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THE EMBEDDABILITY OF A SEMIGROUP—CONDITIONS COMMON TO MAL'CEV AND LAMBEK

BY
GEORGE C. BUSH

Abstract. Two systems of conditions—due to Mal'cev and to Lambek—are known to be necessary and sufficient for a semigroup to be embeddable in a group. This paper shows by means of an example that the conditions common to the two systems are not sufficient to guarantee embeddability.

1. **Introduction.** Both Mal'cev [5], [6] and Lambek [4] have developed necessary and sufficient systems of conditions for the embeddability of a cancellation semigroup in a group. All the conditions consist of a given system of formal equations and a "locked" equation involving some of the same elements. A semigroup satisfies the condition if, whenever elements of the semigroup satisfy all the given equations, the corresponding elements also satisfy the locked equation. The Mal'cev system and the Lambek system each involve an infinite number of conditions. The two systems are developed from different points of view.

The Lambek conditions are described by assigning letters to the sides and angles of an Eulerian polyhedron. Clifford and Preston [2, pp. 319–323] have shown that the conditions that are common to the systems of Mal'cev and Lambek are just those Lambek conditions that are associated with a two-vertex polyhedron. Jackson [3] had previously begun a study of these conditions which his thesis supervisor, I. Halperin, had called the *lunar conditions*, from the appearance of the polyhedron. Jackson had considered the interdependence of the lunar conditions and had developed a line of approach which the author [1] used in his thesis to show that satisfying the set of all the lunar conditions was not sufficient to ensure the embeddability of a semigroup. That part of the thesis was never published. The purpose of this paper is to make available a slightly modified form of these results which take on new interest in the light of Clifford and Preston's work.

2. **The Lambek conditions.** We assign letters to all the sides and all the angles of an Eulerian polyhedron. We shall use capital letters here and hence in the formal statement of Lambek conditions. A typical edge and its corresponding angles is shown in Figure 1. From this edge we form two half-edge equations

$$XA = YB, \qquad XC = YD.$$

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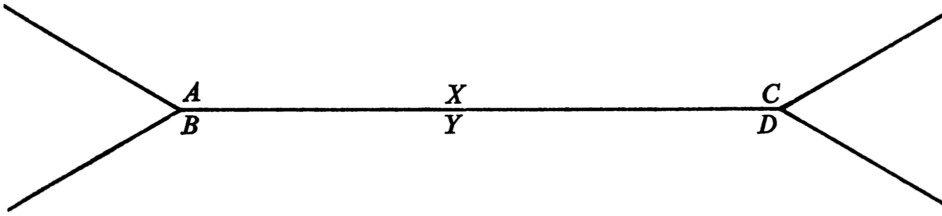

FIGURE 1

To get a Lambek condition corresponding to a given polyhedron we designate one of the half-edge equations as the locked equation and take the remaining ones for the given equations.

In particular, consider the polyhedron represented in Figure 2. From it we obtain a Lambek condition whose given equations are

$$
\begin{aligned}
&X_1A_1 = Y_1B_1, \quad && X_1A_2 = Y_1B_2,\\
&X_2A_2 = Y_2C_2, && X_2A_3 = Y_2C_3,\\
&X_3A_3 = Y_3B_3, && X_3A_1 = Y_3C_1,\\
&X_4B_2 = Y_4C_2, && X_4A_4 = Y_4C_4,\\
&X_5B_1 = Y_5C_1, && X_5A_4 = Y_5B_4,\\
&X_6B_4 = Y_6C_4, &&
\end{aligned} \tag{1}
$$

