

INTEGRAL REPRESENTATIONS FOR CONTINUOUS LINEAR OPERATORS IN THE SETTING OF CONVEX TOPOLOGICAL VECTOR SPACES

BY

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Abstract. Suppose X and Y are locally convex Hausdorff spaces, H is arbitrary and Σ is a ring of subsets of H . The authors prove the analog of the theorem stated in [Abstract 672-372, Notices Amer. Math. Soc. 17 (1970), 188] in this setting. A theory of extended integration on function spaces with Lebesgue and non-Lebesgue type convex topologies is then developed. As applications, integral representations for continuous transformations into Y for the following function spaces F (which have domain H and range X) are obtained: (1) H and Σ are arbitrary, τ is a convex topology on the simple functions over Σ , K is a set function on Σ with values in $L[X, Y]$, and F is the Lebesgue-type space generated by K ; (2) H is a normal space and F is the space of continuous functions each of whose range is totally bounded, with the topology of uniform convergence; (3) H is a locally compact Hausdorff space, F is the space of continuous functions of compact support with the topology of uniform convergence; (4) H is a locally compact Hausdorff space and F is the space of continuous functions with the topology of uniform convergence on compact subsets. In the above X and Y may be replaced by topological Hausdorff spaces under certain additional compensating requirements.

1. Introduction. Let H be a compact Hausdorff space, and let X and Y be locally convex topological vector spaces over the real or complex field where Y is Hausdorff. Let $C(H, X)$ be the space of continuous functions from H into X with the topology of uniform convergence. R. K. Goodrich [6] has given an integral representation theorem for a continuous linear transformation T from $C(H, X)$ into Y . In this paper we extend the aforementioned theorem to the setting where H is an arbitrary space and $C(H, X)$ is replaced by a function space F which is a subset of the closure of the totally bounded functions from H into X under a locally convex topology τ not stronger than the topology of uniform convergence and satisfying certain extension properties relative to simple functions. The operator valued measures which arise in connection with these integral representations are used to generate Lebesgue-type topologies on F and then a representation theorem is given for transformations T from F into a topological space Z under these new

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topologies. These results generalize the works of Easton and Tucker [2] who give a Lebesgue-type integral and a representation theorem for the associated function space in the setting where H is compact, X and Y are normed spaces, and K is of bounded semivariation.

As a corollary we restrict considerations to the setting where H is normal and F is the space of continuous totally bounded functions with the topology of uniform convergence, which extends the results in [3]. Another corollary gives a representation for continuous operators on the space of continuous functions under the topology of uniform convergence on compact subsets where the domain space H of the functions is locally compact.

This paper continues the development in [3] where X and Y were assumed to be normed spaces.

2. Notation and preliminaries. A linear map T from X into Y is (p, q) related if for each continuous seminorm q on Y there is a continuous seminorm p on X and a constant $L_{p,q}$ such that $q(T(x)) \leq L_{p,q}p(x)$. Furthermore if T is (p, q) related we define $|T|_{p,q} = \sup_{x \in X} \{q(T(x)) : p(x) \leq 1\}$. By X^+ we mean the weak sequential completion of X , i.e., the linear subspace of X'' (the bidual of X) such that $x^+ \in X^+$ if and only if there is a sequence in X which converges weakly to x^+ . The topology on X^+ is the topology of uniform convergence on equicontinuous subsets of X' (the continuous dual of X), i.e., the ϵ^{00} topology. We shall let X^ω denote the subspace of X'' which is the weak closure of X . Again the topology is the ϵ^{00} topology.

The development of most of this paper could be carried out with a field of sets $\Sigma \subset P(H)$ and integration in the Stieltjes-Hildebrandt sense [7], i.e., a function f from H into X is said to be integrable with respect to the finitely additive set function K from Σ into $L[X, Y]$ (the space of bounded linear operators from X into Y) if $\lim_\pi \sum [K(E_i^n)](f(t_i^n))$ exists in \bar{Y} , the completion of Y , where $\{E_i^n\}$ is a partition of H over Σ and $t_i^n \in E_i^n$, and where the direction on the net of partitions is refinement. However, Theorems 6.2 and 6.3 require a slight extension of the notion of integration which we now make. Let $\Sigma \subset P(H)$ denote a ring of sets rather than a field. We shall say $\pi = \{E_i^n\}_{i=1}^n$ is a partition of H over Σ provided $\{E_i^n\}_{i=1}^n$ is a collection of pairwise disjoint sets in Σ and $E_n = \sim \bigcup_{i=1}^{n-1} E_i$. The collection of partitions of H over Σ is directed by $\pi = \{E_i\}_{i=1}^n > \pi' = \{F_j\}_{j=1}^m$ if and only if $\bigcup_{j=1}^m F_j \subset \bigcup_{i=1}^n E_i$ and $E_i \cap F_j \neq \emptyset$ implies $E_i \supset F_j$. A function f is integrable provided $\lim_\pi \sum_{i=1}^n [K(E_i^n)](f(t_i^n))$ exists in \bar{Y} .

Let $S(\Sigma, X)$ denote the linear space of X -valued simple functions on H over Σ , i.e., functions of the form $\sum_{i=1}^n \chi_{E_i} \cdot x_i$ where $E_i \in \Sigma$, $i = 1, \dots, n-1$, $E_n = \sim \bigcup_{i=1}^{n-1} E_i$, the complement of $\bigcup_{i=1}^{n-1} E_i$, and where $x_n = \theta_X$. Suppose τ is a topology on $S(\Sigma, X)$ under which it is a topological vector space. Then a finitely additive set function K from Σ into $L[X, Y]$ is said to be quasi-Gowurin with respect to τ if given a neighborhood V of θ_Y there is a neighborhood U of $\theta_{S(\Sigma, X)}$ such that if $\sum \chi_{E_i} \cdot x_i \in U$, then $\sum [K(E_i)](x_i) \in V$. If τ is locally convex and K is quasi-Gowurin, then for each continuous seminorm q on Y there is a continuous seminorm ρ on $S(\Sigma, X)$

and a smallest constant $WK_{\rho,q}$ such that $q(\sum [K(E_i)](x_i)) \leq WK_{\rho,q}\rho(\sum \chi_{E_i} \cdot x_i)$. If X is a locally convex topological vector space and τ is the topology of uniform convergence, then quasi-Gowurin becomes bounded (p, q) variation [6]. We shall use all other definitions and notations defined in [6], except that we shall view $X \subset X^+ \subset X^\omega \subset X''$, where X^ω denotes the weak closure of X .

Let Γ denote the closure of $S(\Sigma, X)$ under the topology of uniform convergence. The following lemma is straightforward and is stated without proof.

