscr{F}$ be a torsion-free coherent analytic sheaf in a neighborhood X of 0 in $\boldsymbol{C}^n$ where S_0 has a representative S. Let codh $\mathscr{F}_x \geqq n-1$ for any $x \in X-S$. If then codh $\mathscr{F}_0 < n-1$, we obtain for the minimal number $\mu = \mu(\mathscr{F}_0)$ of generators of $\mathscr{F}_0$ the estimate $\mu > (n-2) + \operatorname{rank} \mathscr{F} - \dim S$.

(3) Let M be a noetherian module over $\mathscr{O}_0$ without torsion and let $\mathfrak{A} \subset \mathscr{O}_0$ be the ideal of S_0. If, for every prime ideal $\mathfrak{P} \not\supset \mathfrak{A}$, codh $M_{\mathfrak{P}} \geqq n-1$ and codh $M < n-1$, then $\mu(M) > (n-2) + \operatorname{rank} M - \dim \mathfrak{A}$.

Proof. It follows by standard arguments that (2) and (3) are equivalent.

(1) $\Rightarrow$ (2). By shrinking X we may assume that $\mathscr{F}$ has a representation $0 \to \mathscr{K} \to \mathscr{O}^\mu \to \mathscr{F} \to 0$ and rank $\mathscr{K} = \mu - \operatorname{rank} \mathscr{F}$. Since $\mathscr{F}$ is torsion-free, we have $\mathscr{K} = \mathscr{R}_S^0 \mathscr{K}$. Also $\mathscr{K}|X-S$ is locally free. Since, however, codh $\mathscr{K}_0 = \operatorname{codh} \mathscr{F}_0 + 1 < n$, $\mathscr{K}_0$ is not free. Hence by (1) rank $\mathscr{K} + \dim S > n-2$ which gives the estimate.

(2) $\Rightarrow$ (1). Let $\mathscr{F}^*$ denote the sheaf $\mathit{Hom}_{\mathscr{O}}(\mathscr{F}, \mathscr{O})$. By a representation $\mathscr{O}^g \to \mathscr{F}^* \to 0$ we get an exact sequence $0 \to \mathscr{F}^{**} \to \mathscr{O}^g \to \mathscr{G} \to 0$ where $\mathscr{G}$ is without torsion. Now since $\mathscr{F}|X-S$ is locally free, we obtain $\mathscr{F} = \mathscr{R}_S^0 \mathscr{F} = \mathscr{R}_S^0 \mathscr{F}^{**}$. Since however $\mathscr{F}^{**}$ is reflexive, also $\mathscr{R}_S^0 \mathscr{F}^{**} = \mathscr{F}^{**}$. Hence we have the exact sequence $0 \to \mathscr{F} \to \mathscr{O}^g \to \mathscr{G} \to 0$. Now $\mu(\mathscr{G}_0) \leqq g$, and since rank $\mathscr{G} = g - \operatorname{rank} \mathscr{F}$ we get $\mu(\mathscr{G}_0) \leqq (n-2) + \operatorname{rank} \mathscr{G} - \dim S$ by the assumptions in (1). By (2) codh $\mathscr{G}_0 \geqq n-1$, and hence $\mathscr{F}_0$ is free.

2. Let $z_{11}, \ldots, z_{1n}; z_{21}, \ldots, z_{2n}; \ldots; z_{p1}, \ldots, z_{pn}$ be the pn coordinates in $\boldsymbol{C}^N$, $N = pn$ and let $p \leqq n$. Denote by $D_{\nu_1 \ldots \nu_p}$ the determinants of the matrix $(z_{i\nu_j})_{1 \leqq i,j \leqq p}$ where $1 \leqq \nu_1 < \cdots < \nu_p \leqq n$. Let further $\mathscr{D}_p(n)$ be the ideal sheaf on $\boldsymbol{C}^N$ generated by the holomorphic functions $D_{\nu_1 \ldots \nu_p}$. It is proved in [3] and [4] that codh $\mathscr{D}_p(n) \geqq N - (n-p)$ and that $S_p(n) = \operatorname{Supp}(\mathscr{O}/\mathscr{D}_p(n))$ is a perfect analytic subvariety of pure dimension $N - (n-p) - 1$.