and whose locked equation is

$$X_6B_3 = Y_6C_3. \tag{2}$$

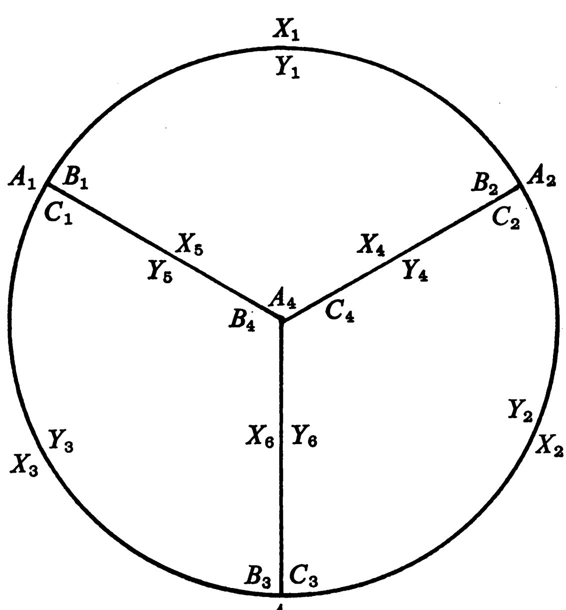

FIGURE 2

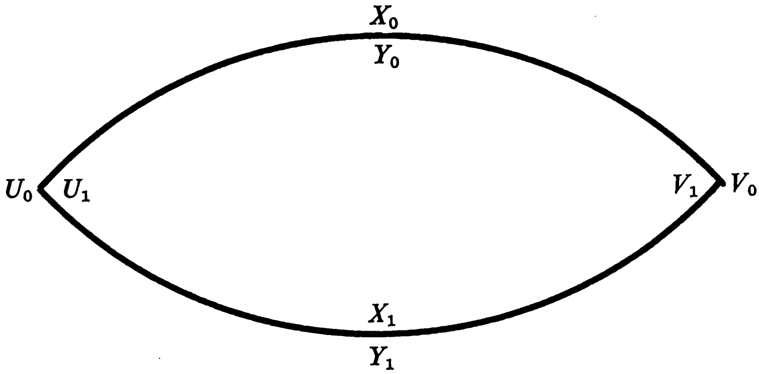

FIGURE 3

To obtain the lunar conditions we consider a polyhedron with 2 vertices and $n+1 \geqq 2$ edges. Because of the symmetry it makes no difference which of the half-edge equations we choose for the locked equation.

The simplest lunar condition can be read from Figure 3. Its given equations are

$$X_0U_0 = Y_0U_1, \qquad X_0V_0 = Y_0V_1, \\ X_1U_1 = Y_1U_0, \tag{3}$$

and its locked equation is

$$X_1V_1 = Y_1V_0. \tag{4}$$

We call this condition C_1. The general lunar condition C_n is suggested by Figure 4. Its given equations are

$$X_iU_i = Y_iU_{i+1}, \qquad X_iV_i = Y_iV_{i+1}, \quad i = 0, 1, \ldots, n-1, \\ X_nU_n = Y_nU_0, \tag{5}$$

and its locked equation is

$$X_nV_n = Y_nV_0. \tag{6}$$

Our goal is to construct a cancellation semigroup that satisfies all the lunar conditions but does not satisfy the Lambek condition (1), (2).

3. **A nonembeddable semigroup.** In this section we adapt a construction of Jackson [3] to obtain a cancellation semigroup that does not satisfy the Lambek condition (1), (2).

Consider the letters

$$x_i, y_i, \qquad i = 1, 2, \ldots, 6, \\ a_j, b_j, c_j \qquad j = 1, 2, 3, 4, \tag{7}$$

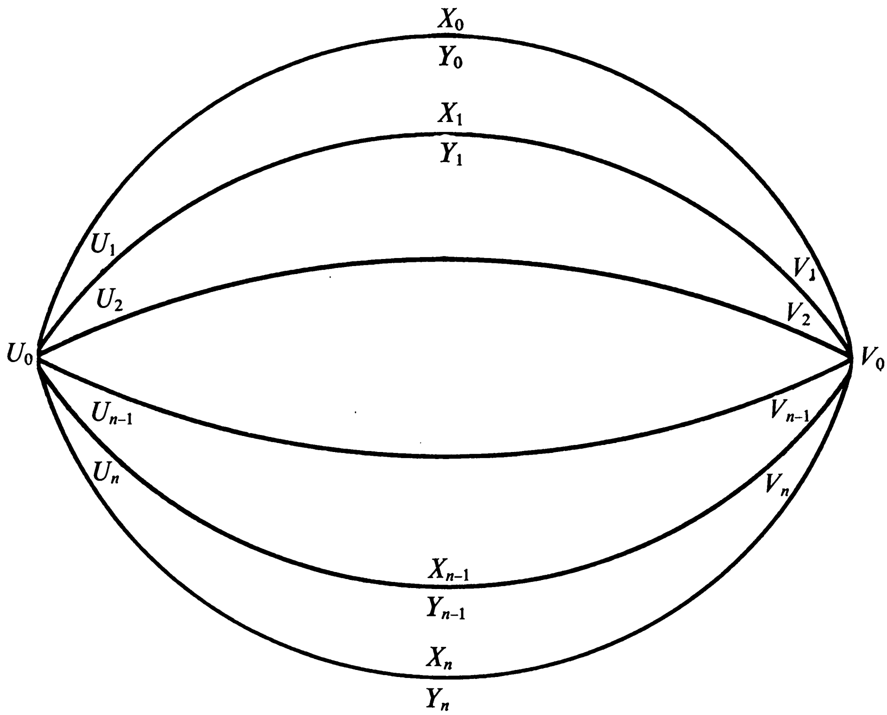

FIGURE 4

and words constructed from these letters. The following pairs of two-letter words are called *corresponding*:

$$\begin{array}{llllll} x_1a_1 & \text{and} & y_1b_1, & x_1a_2 & \text{and} & y_1b_2, \\ x_2a_2 & \text{and} & y_2c_2, & x_2a_3 & \text{and} & y_2c_3, \\ x_3a_3 & \text{and} & y_3b_3, & x_3a_1 & \text{and} & y_3c_1, \\ x_4b_2 & \text{and} & y_4c_2, & x_4a_4 & \text{and} & y_4c_4, \\ x_5b_1 & \text{and} & y_5c_1, & x_5a_4 & \text{and} & y_5b_4, \\ x_6b_4 & \text{and} & y_6c_4. & & & \end{array} \tag{8}$$

The relationship between these pairs and the Lambek condition (1), (2) is obvious.