2.1 LEMMA. *If $f \in \Gamma$, then $\lim_\pi \sum \chi_{E_i} \cdot f(t_i^\pi) = f$, where the convergence is uniform.*

2.2 THEOREM. *Suppose the linear subspace F is contained in Γ and τ is a locally convex topology on $F + S(\Sigma, X)$ which is not stronger than the topology of uniform convergence. If K is quasi-Gowurin with respect to τ , then the linear operator $T(f) = \int dK \cdot f$ exists for each $f \in F$ and is a continuous operator on $F + S(\Sigma, X)$. Furthermore, if T is (ρ, q) related then $WK_{\rho,q} = |T|_{\rho,q}$.*

Proof. Suppose $f \in F$ and q is a continuous seminorm on \bar{Y} . Then there is a continuous seminorm ρ on $F + S(\Sigma, X)$ such that K is (ρ, q) related. Then for partitions π and π' ,

$$q\left(\sum_\pi [K(E_i)](f(t_i)) - \sum_{\pi'} [K(E_j)](f(t_j))\right) \leq WK_{\rho,q}\rho\left(\sum_\pi \chi_{E_i} \cdot f(t_i) - \sum_{\pi'} \chi_{E_j} \cdot f(t_j)\right).$$

Since Γ is not stronger than the topology of uniform convergence, it follows that ρ is continuous on $S(\Sigma, X)$ with the topology of uniform convergence. It follows from Lemma 2.1 that $\{\sum_\pi \chi_{E_i} \cdot f(t_i)\}_\pi$ is Cauchy in the topology of uniform convergence. Therefore, $\{q(\sum_\pi [K(E_i)](f(t_i)))\}_\pi$ is Cauchy and hence $\{\sum_\pi [K(E_i)](f(t_i))\}_\pi$ is Cauchy in Y , from which it follows that $\int dK \cdot f$ exists in \bar{Y} . Furthermore,

$$\begin{aligned} q\left(\int dK \cdot f\right) &= \lim_\pi q\left(\sum_\pi [K(E_i)](f(t_i))\right) \\ &\leq \lim_\pi \sup WK_{\rho,q}\rho\left(\sum_\pi \chi_{E_i} \cdot f(t_i)\right) \\ &= WK_{\rho,q}\rho(f). \end{aligned}$$

Hence T is (ρ, q) related and $|T|_{\rho,q} \leq WK_{\rho,q}$, from which we conclude T is continuous. Finally, we show $|T|_{\rho,q} \geq WK_{\rho,q}$. Suppose $\{E_i\} = \pi$ is a partition of H over Σ , and $\{x_i\} \subset X$. Then

$$q\left(\sum [K(E_i)](x_i)\right) = q\left(T\left(\sum \chi_{E_i} \cdot x_i\right)\right) \leq |T|_{\rho,q}\rho\left(\sum \chi_{E_i} \cdot x_i\right).$$

2.3 COROLLARY. *Suppose F , $S(\Sigma, X)$, τ and K are as in Theorem 2.2. If $\{f_\alpha\} \subset (F + S(\Sigma, X))$ is a net converging to $f \in F + S(\Sigma, X)$, then $\int dK \cdot f_\alpha \rightarrow \int dK \cdot f$.*

3. The main result. Suppose $F \subset \Gamma$ is a locally convex topological space with topology τ not stronger than the topology of uniform convergence and such that there is a ring of sets Σ and a subspace $S'(\Sigma, X) \subset S(\Sigma, X)$ satisfying:

- (i) The topology τ has a locally convex extension to $F+S(\Sigma, X)$.
- (ii) F is contained in the τ closure of $S'(\Sigma, X)$.
- (iii) There is a linear map η from $F+S(\Sigma, X)$ into F^ω which is (ρ, ρ'') related when restricted to $S(\Sigma, X)$, and continuous when restricted to $F+S'(\Sigma, X)$ and such that $\eta(f)=f$ for each $f \in F$.

Since η when restricted to $S(\Sigma, X)$ is (ρ, ρ'') related, for each continuous seminorm ρ there is a constant $|\eta|_\rho$ such that $\rho''(\eta(\sum_{E_i} \chi_{E_i} \cdot x_i)) \leq |\eta|_\rho \rho(\sum_{E_i} \chi_{E_i} \cdot x_i)$.

3.1 THEOREM. *Suppose T is a continuous linear transformation from F into Y . Then, there is an additive set function K on Σ with values in $L[X, Y^\omega]$ which is quasi-Gowurin with respect to τ such that $T(f) = \int dK \cdot f$. Furthermore, if T is (ρ, q) related, then $WK_{\rho, q''} \geq |T|_{\rho, q} \geq WK_{\rho, q''}/|\eta|_\rho$.*

Proof. Let T^ω denote the continuous linear map from F^ω into Y^ω which is the natural extension of T . Define the set function K from Σ into $L[X, Y^\omega]$ by $[K(E)](x) = T^\omega(\eta(\chi_E \cdot x))$ for $x \in X$ and for each $E \in \Sigma$. Suppose q is a continuous seminorm on Y . Since $T^\omega \eta$ is continuous on $S(\Sigma, X)$, there is a continuous seminorm ρ on $F+S(\Sigma, X)$ such that, for $\pi = \{E_i\}$ a partition of H and a corresponding collection $\{x_i\} \subset X$,

$$q''\left(\sum_{\pi} [K(E_i)](x_i)\right) = q''(T^\omega(\eta(\sum \chi_{E_i} \cdot x_i))) \\ \leq |T^\omega|_{\rho'', q''} |\eta|_\rho (\sum \chi_{E_i} \cdot x_i).$$

From which it follows that K is quasi-Gowurin with respect to τ and that $WK_{\rho, q''}/|\eta|_\rho \leq |T|_{\rho, q}$, since $|T^\omega|_{\rho'', q''} = |T|_{\rho, q}$. Suppose $f \in F$. Then (ii) implies there is a net $\{s_\alpha\}$ in $S'(\Sigma, X)$ which converges to f in the τ topology. Since η is continuous on $F+S'(\Sigma, X)$, it follows from Corollary 2.3 that

$$T(f) = T^\omega(\eta(f)) = \lim_{\alpha} T^\omega(\eta(s_\alpha)) = \lim_{\alpha} \int dK \cdot s_\alpha = \int dK \cdot f.$$

Theorem 2.2 implies that if T is (ρ, q) related, then $|T|_{\rho, q} \leq WK_{\rho, q''}$, and the theorem is established.

The following lemma gives a necessary and sufficient condition for η to be continuous on $F+S'(\Sigma, X)$. The proof is analogous to that given for Corollary 2.3 in [3] and is omitted.

3.2 LEMMA. *Suppose the linear map η from $F+S'(\Sigma, X)$ into F^ω satisfies the condition that $\eta(f)=f$ for each $f \in F$. Then a necessary and sufficient condition that η be continuous on $F+S'(\Sigma, X)$ is that if the net $\{s_\alpha\} \subset S'(\Sigma, X)$ converges to $f \in F$ in the τ topology then $\{\eta(s_\alpha)\}$ converges to f in the τ^ω topology (the $\epsilon^{\omega 0}$ topology restricted to F^ω).*

3.3 REMARK. If the map η takes its values in F^+ , then the K given in Theorem 3.1 takes its values in $L[X, Y^+]$. Similarly, if η takes its values in F , then K takes its values in $L[X, Y]$.

4. The extended integral. In this section we develop a Lebesgue-type extension of the integral and give an extended version of Theorem 3.1. We suppose the general setting of §3. Let G denote the τ completion of F . Observe that G is a collection of equivalence classes of nets. The question of when it is possible to identify functions from H into X with equivalence classes is of some interest. In the following three definitions we define certain classes of functions which have natural identifications with equivalence classes in G .