2.1. It can be shown moreover that $\mathscr{D}_p(n)$ has a canonical resolution

$$(*) \qquad 0 \to \mathscr{O}^{g_{n-p}} \to \cdots \to \mathscr{O}^{g_k} \to \cdots \to \mathscr{O}^{g_1} \to \mathscr{O}^{g_0} \to \mathscr{D}_p(n) \to 0$$

with $g_k = \binom{p+k-1}{k}\binom{n}{p+k}$, which is some kind of a generalized Koszul-complex, see [2]. One can construct this resolution inductively and thereby obtain the result of Northcott. By this, one can also show that all the matrices representing the homomorphisms in this sequence are in terms of the coordinates z_{ij}.

2.2. Let now $(a_{ij})_{1\leqq i\leqq p;1\leqq j\leqq n}$ be a matrix of holomorphic functions in a domain X in $\mathbf{C}^M$. Let $\rho\colon X\to \mathbf{C}^N$ be the holomorphic mapping defined by $z_{ij}=a_{ij}$. If $A_{\nu_1\ldots\nu_p}$ denote the subdeterminants of order p of (a_{ij}) we have $A_{\nu_1\ldots\nu_p}=D_{\nu_1\ldots\nu_p}\circ\rho$. Then $\mathscr{A}=\rho^*\mathscr{D}_p(n)$ is the ideal sheaf generated by the functions $A_{\nu_1\ldots\nu_p}$. We prove now

2.3. Let S be the subvariety of $\mathscr{A}$ in X. If $\dim S\leqq M-(n-p)-1$ (this implies that (a_{ij}) has maximal rank p) then $\mathscr{A}$ has a canonical resolution

$$0\to \mathcal{O}_M^{g_{n-p}}\to\cdots\to\mathcal{O}_M^{g_k}\to\cdots\to\mathcal{O}_M^{g_0}\to\mathscr{A}\to 0$$

with $g_k=\binom{p+k-1}{k}\binom{n}{p+k}$. Especially S has pure dimension $M-(n-p)-1$ and is perfect.

Proof. Let $\mathcal{O}_M$ (resp. $\mathcal{O}_N$) denote the structure sheaves on X (resp. $\mathbf{C}^N$). Let $\mathscr{Z}^i=\operatorname{Im}(\mathcal{O}_N^{g_i}\to\mathcal{O}_N^{g_{i-1}})$ and $\mathscr{Z}^{n-p}=\mathcal{O}_N^{g_{n-p}}$, $\mathscr{Z}^0=\mathscr{D}_p(n)$. By 2.1 we have the exact sequences

$$0\to\mathscr{Z}^{i+1}\to\mathcal{O}_N^{g_i}\to\mathscr{Z}^i\to 0,\qquad 0\leqq i\leqq n-p-1.$$

We define $\mathscr{A}^i=\rho^*\mathscr{Z}^i\otimes_{\rho^*\mathcal{O}_N}\mathcal{O}_M$ where $\mathcal{O}_M$ is considered a $\rho^*\mathcal{O}_N$-module by the sheaf morphism $\rho^*\mathcal{O}_N\to\mathcal{O}_M$ induced by ρ. We obtain the exact sequences

$$\mathscr{A}^{i+1}\to\mathcal{O}_M^{g_i}\to\mathscr{A}^i\to 0$$

and

$$0\to\mathscr{A}^{i+1}|X-S\to\mathcal{O}_M^{g_i}|X-S\to\mathscr{A}^i|X-S\to 0$$

since $\rho(S)=S_p(n)$ and $S=\rho^{-1}(S_p(n))$ and since $\rho^*\mathscr{D}_p(n)_x=\mathscr{D}_\rho(n)_{\rho(x)}$ is a free $\rho^*(\mathcal{O}_N)_x=(\mathcal{O}_N)_{\rho(x)}$-module for $x\notin S$. From these we obtain the following diagrams:

$$\begin{array}{ccccccccc}
 & & \mathscr{A}^{i+1} & \longrightarrow & \mathcal{O}_M^{g_i} & \longrightarrow & \mathscr{A}^i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{R}_S^0\mathscr{A}^{i+1} & \longrightarrow & \mathscr{R}_S^0\mathcal{O}_M^{g_i} & \longrightarrow & \mathscr{R}_S^0\mathscr{A}^i & \longrightarrow & \mathscr{R}_S^1\mathscr{A}^{i+1} \longrightarrow \cdots.
\end{array}$$

Using the result 1.1 we can prove by induction that $\mathscr{R}_S^0\mathscr{A}^i=\mathscr{A}^i$ for $1\leqq i\leqq p-n$ and $\mathscr{R}_S^j\mathscr{A}^i=0$ for $1\leqq j<i$. Hence all the sequences $0\to\mathscr{A}^{i+1}\to\mathcal{O}_M^{g_i}\to\mathscr{A}^i\to 0$ are exact and 2.3 has been proved.

3. From 2.3 the main result follows now very easily.

3.1. Theorem. *Let X be a neighborhood of* 0 *in $\mathbf{C}^m$ and S a subvariety in X. Assume $\mathscr{F}$ is a coherent analytic sheaf on X having a representation* $0\to\mathcal{O}^p\xrightarrow[\alpha]{}\mathcal{O}^n\to\mathscr{F}\to 0$. *If $\mathscr{F}|X-S$ is locally free, $\mathscr{F}=\mathscr{R}_S^0\mathscr{F}$ and* $\dim_0 S+\operatorname{rank}\mathscr{F}\leqq m-2$, *then $\mathscr{F}_0$ is free.*

Proof. Let (a_{ij}) be the matrix by which α is determined. (a_{ij}) has maximal rank. Let $\mathscr{A}$ be as in §2. Then $\operatorname{Supp}(\mathcal{O}/\mathscr{A})\subset S$ since $\mathscr{F}|X-S$ is locally free, by a well-known fact. Hence $\dim\operatorname{Supp}(\mathcal{O}/\mathscr{A})\leqq m-2-(n-p)$ because $n-p=\operatorname{rank}\mathscr{F}$. By

2.3, codh $\mathscr{A} \geqq m-(n-p)$ and, by 1.1, $\mathscr{A}=\mathscr{H}_S^0\mathscr{A}=\mathscr{O}$ and hence Supp $(\mathscr{O}/\mathscr{A})=\varnothing$. Hence $\mathscr{F}_0$ is free.

3.2. Example. Let $\mathscr{T}$ be the ideal sheaf of the subvariety

$$A = \{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\}$$

in C^4 [5, p. 86]. For $x \in A-\{0\}$ we have codh $\mathscr{T}_x=3$ and, since $\mathscr{H}^0_{\{0\}}\mathscr{T}=\mathscr{T}$, codh $\mathscr{T}_0 \geqq 2$ by 1.1. Since however $\mathscr{H}^0_{\{0\}}(\mathscr{O}/\mathscr{T}) \neq \mathscr{O}/\mathscr{T}$, we have codh $\mathscr{T}_0=2$ also by 1.1. Hence

$$\begin{aligned} \operatorname{codh} \mathscr{T}_x &= 4, \qquad x \notin A, \\ &= 3, \qquad x \in A-\{0\}, \\ &= 2, \qquad x = 0. \end{aligned}$$

Let $\mu(\mathscr{T}_0)$ be the minimal number of generators of $\mathscr{T}_0$. By (1) $\Rightarrow$ (2) in 1.2, we obtain $\mu > (4-2)+1+0=3$ for $S=\{0\}$; on the other hand, z_1z_3, z_1z_4, z_2z_3, z_2z_4 are four generators of $\mathscr{T}_0$. Hence $\mu(\mathscr{T}_0)=4$.

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