We shall usually employ Greek letters to represent words formed from the letters (7). If $\alpha = \cdots mn \cdots$ is a word in which mn is one member of a corresponding pair in (8), the word formed by replacing mn by the other member of the corresponding pair is said to be obtained from α by an elementary transformation. We can use elementary transformations to define an equivalence relation on the set of all finite words from (7). We say that α and β are equivalent ($\alpha \sim \beta$) if β can be obtained from α by a finite number (possibly 0) of elementary transformations.

We denote by α' the equivalence class containing the word α. We shall define a binary operation under which the quotient set forms a cancellation semigroup. The product $\alpha\beta$ of words α and β is formed by juxtaposition—the letters of α are written followed by the letters of β. If $\alpha \sim \beta$ and $\gamma \sim \delta$ then $\alpha\gamma \sim \beta\delta$. This allows us to define a binary operation

$$\alpha'\beta' = (\alpha\beta)' \tag{9}$$

which is clearly associative.

Before proving the cancellation laws we examine some important properties of the corresponding pairs (8). If we call the letters x_i and y_i L-letters and the letters a_j, b_j, c_j R-letters, we see that in the corresponding pairs no L-letter appears as a right-hand factor and no R-letter appears as a left-hand factor. We also observe that an elementary transformation cannot change the number of letters in a word. We call the number of letters in a word α the *length* of the word and denote it by $\|\alpha\|$. Similarly $\|\alpha'\|$ denotes the length of the class α', that is the common length of all the words in α'.

We see from the above remarks that the kth letter in all the words of a class are either all L-letters or all R-letters. This means that if $\alpha \sim \beta$ we can partition α into pairs of letters forming words in (8) and single letters that remain unchanged in the passage from α to β. We can then operate on the pairs independently to reach β from α.

These considerations enable us to prove the cancellation law.

LEMMA 1. *If $\alpha\gamma \sim \beta\gamma$ or $\gamma\alpha \sim \gamma\beta$ then $\alpha \sim \beta$.*

Proof. Suppose

$$\alpha\gamma \sim \beta\gamma. \tag{10}$$

We have just seen that $\|\alpha\gamma\| = \|\beta\gamma\|$ and hence $\|\alpha\| = \|\beta\|$. Thus (10) can be written as

$$\alpha_1\alpha_2\cdots\alpha_m\gamma_1\gamma_2\cdots\gamma_n \sim \beta_1\beta_2\cdots\beta_m\gamma_1\gamma_2\cdots\gamma_n \tag{11}$$

where $\alpha_i, \beta_i, \gamma_i$ represent the ith letter in α, β, γ respectively.

If $\alpha_m\gamma_1$ is not a member of a corresponding pair in (8), then each elementary transformation acts only on α and its successors or only on γ and its successors. If we consider only the elementary transformations that act on α and its successors we get $\alpha \sim \beta$.

If $\alpha_m\gamma_1$ is a member of a corresponding pair, then $\alpha_m\gamma_1 \sim \beta_m\gamma_1$ and an inspection of (8) shows that this means $\alpha_m = \beta_m$. Then $\alpha_{m-1}\alpha_m$ is not a member of a corresponding pair and as before we have $\alpha_1\alpha_2\cdots\alpha_{m-1} \sim \beta_1\beta_2\cdots\beta_{m-1}$ and hence $\alpha \sim \beta$.

This proves the right cancellation law. The left cancellation law is proved similarly.

COROLLARY 2. *If $\alpha'\gamma' = \beta'\gamma'$ or $\gamma'\alpha' = \gamma'\beta'$ then $\alpha' = \beta'$.*

We have now shown that the quotient set forms a cancellation semigroup which we denote by $\mathfrak{S}$. If we include the empty word in our consideration, the class of the empty word is an identity for $\mathfrak{S}$.

PROPOSITION 3. *The semigroup $\mathfrak{S}$ cannot be embedded in a group.*

Proof. If we write x_i' for the equivalence class containing the word with one letter x_i, we see that the classes $x_i', y_i', i=1, 2, \ldots, 6; a_j', b_j', c_j', j=1, 2, 3, 4,$ are elements of $\mathfrak{S}$ that satisfy equations (1). But an examination of (8) shows that equation (2) is not satisfied. Thus the Lambek condition is not satisfied and $\mathfrak{S}$ is not embeddable.