4.1 DEFINITION. Let $G(\tau)$ denote the space of functions f on H with values in X such that all nets of simple functions of the form $\{\sum_{\pi} \chi_{E_i} \cdot f(t_i)\}_{\pi}$, where $t_i \in E_i$, are Cauchy and Cauchy equivalent in the τ topology.

Observe that $\Gamma \subset G(\tau)$ because the topology of uniform convergence is stronger than τ . The identification of $G(\tau)$ as a subspace of G is the natural one.

4.2 DEFINITION. A function f on H with values in X is said to be an allowable σ -totally bounded function if its range is the union of a nested countable collection of sets $\{B_n\}_{n=1}^{\infty}$ ordered as to n , such that each B_n is totally bounded and such that $\chi_{f^{-1}[B_n]} \cdot f \in \Gamma$. The collection $\{H_n\} = \{f^{-1}[B_n]\}$ is called a σ -totally bounded decomposition of H with respect to f (we assume $H_n \in \Sigma$ for each n).

Let Λ denote the directed set $\{(V, n) : V \text{ is a neighborhood of } \theta_{\Gamma} \text{ and } n \text{ is an integer}\}$ whose direction is given by $(V, n) > (U, m)$ if $U \supset V$ and $n \geq m$.

4.3 DEFINITION. Let $G^{\sigma}(\tau)$ denote the space of allowable σ -totally bounded functions such that if $\{H_n\}$ and $\{F_n\}$ are two σ -totally bounded decompositions of H with respect to f and if $\{s_{\alpha}\}_{\Lambda}$ and $\{t_{\alpha}\}_{\Lambda}$ are nets of functions in F such that for $\alpha = (V, n)$, $s_{\alpha} - \chi_{H_n} \cdot f$ and $t_{\alpha} - \chi_{F_n} \cdot f$ are in V , then $\{s_{\alpha}\}$ and $\{t_{\alpha}\}$ are Cauchy and Cauchy equivalent.

For each $f \in G^{\sigma}(\tau)$ we identify f with the equivalence class in G determined by Definition 4.3.

4.4 DEFINITION. Suppose K is quasi-Gowurin with respect to τ . Then, for $f \in G$ the extended integral of f with respect to K is denoted by $E \int dK \cdot f$ and is defined by $\lim_{\alpha} \int dK \cdot f_{\alpha} \in \bar{Y}$ where $\{f_{\alpha}\} \subset F$ is a net which converges to f in the τ topology.

The following lemma guarantees that the extended integral is well defined and is consistent with the integral on $G(\tau)$.

4.5 LEMMA. Suppose that K is quasi-Gowurin with respect to τ . Then,

- (i) if $\{f_{\alpha}\}$ and $\{g_{\alpha}\}$ are Cauchy equivalent in F , then $\{\int dK \cdot f_{\alpha}\}$ and $\{\int dK \cdot g_{\alpha}\}$ are Cauchy equivalent,
- (ii) if $f \in G$, the extended integral $E \int dK \cdot f$ exists,
- (iii) if $f \in G(\tau)$, then $E \int dK \cdot f = \int dK \cdot f$.

The proof of this lemma follows from Theorem 2.2.

4.6 THEOREM. Suppose K is a finitely additive set function on Σ with values in $L[X, Y]$ which is quasi-Gowurin with respect to τ . Then the map $T(f) = E \int dK \cdot f$ is a continuous linear operator from G into \bar{Y} . Furthermore, if T is (ρ, q) related, then $|T|_{\rho, q} = WK_{\rho, q}$.

The proof follows from Theorem 2.2 and Lemma 4.5.

4.7 THEOREM. *Suppose T is a continuous linear operator from G into Y . Then, there is a finitely additive set function K on Σ with values in $L[X, Y^\omega]$ which is quasi-Gowurin with respect to τ such that $T(f) = E \int dK \cdot f$ for $f \in G$. Furthermore, if T is (ρ, q) related, then $WK_{\rho, q} \geq |T|_{\rho, q} \geq WK_{\rho, q} / |\theta|_\rho$.*

Proof. Since T is a continuous linear operator from F into Y , it follows from Theorem 3.1 that there is a set function K which satisfies the above conditions such that $T(f) = \int dK \cdot f$ for $f \in F$. Since T is continuous, it follows from Lemma 4.5(ii) that $T(f) = E \int dK \cdot f$ for all $f \in G$.

4.8 REMARK. If the η map of §3 takes its values in F^+ or F then K in Theorem 4.7 takes its values in $L[X, Y^+]$ or $L[X, Y]$ respectively.

5. A Lebesgue-type topology. In this section we define a Lebesgue-type topology on a space of simple functions and complete this space into what we shall call L_K , the analog of the Lebesgue integrable functions in the classical setting. A characterization of the continuous linear operators from L_K into a complete locally convex space is then given. In the special case that the topology τ considered in this section is the topology of uniform convergence, we have an extension of the results in [2] and of the corresponding results in [3] which are given in the setting of linear normed spaces.

Let X , Y , and H be as in §1. Let $\Sigma \subset P(H)$ be a ring of sets and let τ denote a locally convex topology on $S(\Sigma, X)$ which is not stronger than the topology of uniform convergence. Suppose K is a finitely additive set function on Σ with values in $L[X, Y]$ which is quasi-Gowurin with respect to τ . For each continuous seminorm q on Y define the seminorm ρ_q on $S(\Sigma, X)$ by

$$\rho_q \left(\sum_i \chi_{E_i} \cdot x_i \right) = \sup \left\{ q \left(\sum_{ij} [K(E_i \cap F_j)](\alpha_j x_i) \right) \right\}$$

where the supremum is taken over all partitions $\{F_j\}$ of H over Σ and corresponding collections of scalars $\{\alpha_j\}$ such that $|\alpha_j| \leq 1$ for each j . Since τ is not stronger than the topology of uniform convergence the supremum exists for each q . Two simple functions s and s' are said to be equivalent if $\rho_q(s - s') = 0$ for each continuous seminorm on Y . Let $S(\Sigma, X)_L$ denote the resulting space.

5.1 DEFINITION. Let κ denote the locally convex topology generated on $S(\Sigma, X)_L$ by the seminorms ρ_q . Let L_K denote the completion of $S(\Sigma, X)_L$ under the κ topology. As is the usual practice, if a function is identified with an equivalence class in L_K , we do not distinguish between the function and the equivalence class. In this setting we give a more general criterion than those given in §4 with which to identify a function with an equivalence class in L_K . Suppose f is an allowable σ -totally bounded function on H with values in X . If there is a $y \in \bar{Y}$ such that for each σ -totally bounded decomposition $\{H_n\}$ of H , $\lim_n \int_{H_n} dK \cdot f = y$, then f is viewed as being in L_K and is identified with the equivalence class

containing the net of simple functions $\{s_\alpha\}_\Lambda$ (where Λ is defined as in §4) defined by $s_\alpha = \sum \chi_{E_i^\alpha} \cdot f(t_i)$ for $\alpha = (V, n)$ and where $\{E_i\}$ is a partition of H_n over Σ such that $f[E_i^\alpha] - f(t_i) \subset V$, with $t_i \in E_i$.

5.2 LEMMA. *The topology of uniform convergence on $S(\Sigma, X)$ is stronger than the κ topology on $S(\Sigma, X)$.*

Proof. Suppose q is a continuous seminorm on Y . Since K is quasi-Gowurin with respect to τ , there is a seminorm ρ continuous in the τ topology such that, for $s = \sum \chi_{E_i} \cdot x_i \in S(\Sigma, X)$,

$$\begin{aligned} \rho_q(s) &= \sup \left\{ q \left(\sum_{ij} [K(E_i \cap F_j)](\alpha_j x_i) \right) \right\} \\ &\leq \sup \left\{ WK_{\rho, q} \rho \left(\sum_{ij} \chi_{E_i \cap F_j} \alpha_j x_i \right) \right\}. \end{aligned}$$

Since τ is not stronger than the topology of uniform convergence there is a continuous seminorm p on X such that the corresponding seminorm ρ_p on $S(\Sigma, X)$ dominates ρ . Therefore,

$$\begin{aligned} \rho_q(s) &\leq \sup \left\{ WK_{\rho, q} \rho_p \left(\sum_{ij} \chi_{E_i \cap F_j} \alpha_j x_i \right) \right\} \\ &\leq WK_{\rho, q} \max_i p(x_i) = WK_{\rho, q} \rho_p(s). \end{aligned}$$

Hence for $f \in \Gamma$ and q a continuous seminorm on Y , $\rho_q(f) = \sup \{ q(\sum \alpha_i \int_{E_i} dK \cdot f) \}$ where the supremum is taken over partitions $\{E_i\}$ of H over Σ and corresponding scalars $\{\alpha_i\}$ such that $|\alpha_i| \leq 1$ for all i . By $\int_E dK \cdot f$ we mean $\int dK(f \cdot \chi_E)$ which is well defined for $f \in \Gamma$. Hence, it follows that if $f \in L_K$, and if q is a continuous seminorm on Y , then $\rho_q(f) = \sup \{ q(\sum \alpha_i \int_{E_i} dK \cdot f) \}$.

Suppose Z is a locally convex space. In the following definition we give the analog of the concept of strongly Lipschitz in [2].

5.3 DEFINITION. A set function G on Σ with values in $L[X, Z]$ is said to be (q, ν) strongly Lipschitz with respect to K if given a partition $\{E_i\}$ of H over Σ and a corresponding collection $\{x_i\} \subset X$, then there is a pairing (q, ν) with constants $P_{q, \nu}$ such that

$$\nu \left(\sum_i [G(E_i)](x_i) \right) \leq P_{q, \nu} \sup \left\{ \sum_{ij} [K(E_i \cap F_j)](\alpha_j x_i) \right\}$$

where the supremum is taken over partitions $\{F_j\}$ of H over Σ and corresponding collections of scalars $\{\alpha_j\}$ for which $|\alpha_j| \leq 1$.

5.4 LEMMA. *A set function G on Σ with values in $L[X, Z]$ is (q, ν) strongly Lipschitz if and only if G is quasi-Gowurin with respect to κ . Furthermore, if G is (q, ν) related, then $WG_{\rho_q, \nu} = P_{q, \nu}$.*

The proof is straightforward and is therefore omitted.

5.5 THEOREM. Suppose that T is a continuous linear operator from L_K into Z . Then, there is an additive set function G on Σ with values in $L[X, Z]$ which is (q, ν) strongly Lipschitz with respect to K such that for $f \in L_K$, $T(f) = E \int dG \cdot f$. Furthermore, if T is (ρ_q, ν) related, then $|T|_{\rho_q, \nu} = P_{q, \nu}$.

Proof. The theorem follows from Theorem 4.7, Remark 4.8, and Lemma 5.4.

If Z is a complete locally convex space, then the above theorem and Theorem 4.6 yield the following characterization of the bounded linear operators from L_K into Z .

5.6 COROLLARY. Suppose T is a linear transformation from L_K into Z . Then T is continuous if and only if there is a finitely additive set function G which is (q, ν) strongly Lipschitz with respect to K such that $T(f) = E \int dG \cdot f$ for $f \in L_K$. Furthermore, if T is (ρ_q, ν) related, then $|T|_{\rho_q, \nu} = P_{q, \nu}$.

6. Some other applications. As before X and Y shall denote locally convex spaces. In §§7, 8 and 9 we shall establish the following special cases of Theorems 3.1 and 4.7 which in the classical settings have been of particular interest. For H an arbitrary normal space we shall let CTB denote the space of totally bounded continuous functions from H into X with the topology of uniform convergence and shall let CBR denote the space of bounded continuous real valued functions with the topology of uniform convergence. If H is a locally compact Hausdorff space C_c denotes the space of continuous X -valued functions of compact support and C_0 denotes the space of continuous X -valued functions which vanish at infinity, both with the topology of uniform convergence. The space of continuous real-valued functions of compact support with the topology of uniform convergence is denoted by $C_c R$. Finally, if H is locally compact, C shall denote the space of continuous X -valued functions from H with the topology of uniform convergence on compact subsets of H .

6.1 THEOREM. Let H denote a normal topological space and let Σ denote the field of sets generated by the closed subsets of H . If T is a continuous linear operator from CTB into Y , then there is a unique finitely additive weakly regular set function K on Σ with values in $L[X, Y^\omega]$ which is of bounded (p, q) variation such that $T(f) = \int dK \cdot f$. Furthermore, if T is (p, q) related, then $WK_{p, q} = |T|_{p, q}$.

When we say T is (p, q) related we mean T is (ρ_p, q) related where ρ_p is the continuous seminorm on CTB generated by the continuous seminorm p on X .

For the purpose of the next theorem we extend the notion of bounded (p, q) variation to additive set functions defined over a ring as follows: The finitely additive set function K from the ring Σ into $L[X, Y]$ is of bounded (p, q) variation if there is a pairing (p, q) such that, for each partition $\{E_i\}_{i=1}^n$ of H over Σ and each collection $\{x_i\}_{i=1}^n$ with $x_n = \theta$, $q(\sum_{i=1}^n [K(E_i)](x_i)) \leq L \max_{1 \leq i \leq n} p(x_i)$ where the constant L depends on p and q . Furthermore, the (p, q) variation of K , denoted by $WK_{p, q}$, is given by the infimum of all such L .