4. **Satisfying the lunar conditions.** Suppose that α_i, β_i, γ_i, δ_i are words formed from the letters (7) such that

$$(12) \qquad \begin{aligned} &\alpha_i'\gamma_i' = \beta_i'\gamma_{i+1}', \qquad \alpha_i'\delta_i' = \beta_i'\delta_{i+1}', \qquad i = 0, 1, \ldots, n-1, \\ &\alpha_n'\gamma_n' = \beta_n'\gamma_0'. \end{aligned}$$

Our goal is to prove that in this case we must have

$$(13) \qquad \alpha_n'\delta_n' = \beta_n'\delta_0'.$$

We begin by considering the possibility of cancelling on the left in equations (12). If $\alpha_i\gamma_i$ begins with an LL-pair, an RR-pair or an RL-pair, then $\beta_i\gamma_{i+1}$ begins with the same letter and it can be cancelled. If $\alpha_i\gamma_i$ begins with an LR-pair, then $\beta_i\gamma_{i+1}$ begins with the same pair or a corresponding pair and the pair can be cancelled. Similar considerations apply to the equations involving the δ_i. We shall assume that the equations in (12) have been simplified by carrying out all cancellation that is possible on the left without violating the following conditions:

(a) If $i=0, 1, \ldots, n-1$, the same cancellation is carried out on the equations involving γ_i and δ_i.

(b) No cancellation is permitted to remove a letter from any γ_i or δ_i.

If we can obtain the simplified form of (13) from the simplified form of (12), we can multiply on the left by the cancelled factors to obtain the original (13). Thus there is no loss of generality in assuming that in system (12) all the above cancellation has been performed. Under this assumption we see that for each $i=0, 1, \ldots, n$, either α_i or β_i is either empty or consists of a single L-letter.

Our proof involves the consideration of three cases.

1. $\|\alpha_i'\|=0$ or $\|\beta_i'\|=0$ for some i.
2. $\|\alpha_i'\|=\|\beta_i'\|=1$ for all i.
3. For all i, $\|\alpha_i'\|>0$, $\|\beta_i'\|>0$; and for some j, $\|\alpha_j'\|\neq\|\beta_j'\|$.

The proof in Cases 1 and 2 is by induction. We show that C_1 is satisfied and that C_n is implied by C_{n-1}. For Case 3 we show that further cancellation is possible to put it into a form where the results of Case 1 apply.

For Case 2 it is convenient to carry out cancellation on the right as well as on the left. But for Case 3 it is more advantageous to multiply on the right to ensure that no γ_i or δ_i is empty. We shall describe these techniques when we require them.

Case 1. $\|\alpha_i'\|=0$ or $\|\beta_i'\|=0$ for some i.

We observe that equations (12) can be rewritten in the form

$$\beta'_{n-i}\gamma'_{n+1-i} = \alpha'_{n-i}\gamma'_{n-i}, \qquad \beta'_{n-i}\delta'_{n+1-i} = \alpha'_{n-i}\delta'_{n-i}, \qquad i = 1, 2, \ldots, n,$$
$$\beta'_n\gamma'_0 = \alpha'_n\gamma'_n.$$

From these the required conclusion is $\beta'_n\delta'_0 = \alpha'_n\delta'_n$ which is the same as (13). From this we see that we do not have to treat α'_i and β'_i separately in the proof.

If $\|\alpha'_i\| = 0$ for $i \neq 0$, n then we have among the equations in (12)

$$\alpha'_{i-1}\gamma'_{i-1} = \beta'_{i-1}\gamma'_i, \qquad \alpha'_{i-1}\delta'_{i-1} = \beta'_{i-1}\delta'_i,$$
$$\gamma'_i = \beta'_i\gamma'_{i+1}, \qquad \delta'_i = \beta'_i\delta'_{i+1},$$

and these reduce to

$$\alpha'_{i-1}\gamma'_{i-1} = \beta'_{i-1}\beta'_i\gamma'_{i+1}, \qquad \alpha'_{i-1}\delta'_{i-1} = \beta'_{i-1}\beta'_i\delta'_{i+1}.$$

But this reduces (12) to the "given" equations for lunar condition C_{n-1}. Thus if C_{n-1} is satisfied we have $\alpha'_n\delta'_n = \beta'_n\delta'_0$ and so C_n is satisfied.

Minor modifications are needed if $i=0$ or $i=n$.

If $\|\alpha'_0\| = 0$, (12) includes

$$\gamma'_0 = \beta'_0\gamma'_1, \qquad \delta'_0 = \beta'_0\delta'_1,$$
$$\alpha'_n\gamma'_n = \beta'_n\gamma'_0,$$

and so reduces to

$$\alpha'_1\gamma'_1 = \beta'_1\gamma'_2, \qquad \alpha'_1\delta'_1 = \beta'_1\delta'_2,$$
$$\cdots$$
$$\alpha'_n\gamma'_n = \beta'_n\beta'_0\gamma'_1.$$

Then C_{n-1} implies that $\alpha'_n\delta'_n = \beta'_n\beta'_0\delta'_1$. But $\beta'_0\delta'_1 = \delta'_0$, and so we have $\alpha'_n\delta'_n = \beta'_n\delta'_0$, the required conclusion for C_n.