6.2 THEOREM. *Let H denote a locally compact Hausdorff space and let Σ_c denote the ring generated by all compact subsets of H . If T is a continuous linear operator from C_c into Y , then there is a unique finitely additive weakly regular set function K on Σ_c with values in $L[X, Y^\omega]$ which is of bounded (p, q'') variation such that $T(f) = \int dK \cdot f$ for all $f \in C_c$. Furthermore, if T is (p, q) related then $WK_{p,q''} = |T|_{p,q}$.*

6.3 THEOREM. *Let H and Σ_c be as in Theorem 6.2. If T is a continuous linear operator from C_0 into Y , then there is a unique finitely additive weakly regular set function K on Σ_c with values in $L[X, Y]$ which is of bounded (p, q'') variation such that $T(f) = E \int dK \cdot f$ for $f \in C_0$. Furthermore, if T is (p, q) related, then $WK_{p,q''} = |T|_{p,q}$.*

6.4 DEFINITION. Suppose Σ_c is as above and K is a finitely additive set function on Σ_c with values in $L[X, Y]$. To say K is of bounded (p, q) variation with compact q -support means, given a continuous seminorm q on Y , there are a continuous seminorm p on X , a constant $WK_{p,q}$, and a compact set M_q such that, for each x , $q([K(E)](x))$ is a real-valued set function with support M_q and such that $q(\sum [K(E_i)](x_i)) \leq WK_{p,q} \max \{p(x_i) : E_i \cap M_q \neq \emptyset\}$. Such a set function is said to be of bounded (p, q) variation with compact support if the compact set M_q is independent of the seminorm q .

6.5 THEOREM. *Let H and Σ_c be as in Theorem 6.2. If T is a continuous linear operator from C into Y , then there is a unique finitely additive weakly regular set function K on Σ_c with values in $L[X, Y^\omega]$ which is of bounded (p, q'') variation with compact q'' -support such that for $f \in C$, $T(f) = E \int dK \cdot f$. Furthermore, if T is (p, q) related, then $WK_{p,q''} = |T|_{p,q}$.*

6.6 THEOREM. *Suppose H and Σ_c are as in Theorem 6.2 and Y is a normed space. If T is a continuous linear operator from C into Y , then there is a unique finitely additive weakly regular set function K on Σ_c with values in $L[X, Y^\omega]$ which is of bounded $(p, \|\cdot\|_{Y^\omega})$ variation with compact support such that for $f \in C$, $T(f) = \int dK \cdot f$. Furthermore, if T is $(p, \|\cdot\|_{Y^\omega})$ related then $WK_{p, \|\cdot\|_{Y^\omega}} = |T|_{p, \|\cdot\|_{Y^\omega}}$.*

We conclude this section by observing that the results obtained in §3 of [3] which include generalizations of the results of Hildebrandt [7] and Fihtengol'c and Kantorovič [5] can be obtained in this setting by using Theorem 3.1 of this paper as Theorem 2.2 is used in [3].

7. Proof of Theorem 6.1. We show that Theorem 6.1 follows as a corollary of Theorem 3.1. To do so it is necessary to identify the subspace $S'(\Sigma, X)$ and to construct the η map of Theorem 3.1. The required space $S'(\Sigma, X)$ is the space of simple functions which are representable as simple functions over $\{E_i\}_{i=1}^n$ where $\bigcup_{i=1}^n E_i$ is a closed set and where $\bigcup_{i=1}^n E_i = H$. Suppose $E \subset \Sigma$. Consider the directed set $\Lambda(E) = \{(F, V) : F \subset E, F \text{ closed and } E \subset V, V \text{ open}\}$ where the direction is given by $(F, V) > (F', V')$ if and only if $F' \subset F$ and $V \subset V'$. Define the net of

functions $\{\phi_\alpha\}_{\Lambda(E)}$ (using Urysohn's lemma) such that for $\alpha=(F, V)$, ϕ_α is a continuous function whose values are taken in $[0, 1]$ which is 1 on F and 0 on the complement of V . In the following lemma we establish the preliminaries necessary to define the η map.

7.1 LEMMA. *Suppose E and $\{\phi_\alpha\}_{\Lambda(E)}$ are as above. Then, for each vector $x \in X$, $\{\phi_\alpha \cdot x\}_{\Lambda(E)}$ is weakly Cauchy in CTB . Furthermore, if $\{\phi_\alpha\}_{\Lambda(E)}$ and $\{\psi_\alpha\}_{\Lambda(E)}$ are any two nets picked in the above fashion, then $\{\phi_\alpha \cdot x\}$ and $\{\psi_\alpha \cdot x\}$ are weakly Cauchy equivalent in CTB .*

Proof. It is sufficient to show that $\{\phi_\alpha\}$ is weakly Cauchy in the space CR of bounded continuous real valued functions on H . Suppose $c' \in CR'$. Then theorem [1, p. 262] implies there is a regular bounded additive set function $\mu_{c'}$ such that $\langle c', f \rangle = \int f d\mu_{c'}$. Therefore, given $\varepsilon > 0$, there is a closed set $F_\varepsilon \subset E$ and an open set $V_\varepsilon \supset E$ such that $\|\mu\|(V_\varepsilon \setminus E)$ and $\|\mu\|(E \setminus F_\varepsilon)$ are both smaller than $\varepsilon/2$. If (F, V) , $(F', V') > (F_\varepsilon, V_\varepsilon)$, then

$$\begin{aligned} \|\mu\|(V \cup V' \setminus F \cap F') &\leq \|\mu\|(V_\varepsilon \setminus F_\varepsilon) \\ &= \|\mu\|(V_\varepsilon \setminus E) + \|\mu\|(E \setminus F_\varepsilon) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, for $\alpha=(F, V)$ and $\alpha'=(F', V')$ and $\alpha, \alpha' > \alpha_\varepsilon=(F_\varepsilon, V_\varepsilon)$, ϕ_α differs from $\phi_{\alpha'}$ on a set of $\|\mu\|$ -measure less than ε . Therefore, for $\alpha, \alpha' > \alpha_\varepsilon$,

$$|\langle c', \phi_\alpha \rangle - \langle c', \phi_{\alpha'} \rangle| = \left| \int (\phi_\alpha - \phi_{\alpha'}) d\mu \right| < \varepsilon,$$

and it follows that $\{\langle c', \phi_\alpha \rangle\}$ is Cauchy. Hence, $\{\phi_\alpha\}$ is weakly Cauchy in CR' . That $\{\phi_\alpha\}_{\Lambda(E)}$ and $\{\psi_\alpha\}_{\Lambda(E)}$ are weakly Cauchy equivalent follows by an analogous argument.

For each $E \in \Sigma$ and corresponding set of functions $\{\phi_\alpha\}_{\Lambda(E)}$, and each $x \in X$, we denote by $\chi(E, x)$ the weak limit of $\{\phi_\alpha \cdot x\}_{\Lambda(E)}$. Define the linear map η from $S(\Sigma, X) + CTB$ into $(CTB)^\omega$ by $\eta(f) = f$ for $f \in CTB$ and $\eta(\sum_i \chi_{E_i} \cdot x_i) = \sum_i \chi(E_i \cdot x_i)$. Since the decomposition of a function into a continuous function and a simple function is unique modulo constant functions, the linear map η is well defined. In the following lemma we establish that the η map has the desired properties and thus complete the proof of 6.1 except for the uniqueness of K .

7.2 LEMMA. *Suppose η is as above.*

- (i) *For each continuous seminorm p on X , $|\eta|_{\rho_p} = 1$.*
- (ii) *The space CTB is a subset of the closure of $S'(\Sigma, X)$ under the topology of uniform convergence.*
- (iii) *The linear map η is a continuous linear map from $S'(\Sigma, X) + CTB$ into $(CTB)^\omega$.*

Proof (i). We show that for each continuous seminorm p on X , $\rho_p(\eta(\sum_i \chi_{E_i} \cdot x_i)) \leq \rho_p(\sum_i \chi_{E_i} \cdot x_i)$. The simple function $\sum_{i=1}^n \chi_{E_i} \cdot x_i$ in $S(\Sigma, X)$ implies that there are nets $\{\phi_{i\alpha_i} \cdot x_i\}_{\Lambda(E_i)}$, $i=1, \dots, n$, defined as in this section, which are weakly convergent to $\eta(\chi_{E_i} \cdot x_i)$, $i=1, \dots, n$, respectively. Let Λ denote the directed set defined

by $\Lambda(E_1) \times \Lambda(E_2) \times \cdots \times \Lambda(E_n)$. Define the nets $\{\phi_{i\alpha}\}$, $i=1, \dots, n$, over Λ by $\phi_{i\alpha} = \phi_{i(\alpha_1, \dots, \alpha_n)} = \phi_{i\alpha_i}$. Then, $\{\phi_{i\alpha} \cdot x_i\}_\Lambda$ converges weakly to $\eta(\chi_{E_i} \cdot x_i)$. We now construct nets $\{h_{i\alpha}\}_\Lambda$, $i=1, \dots, n$, in the fashion of Waymott [12] such that for each α , $0 \leq \sum h_{i\alpha}(t) \leq 1$ for $t \in H$ and such that, for each i , $\{h_{i\alpha} \cdot x_i\}_\Lambda$ converges weakly to $\eta(\chi_{E_i} \cdot x_i)$. For each $\alpha \in \Lambda$ define $h_{1\alpha} = \phi_{1\alpha}$, $h_{2\alpha} = \phi_{2\alpha} - \min(\phi_{2\alpha}, h_{1\alpha})$, and in general $h_{j\alpha} = \phi_{j\alpha} - \min(\phi_{j\alpha}, \sum_{i=1}^{j-1} h_{i\alpha})$. Then, $0 \leq \sum_{i=1}^n h_{i\alpha}(t) \leq 1$ for $t \in H$. Choose j . Then for fixed α , $h_{j\alpha}$ differs from $\phi_{j\alpha}$ only on $\bigcup_{i \neq j} (V_i \cap V_j)$ where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\alpha_i = (F_i, V_i)$, $i=1, \dots, n$. Suppose $c' \in (CBR)'$. As in 7.1 we let $\mu_{c'}$ denote the measure corresponding to c' . Then,

$$\begin{aligned} \|\mu_{c'}\| \left(\bigcup_{i \neq j} (V_i \cap V_j) \right) &\leq \sum_{i \neq j} \|\mu_{c'}\|(V_i \cap V_j) \\ &= \sum_{i \neq j} \|\mu_{c'}\|((V_i \cap V_j \cap E_i) \cup (V_i \cap V_j \cap \tilde{E}_i)) \\ &\leq \sum_{i \neq j} [\|\mu_{c'}\|(V_j \cap \tilde{E}_j) + \|\mu_{c'}\|(V_i \cap \tilde{E}_i)]. \end{aligned}$$

The last inequality follows from the fact that $\|\mu_{c'}\|$ is subadditive and the fact that $E_i \subset \tilde{E}_j$, the complement of E_j . As $\alpha = (\alpha_1, \dots, \alpha_n)$ runs through the directed set Λ , $\alpha_i = (F_i, V_i)$ runs through the directed set $\Lambda(E_i)$ for each i . Hence, for each i , $\|\mu_{c'}\|(V_i \cap \tilde{E}_i)$ converges to zero. Therefore,

$$\begin{aligned} \lim_\alpha |\langle c', \phi_{h\alpha} - h_{j\alpha} \rangle| &= \lim_\alpha \left| \int (\phi_{j\alpha} - h_{j\alpha}) d\mu_{c'} \right| \\ &\leq \lim_\alpha \sup \left(\sum_{i \neq j} [\|\mu_{c'}\|(V_j \cap \tilde{E}_j) + \|\mu_{c'}\|(V_i \cap \tilde{E}_i)] \right) = 0. \end{aligned}$$

Hence, $\{\phi_{j\alpha}\}$ and $\{h_{j\alpha}\}$ are weakly Cauchy equivalent from which it follows that $\{h_{j\alpha} \cdot x_j\}_\Lambda$ converges weakly to $\phi(\chi_{E_i} \cdot x_i)$. Observe that

$$\rho_p'' \left(\eta \left(\sum_{i=1}^n \chi_{E_i} \cdot x_i \right) \right) = \sup \left\{ \lim_\alpha \left| \left\langle c', \sum_{i=1}^n h_{i\alpha} \cdot x_i \right\rangle \right| : c' \in B_p^0 \right\}$$

where $B_p^0 = \{c' \in (CTB)' : |\langle c', f \rangle| \leq 1 \text{ for all } f \in CTB \text{ such that } \rho_p(f) \leq 1\}$. Therefore,

$$\begin{aligned} \rho_p'' \left(\eta \left(\sum_{i=1}^n \chi_{E_i} \cdot x_i \right) \right) &\leq \lim_\alpha \sup \rho_p \left(\sum_{i=1}^n h_{i\alpha} \cdot x_i \right) \\ &= \lim_\alpha \sup \left(\sup_{t \in H} p \left(\sum_{i=1}^n h_{i\alpha}(t) \cdot x_i \right) \right) \\ &\leq \lim_\alpha \sup \left(\left(\sup_{t \in H} \sum_{i=1}^n h_{i\alpha}(t) \right) \left(\max_{1 \leq i \leq n} p(x_i) \right) \right) \\ &= \max_{1 \leq i \leq n} p(x_i) = \rho_p \left(\sum_{i=1}^n \chi_{E_i} \cdot x_i \right). \end{aligned}$$

Proof of (ii). Suppose $f \in CTB$ and suppose U is an open neighborhood of θ_X . Since the range of f is totally bounded, there exists a finite number of points $\{x_i\}_{i=1}^n$ such that $\{\frac{1}{2}U + x_i\}_{i=1}^n$ covers the range of f . Hence $\{\frac{1}{2}\bar{U} + x_i\}_{i=1}^n$ covers the range of f . Let $F_i = f^{-1}[\frac{1}{2}\bar{U} + x_i]$ for each i . Define $E_1 = F_1$, $E_2 = F_2 - E_1$, $E_3 = F_3 - E_1 \cup E_2$, and in general let $E_j = F_j - \bigcup_{i=1}^{j-1} E_i$. Finally, let $S_U = \sum f(t_i)\chi_{E_i}$ where $t_i \in E_i$. We note that $S_U \in S'(\Sigma, X)$ and $S_U(t) - f(t) \in U$ for each t .

Proof of (iii). Our plan is to use Lemma 3.2 to establish this result. In order to do so, it is convenient to establish the following sublemma, which extends the corresponding results in [3] and [11].

7.3 SUBLEMMA. Suppose S is a locally convex space. A sufficient condition to guarantee that a net $\{s_\alpha\}_\Lambda$ in S^ω converges to $s \in S$ is that for each α there is a net $\{s_{\alpha,\beta}\}_{\beta \in \Lambda(\alpha)}$ such that

- (a) for each $\alpha \in \Lambda$, $\{s_{\alpha,\beta}\}_{\beta \in \Lambda(\alpha)}$ converges weakly to s_α^ω ,
- (b) $s_{\alpha,\beta} - s$ converges to θ_S as α runs through the directed set Λ .