If $\|\alpha'_n\| = 0$, (12) includes

$$\alpha'_{n-1}\gamma'_{n-1} = \beta'_{n-1}\gamma'_n, \qquad \alpha'_{n-1}\delta'_{n-1} = \beta'_{n-1}\delta'_n,$$
$$\gamma'_n = \beta'_n\gamma'_0.$$

If we replace the first of these by $\alpha'_{n-1}\gamma'_{n-1} = \beta'_{n-1}\beta'_n\gamma'_0$ and ignore the other two, then C_{n-1} implies that $\alpha'_{n-1}\delta'_{n-1} = \beta'_{n-1}\beta'_n\delta'_0$. But we have $\alpha'_{n-1}\delta'_{n-1} = \beta'_{n-1}\delta'_n$ and so by cancellation we have $\delta'_n = \beta'_n\delta'_0$, the required conclusion for C_n.

Thus in Case 1, the lunar condition C_{n-1} implies the lunar condition C_n.

It remains to verify that C_1 is satisfied. The two cases $\|\alpha'_0\| = 0$ and $\|\alpha'_1\| = 0$ are similar to the cases $\|\alpha'_0\| = 0$ and $\|\alpha'_n\| = 0$ above. We omit the details.

Case 2. $\|\alpha'_i\| = \|\beta'_i\| = 1$ for all i.

In this case it is convenient to cancel on the right as well as on the left. Similar considerations apply. For all the discussion of Case 2 we shall assume that all possible cancellation on the right has been performed, subject to the restriction that the same cancellation is applied to δ_i each time it appears and similarly for γ_i, and furthermore no cancellation on the right is allowed to modify any α_i or β_i.

For each i we must have $\alpha_i \neq \beta_i$ or else we would have cancelled again on the left to get Case 1. This means that for any choice of representative words for the classes appearing in any equation in (12) the first two letters from each side form a corresponding pair of words. Furthermore this means that all the rest of the word in each case can be removed by cancellation on the right—cancellation that satisfies the restrictions above. We are now dealing with the case where $\|\alpha_i\| = \|\beta_i\| = \|\gamma_i\| = \|\delta_i\| = 1$ for all i.

If for some i we have $\gamma_i = \delta_i$ then equations (12) include

$$\alpha'_{i-1}\gamma'_{i-1} = \beta'_{i-1}\delta'_i, \qquad \alpha'_{i-1}\delta'_{i-1} = \beta'_{i-1}\delta'_i,$$

from which we conclude $\gamma_{i-1} = \delta_{i-1}$. They also include

$$\alpha'_i\delta'_i = \beta'_i\gamma'_{i+1}, \qquad \alpha'_i\delta'_i = \beta'_i\delta'_{i+1},$$

from which we conclude that $\gamma_{i+1} = \delta_{i+1}$. In this way we get $\gamma_i = \delta_i$ for all i and hence $\alpha'_n\gamma'_n = \beta'_n\gamma'_0$ in (12) is the same as the required conclusion $\alpha'_n\delta'_n = \beta'_n\delta'_0$. Thus C_n is satisfied if some $\gamma_i = \delta_i$.

In the case in which $\gamma_i \neq \delta_i$ for all i we consider the equations

$$\begin{aligned} \alpha'_0\gamma'_0 &= \beta'_0\gamma'_1, & \alpha'_0\delta'_0 &= \beta'_0\delta'_1, \\ \alpha'_1\gamma'_1 &= \beta'_1\gamma'_2, & \alpha'_1\delta'_1 &= \beta'_1\delta'_2, \end{aligned}$$

from (12). Representative words from the two sides of any of these equations form a corresponding pair. The structure of the set of corresponding pairs shows that if β_0 is determined, then γ_1 and δ_1 are determined except for their order. Also an inspection of the corresponding pairs shows that $\alpha_1 = \beta_0$. Then (12) includes the equations

$$\begin{aligned} \alpha'_0\gamma'_0 &= \beta'_0\gamma'_1, & \alpha'_0\delta'_0 &= \beta'_0\delta'_1, \\ \beta'_0\gamma'_1 &= \beta'_1\gamma'_2, & \beta'_0\delta'_1 &= \beta'_1\delta'_2, \end{aligned}$$

which reduce to

$$\alpha'_0\gamma'_0 = \beta'_1\gamma'_2, \qquad \alpha'_0\delta'_0 = \beta'_1\delta'_2.$$

The equations (12) are now in a form to which C_{n-1} may be applied to draw the required conclusion $\alpha'_n\delta'_n = \beta'_n\delta'_0$. Thus we see that in Case 2, condition C_{n-1} implies condition C_n.