Proof. It is sufficient to show that for each continuous seminorm ρ on S that $\rho''(s_\alpha^\omega - s) \rightarrow 0$. Choose $\varepsilon > 0$. Since $s_{\alpha,\beta} - s$ converges to θ_S then there is an $\alpha_0 \in \Lambda$ such that $\alpha > \alpha_0$ implies $\rho(s_{\alpha,\beta} - s) < \varepsilon$. Choose $\alpha > \alpha_0$. Then

$$\rho''(s_\alpha^\omega - s) = \sup \left\{ \lim_{\beta} |\langle s', s_{\alpha,\beta} - s \rangle| : s' \in B_\rho^\circ \right\} \leq \limsup_{\beta} \rho(s_{\alpha,\beta} - s) < \varepsilon.$$

We continue with the proof of (iii). Suppose $\{s_\alpha\}_\Lambda = \{\sum_{i=1}^{n(\alpha)} \chi_{E_i^\alpha} \cdot x_i^\alpha\}$ is a net in $S'(\Sigma, X)$ which converges to $f \in CTB$ in the topology of uniform convergence. Our plan is to construct for each $\alpha \in \Lambda$ a net $\{s_{\alpha,\beta}\}_{\beta \in \Lambda(\alpha)}$ which converges weakly to $\eta(s_\alpha)$ so that Sublemma 7.3 is satisfied in order to apply Lemma 3.2. Fix α . Since $s_\alpha \in S'(\Sigma, X)$, $A_j^\alpha = \bigcup_{i=1}^j E_i^\alpha$ is a closed set. For each j we define the directed set

$$\Lambda(A_j^\alpha) = \{(A_j^\alpha, V) : V \text{ is an open set containing } A_j^\alpha\}$$

where the direction is given by inclusion. For each j we define a net of continuous functions $\{\phi_{j,\beta_j}\}_{\beta_j \in \Lambda(A_j^\alpha)}$ such that, for each j and for $\beta_j = (A_j^\alpha, V)$, ϕ_{j,β_j} takes its values in $[0, 1]$, is 1 on A_j^α and is 0 on \bar{V} . Then for each j , $\{\phi_{j,\beta_j}\}_{\beta_j \in \Lambda(A_j^\alpha)}$ converges in measure to $\chi_{A_j^\alpha}$ for every bounded regular finitely additive set function on Σ , and hence is weakly Cauchy in CBR . Define the directed set $\Lambda(\alpha)$ to be $\Lambda(A_1^\alpha) \times \cdots \times \Lambda(A_{n(\alpha)}^\alpha)$. For each j we define the net of open sets $\{V_{j\beta}\}_{\beta \in \Lambda(\alpha)}$ by $V_{j\beta} = V$ where $\beta_j = (A_j^\alpha, V)$ for $\beta = (\beta_1, \dots, \beta_j, \dots, \beta_{n(\alpha)})$. For each $\beta \in \Lambda(\alpha)$ define $\{h_{j\beta}\}_{j=1}^{n(\alpha)}$ by $h_{1\beta} = \phi_{1\beta}$ and $h_{j\beta} = \phi_{j\beta} - \min(\phi_{j\beta}, \sum_{i=1}^{j-1} h_{i\beta})$. Then for each $\beta \in \Lambda(\alpha)$, $\{h_{j\beta}\}_{j=1}^{n(\alpha)}$ is a partition of unity subordinate to $\{V_{j\beta}\}_{j=1}^{n(\alpha)}$. Next we show that, for each j , $\{h_{j\beta}\}_{\beta \in \Lambda(\alpha)}$ converges in measure to χ_{E_j} for every bounded regular finitely additive set function on Σ . Suppose μ is such a set function and suppose $\varepsilon > 0$. For each β , $h_{j\beta}$ differs from χ_{E_j} on $(E_j^\alpha \cap (\bigcup_{i=1}^{j-1} V_{i\beta})) \cup (V_{j\beta} \setminus A_j^\alpha)$. For each $i < j$, $\|\mu\|(E_j^\alpha \cap V_{i\beta}) < \|\mu\|(\bar{A}_i^\alpha \cap V_{i\beta})$ which converges to zero as β runs through $\Lambda(\alpha)$ because of the regularity of μ and because of the manner in which $\{V_{j\beta}\}_{\beta \in \Lambda(\alpha)}$ is constructed. For the same reason,

$\|\mu\|(V_{j\beta} \setminus A_j^\alpha)$ converges to zero. Hence, it follows that $\{h_{j\beta}\}_{\beta \in \Lambda(\alpha)}$ converges to $\chi_{E_j^\alpha}$ in measure for all bounded regular finitely additive set functions. Hence, it follows that for each i , $\{h_{i\beta} \cdot x_i^\alpha\}_{\beta \in \Lambda(\alpha)}$ converges weakly to $\chi(E_i^\alpha, x_i^\alpha)$ and hence that $\{\sum_{i=1}^{n(\alpha)} h_{i\beta} \cdot x_i^\alpha\}_{\beta \in \Lambda(\alpha)}$ converges weakly to $\sum_{i=1}^{n(\alpha)} \chi(E_i^\alpha, x_i^\alpha) = \eta(\sum_{i=1}^{n(\alpha)} \chi_{E_i^\alpha} \cdot x_i^\alpha)$. For each $\beta \in \Lambda(\alpha)$ define $s_{\alpha, \beta} = \sum_{i=1}^{n(\alpha)} h_{i\beta} \cdot x_i^\alpha$. Then, $\{s_{\alpha, \beta}\}_{\beta \in \Lambda(\alpha)}$, $\alpha \in \Lambda$, satisfies condition (a) of Sublemma 7.3. Suppose p is a continuous seminorm on X and suppose $\varepsilon > 0$. Since $\{s_\alpha\}_\Lambda$ converges to f in the topology of uniform convergence, it follows that there is an α_0 such that $\alpha > \alpha_0$ implies $\rho_p(s_\alpha - f) < \varepsilon$. Suppose $\alpha > \alpha_0$. Then

$$\begin{aligned} \rho_p(s_{\alpha, \beta} - f) &= \sup_{t \in H} p(f(t) - \sum h_{i\beta}(t) \cdot x_i^\alpha) \\ &= \max_{1 \leq i \leq n(\alpha)} \sup_{t \in E_i^\alpha} p(\sum h_{i\beta}(t)(f(t) - x_i^\alpha)) \\ &= \max_{1 \leq i \leq n(\alpha)} \sup_{t \in E_i^\alpha} p(f(t) - x_i^\alpha) = \rho_p(s_\alpha - f) < \varepsilon. \end{aligned}$$

Hence it follows that $\lim_\alpha \rho_p(s_{\alpha, \beta} - f) = 0$. Therefore, $\lim_\alpha s_{\alpha, \beta} - s = \theta_s$, from which it follows that (b) of Sublemma 7.3 is satisfied. Hence, $\{\eta(s_\alpha)\}_\Lambda$ converges to f in CTB . Therefore, Lemma 3.2 implies η is continuous on $S'(\Sigma, X) + CTB$.