For the case C_1 the equations (12) can be reduced as above to

$$\begin{aligned} \alpha'_0\gamma'_0 &= \beta'_0\gamma'_1, & \alpha'_0\delta'_0 &= \beta'_0\delta'_1, \\ \alpha'_1\gamma'_1 &= \beta'_1\gamma'_0. \end{aligned}$$

An inspection of (8) shows that a pair of R-elements γ_0, γ_1 can be matched with only one pair of L-elements so that $\alpha_1 = \beta_0$ and $\beta_1 = \alpha_0$. Thus from $\alpha'_0\delta'_0 = \beta'_0\delta'_1$ we obtain $\alpha'_1\delta'_1 = \beta'_1\delta'_0$, the required conclusion for C_1. This completes the proof of Case 2.

Case 3. For all i, $\|\alpha_i'\|>0$, $\|\beta_i'\|>0$ and for some j, $\|\alpha_j'\|\neq\|\beta_j'\|$.

In this case it is convenient to have $\|\gamma_i\|\geqq 2$, $\|\delta_i\|\geqq 2$ for all $i=0, 1, \ldots, n$. If this is not true we multiply each equation in (12) on the right by σ' where $\|\sigma'\|\geqq 2$. If it follows from the resulting equations that $\alpha_n'\delta_n'\sigma'=\beta_n'\delta_0'\sigma'$, then by cancellation we have $\alpha_n'\delta_n'=\beta_n'\delta_0'$ as required. This means that we can without loss of generality assume that $\|\gamma_i\|\geqq 2$ and $\|\delta_i\|\geqq 2$ for each i.

It simplifies the notation in our proof for this case if we adopt the convention that subscripts are read modulo $n+1$, i.e. $\alpha_{n+1}=\alpha_0$, $\beta_{n+3}=\beta_2$, etc.

LEMMA 4. *There are integers j and k such that* $\|\alpha_j\|>\|\beta_j\|=1$, $\|\alpha_{j+l}\|=\|\beta_{j+l}\|=1$, $0<l<k$, $1=\|\alpha_{j+k}\|<\|\beta_{j+k}\|$.

Proof. We know that for all i, $\|\alpha_i\gamma_i\|=\|\beta_i\gamma_{i+1}\|$; that is,

$$\|\alpha_i\|+\|\gamma_i\| = \|\beta_i\|+\|\gamma_{i+1}\|.$$

If $\|\alpha_i\|\geqq\|\beta_i\|$ for all i, then $\|\gamma_i\|\leqq\|\gamma_{i+1}\|$. If also $\|\alpha_j\|>\|\beta_j\|$ for some j, then $\|\gamma_j\|<\|\gamma_{j+1}\|$. Then $\|\gamma_0\|\leqq\cdots\leqq\|\gamma_j\|<\|\gamma_{j+1}\|\leqq\cdots\leqq\|\gamma_{n+1}\|=\|\gamma_0\|$. This is a contradiction, and hence if $\|\alpha_i\|\neq\|\beta_i\|$ for some i, then we have $\|\alpha_j\|>\|\beta_j\|$ for some j and $\|\alpha_m\|<\|\beta_m\|$ for some m.

Let j' be the smallest nonnegative integer such that $\|\alpha_{j'}\|>\|\beta_{j'}\|$. Let k' be the smallest positive integer such that $\|\alpha_{j'+k'}\|\neq\|\beta_{j'+k'}\|$. If $\|\alpha_{j'+k'}\|<\|\beta_{j'+k'}\|$ we take $j=j'$ and $k=k'$. Otherwise we begin again, calling $\alpha_{j'+k'}$ a new $\alpha_{j'}$. We have shown that we cannot always have $\|\alpha_i\|\geqq\|\beta_i\|$ and so we eventually get all our required results except for $\|\beta_j\|=1$ and $\|\alpha_{j+k}\|=1$. But this follows immediately from the result of cancellation in equations (12).

Consider the following portion of equations (12) with j and k chosen as in Lemma 4:

$$(14)\qquad \begin{array}{rclrcl} \alpha_j'\gamma_j' &=& \beta_j'\gamma_{j+1}', & \alpha_j'\delta_j' &=& \beta_j'\delta_{j+1}', \\ \alpha_{j+1}'\gamma_{j+1}' &=& \beta_{j+1}'\gamma_{j+2}', & \alpha_{j+1}'\delta_{j+1}' &=& \beta_{j+1}'\delta_{j+2}', \\ & & & \cdots & & \\ \alpha_{j+k-1}'\gamma_{j+k-1}' &=& \beta_{j+k-1}'\gamma_{j+k}', & \alpha_{j+k-1}'\delta_{j+k-1}' &=& \beta_{j+k-1}'\delta_{j+k}', \\ \alpha_{j+k}'\gamma_{j+k}' &=& \beta_{j+k}'\gamma_{j+k+1}', & \alpha_{j+k}'\delta_{j+k}' &=& \beta_{j+k}'\delta_{j+k+1}'. \end{array}$$

The two sides of each equation in (14) must be represented by words beginning with a corresponding pair from (8), for otherwise a further cancellation would have been carried out. We now cancel these corresponding pairs even though this violates our earlier restriction on the extent of cancellation on the left. This new cancellation completely removes $\alpha_{j+1}, \ldots, \alpha_{j+k}$ and $\beta_j, \beta_{j+1}, \ldots, \beta_{j+k-1}$. It also removes the first letter from $\gamma_{j+1}, \ldots, \gamma_{j+k}$ and from $\delta_{j+1}, \ldots, \delta_{j+k}$.

We now introduce new notation to facilitate consideration of letters within words. We shall write $\gamma_{i,j}$ and $\delta_{i,j}$ for the jth letter in γ_i and δ_i respectively.

We next prove an important property of the $\gamma_{i,1}$ and $\delta_{i,1}$.

LEMMA 5. *With j and k as in Lemma 4 we have $\gamma_{i,1}=\delta_{i,1}$ for $i\equiv j+1, j+2, \ldots, j+k$.*

Proof. If $j<n$ we consider the equations

$$\alpha_j'\gamma_j' = \beta_j'\gamma_{j+1}', \qquad \alpha_j'\delta_j' = \beta_j'\delta_{j+1}'.$$

Since $\|\alpha_j\| > \|\beta_j\| = 1$ we see that $\alpha_{j,1}\alpha_{j,2}$ and $\beta_j\gamma_{j+1,1}$ are distinct but form a corresponding pair. The same is true for $\alpha_{j,1}\alpha_{j,2}$ and $\beta_j\delta_{j+1,1}$. Then an examination of the corresponding pairs (8) shows that $\gamma_{j+1,1}=\delta_{j+1,1}$.

If $k=1$ we are through. If $k>1$ and $j+1<n$ we consider

$$\alpha_{j+1}'\gamma_{j+1}' = \beta_{j+1}'\gamma_{j+2}', \qquad \alpha_{j+1}'\delta_{j+1}' = \beta_{j+1}'\delta_{j+2}'.$$

Since $\|\alpha_{j+1}\| = \|\beta_{j+1}\| = 1$ we have the corresponding pairs

$$\alpha_{j+1}\gamma_{j+1,1} \text{ and } \beta_{j+1}\gamma_{j+2,1}, \qquad \alpha_{j+1}\delta_{j+1,1} \text{ and } \beta_{j+1}\delta_{j+2,1}.$$

But $\gamma_{j+1,1}=\delta_{j+1,1}$ and so $\gamma_{j+2,1}=\delta_{j+2,1}$.

If $j+k\leqq n$ we can continue this process up to $\gamma_{j+k,1}=\delta_{j+k,1}$ and the proof is complete.

If $j+k>n$, we continue the above process up to $\gamma_{n,1}=\delta_{n,1}$. We then begin again with the equations

$$\alpha_{j+k}'\gamma_{j+k}' = \beta_{j+k}'\gamma_{j+k+1}', \qquad \alpha_{j+k}'\delta_{j+k}' = \beta_{j+k}'\delta_{j+k+1}'.$$

Here $\|\beta_{j+k}\| > \|\alpha_{j+k}\| = 1$ and arguments similar to the above show that $\gamma_{j+k,1} = \delta_{j+k,1}, \gamma_{j+k-1,1}=\delta_{j+k-1,1}, \ldots, \gamma_{0,1}=\delta_{0,1}$. This completes the proof of the lemma.

When we have carried out this latest cancellation we have put (12) into a form where α_{j+1} has been completely removed but the general structure of (12) has been preserved. This means that we have put (12) into a form in which the results of our Case 1 (some $\|\alpha_i\|=0$) apply. But the conclusions we draw may not be immediately in the required form if α_0 or α_n have been involved in the latest cancellation.

The various possibilities are summarized in Table 1 where we have included some information about corresponding pairs involved in the latest cancellation and some results from Lemma 5. These columns are by no means complete; we have included only those facts that are needed in our later calculations. We write $\gamma_i^{(1)}$ to denote the result of deleting $\gamma_{i,1}$ from γ_i, and $\gamma_i^{(1)\prime}$ to denote the class containing $\gamma_i^{(1)}$. Similarly $\gamma_i^{(2)}$ is the word obtained from γ_i by deleting $\gamma_{i,1}\gamma_{i,2}$ and $\gamma_i^{(2)\prime}$ is its class.

We observe that in cases (i) and (ii) of Table 1 the required conclusion for C_n follows immediately. In cases (iii) and (iv) we have

$$\alpha_n'\delta_n' = \alpha_n'\delta_{n,1}'\delta_n^{(1)\prime} = \alpha_n'\gamma_{n,1}'\delta_n^{(1)\prime} = \beta_{n,1}'\beta_{n,2}'\beta_n^{(2)\prime}\delta_0' = \beta_n'\delta_0'$$

as required for C_n. Cases (v) and (vi) lead to

$$\begin{aligned}\alpha_n'\delta_n' &= \alpha_n'\delta_{n,1}'\delta_n^{(1)\prime} = \alpha_n'\gamma_{n,1}'\delta_n^{(1)\prime}\\ &= \beta_n'\gamma_{0,1}'\delta_0^{(1)\prime} = \beta_n'\delta_{0,1}'\delta_0^{(1)\prime} = \beta_n'\delta_0'\end{aligned}$$

and cases (vii) and (viii) to

$$\alpha'_n\delta'_n = \alpha'_{n,1}\alpha'_{n,2}\alpha_n^{(2)'}\delta'_n = \beta'_n\gamma'_{0,1}\delta_0^{(1)'} = \beta'_n\delta'_{0,1}\delta_0^{(1)'} = \beta'_n\delta'_0,$$

the required conclusion for C_n.

TABLE 1

	Conditions on j and k	Cancelled Form of Equations (12)	Conclusion from Case 1	Corresponding Pairs Cancelled	Results from Lemma 5
(i)	$0 < j < n$ $j+k < n$	$\alpha'_0\gamma'_0 = \beta'_0\gamma'_1$ $\alpha'_0\delta'_0 = \beta'_0\delta'_1$ $\cdots$ $\alpha'_n\gamma'_n = \beta'_n\gamma'_0$	$\alpha'_n\delta'_n = \beta'_n\delta'_0$		
(ii)	$j = 0$ $j+k < n$	$\alpha_0^{(2)'}\gamma'_0 = \gamma_1^{(1)'}$ $\alpha_0^{(2)'}\delta'_0 = \delta_1^{(1)'}$ $\cdots$ $\alpha'_n\gamma'_n = \beta'_n\gamma'_0$			
(iii)	$0 < j < n$ $j+k = n$	$\alpha'_0\gamma'_0 = \beta'_0\gamma'_1$ $\alpha'_0\delta'_0 = \beta'_0\delta'_1$ $\cdots$ $\gamma_n^{(1)'} = \beta_n^{(2)'}\gamma'_0$	$\delta_n^{(1)'} = \beta_n^{(2)'}\delta'_0$	$\alpha'_n\gamma'_{n,1} = \beta'_{n,1}\beta'_{n,2}$	$\gamma_{n,1} = \delta_{n,1}$
(iv)	$j = 0$ $j+k = n$	$\alpha_0^{(2)'}\gamma'_0 = \gamma_1^{(1)'}$ $\alpha_0^{(2)'}\delta'_0 = \delta_1^{(1)'}$ $\cdots$ $\gamma_n^{(1)'} = \beta_n^{(2)'}\gamma'_0$			
(v)	$0 < j < n$ $j+k = n+1$	$\gamma_0^{(1)'} = \beta_0^{(2)'}\gamma'_1$ $\delta_0^{(1)'} = \beta_0^{(2)'}\delta'_1$ $\cdots$ $\gamma_n^{(1)'} = \gamma_0^{(1)'}$	$\delta'_n = \delta'_0$	$\alpha'_n\gamma'_{n,1} = \beta'_n\gamma'_{0,1}$	$\gamma_{n,1} = \delta_{n,1}$ $\gamma_{0,1} = \delta_{0,1}$
(vi)	$0 < j < n$ $j+k > n+1$	$\gamma_0^{(1)'} = \gamma_1^{(1)'}$ $\delta_0^{(1)'} = \delta_1^{(1)'}$ $\cdots$ $\gamma_n^{(1)'} = \gamma_0^{(1)'}$			
(vii)	$j = n$ $j+k = n+1$	$\gamma_0^{(1)'} = \beta_0^{(2)'}\gamma'_1$ $\delta_0^{(1)'} = \beta_0^{(2)'}\delta'_1$ $\cdots$ $\alpha_n^{(2)'}\gamma'_n = \gamma_0^{(1)'}$	$\alpha_n^{(2)'}\delta'_n = \delta_0^{(1)'}$	$\alpha'_{n,1}\alpha'_{n,2} = \beta'_n\gamma'_{0,1}$	$\gamma_{0,1} = \delta_{0,1}$
(viii)	$j = n$ $j+k > n+1$	$\gamma_0^{(1)'} = \gamma_1^{(1)'}$ $\delta_0^{(1)'} = \delta_1^{(1)'}$ $\cdots$ $\alpha_n^{(2)'}\gamma'_n = \gamma_0^{(1)'}$			

Thus in each of the subcases of Case 3, the lunar condition C_n is satisfied.

The lunar condition C_1 does not require any special consideration. It is obtained from C_1 for Case 1 as in cases (iv) and (vii) of Table 1.

We have now proved that C_1 holds in each of Cases 1, 2 and 3. We have also proved that if C_n holds then C_{n+1} also holds in each of these cases. Thus we have completed the proof of the following proposition.

PROPOSITION 6. *The semigroup $\mathfrak{S}$ satisfies all of the lunar conditions.*

If we compare Propositions 3 and 6 we see that we have reached our goal.

THEOREM 7. *The lunar conditions are not sufficient to ensure the embeddability of a semigroup in a group.*

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BEIRUT COLLEGE FOR WOMEN,
BEIRUT, LEBANON