7.4 LEMMA. Suppose K_1 and K_2 are weakly regular and give the same transformation on CTB . Then $K_1(E) = K_2(E)$ for all $E \in \Sigma$.

Proof. For each $y' \in Y'$ and $x \in X$, $\langle y', [K(\cdot)](x) \rangle$ is a bounded regular finitely additive set function on Σ and generates a continuous linear functional on CBR . Theorem [1, p. 262] implies such set functions are unique. Thus, if K_1 and K_2 generate the same transformation on CTB , then for each y', x , $\langle y', [K_1(\cdot) - K_2(\cdot)](x) \rangle$ generates the zero functional on CBR . Hence for each E , $[K_1(E) - K_2(E)](x) = \theta_Y$ which in turn implies $K_1(E) - K_2(E) = \theta_{L(X, Y^{\omega})}$.

8. Proofs of Theorems 6.2 and 6.3. The proofs in this section mirror those of §7 and hence only brief indications of proofs are given in this section. As before, our plan is to use Theorem 3.1, and therefore, we must identify the map η and the collection $S'(\Sigma_c, X)$. In this setting $S'(\Sigma_c, X)$ denotes the space of simple functions $\sum_{i=1}^n \chi_{E_i} \cdot x_i$ such that $\bigcup_{i=1}^j E_i$ is a compact subset of H for $j < n$. Let K be a compact subset of H and let Σ_K denote the field generated by the closed subsets of K . Then $\bigcup_{K \subset H} \Sigma_K$ is a ring in H which contains all of the compact sets in H . Hence if Σ_c denotes the ring generated by the compact subsets of H , we have $\Sigma_c \subset \bigcup_{K \subset H} \Sigma_K$. On the other hand if $F \in \bigcup_{K \subset H} \Sigma_K$, then $F \in \Sigma_K$ for some K . We observe that $\mathfrak{A}_K = \{E \cap K \mid E \in \Sigma_c\}$ is a field of sets satisfying $\mathfrak{A}_K \subset \Sigma_c$ which contains all the closed sets in K . Hence $\Sigma_K \subset \mathfrak{A}_K$. This implies $F \in \mathfrak{A}_K$ and consequently $\Sigma_c = \bigcup_{K \subset H} \Sigma_K$. It follows that for each E in Σ_c there is an open set O_E containing E which has compact closure. Define the directed set $\Lambda(E)$ by

$$\Lambda(E) = \{(F, V) : F \subset E \text{ is closed, } E \subset V \subset O_E \text{ is open}\}$$

where the direction is given by $(F', V') > (F, V)$ means $F \subset F'$ and $V' \subset V$. For each $\alpha = (F, V) \in \Lambda(E)$ define (using Urysohn's lemma) a continuous function ϕ_α which has its range in $[0, 1]$ and such that $\phi_\alpha \equiv 1$ on F and $\phi_\alpha \equiv 0$ on \tilde{V} . Suppose T is a linear functional on $C_c R$. As in [8, p. 139], it is possible to decompose T into the difference of two positive linear functions and hence by Theorem 2.14 [9, p. 40] there is a unique Borel measure which is outer regular and inner regular on Borel sets of compact closure such that $T(f) = \int f d\mu$ for $f \in C_c R$. Then proceeding as in the proof of Lemma 7.1, we have the following lemma.

8.1 LEMMA. *For each $E \in \Sigma_c$, the net $\{\phi_\alpha\}_{\alpha \in \Lambda(E)}$ is weakly convergent. Furthermore, if $\Lambda'(E)$ is a directed set which is cofinal with $\Lambda(E)$ and the net $\{\psi_\alpha\}_{\alpha \in \Lambda'(E)}$ is constructed in a fashion analogous to that of $\{\phi_\alpha\}_{\alpha \in \Lambda(E)}$ then the two nets are weakly Cauchy equivalent.*

For each E , since $\{\phi_\alpha\}_{\alpha \in \Lambda(E)}$ is weakly Cauchy in $C_c R$, then $\{\phi_\alpha \cdot x\}_{\alpha \in \Lambda(E)}$ is weakly Cauchy in C_c for each $x \in X$. As before, we let $\chi(E, x)$ denote the weak limit of $\{\phi_\alpha \cdot x\}_{\alpha \in \Lambda(E)}$. Define the linear map η from $C_c + S'(\Sigma_c, X)$ into C_c^ω by $\eta(f + \sum_i \chi_{E_i} \cdot x_i) = f + \sum_i \chi(E_i, x_i)$ for $f \in C_c$ and $\sum_i \chi_{E_i} \cdot x_i \in S'(\Sigma_c, X)$.

8.2 LEMMA. *Suppose η and $S'(\Sigma_c, X)$ are as above. Then,*

(i) *the space C_c is in the closure of $S'(\Sigma_c, X)$ under the topology of uniform convergence,*

(ii) *the linear map η is continuous on the space $C_c + S'(\Sigma_c, X)$,*

(iii) *the linear map η is continuous on the space $S'(\Sigma_c, X)$.*

Furthermore, for each continuous seminorm p on X , $|\eta|_{\rho_p} = 1$.

The proof of this lemma follows as the proof of Lemma 7.2 by using the above characterization of linear functions and by observing that in this setting the function ϕ_α can be constructed to have compact support.

Theorem 6.2 now follows from Theorem 3.1 except for the uniqueness of K . However, the uniqueness of K follows from Theorem 2.14 [9, p. 40] in the same way the uniqueness of K was obtained using Theorem 2 [1, p. 262].

Theorem 6.3 now follows from Theorems 6.2 and 4.7 since C_0 is the closure of C_c in the topology of uniform convergence.

9. Proofs of Theorems 6.5 and 6.6. Theorem 6.5 is first proved for C_c under the topology of uniform convergence on compact subsets. We remark that this topology is a weaker topology than that used in 6.2. Also note that the set function K in Theorems 6.5 and 6.6 is of bounded semivariation on a ring and cannot in general be extended to a set function of bounded semivariation on the field generated by the ring.

9.1 LEMMA. *Suppose H and Σ_c are as in Theorem 6.2. If T is a continuous linear map from C_c into Y , then there is a unique finitely additive set function K on Σ_c with values in $L[X, Y^\omega]$ which is of bounded (p, q) variation with compact q -support such that $T(f) = \int dK \cdot f$. Furthermore, if T is (ρ, q) related then $WK_{\rho, q} = |T|_{\rho, q}$.*

Proof. Define $S'(\Sigma_c, X)$ as in §8 and for each $E \in \Sigma_c$ construct the net $\{\phi_\alpha\}_{\alpha \in \Lambda(E)}$ as in that section. That $\{\phi_\alpha\}_{\alpha \in \Lambda(E)}$ is weakly Cauchy follows from Theorem 4.10.1 in [4, p. 203]. Define η as in §8. That Lemma 8.2 is true in this setting follows as indicated in §8 using Theorem 4.10.1 in [4] in place of Theorem 2.14 in [9]. Hence, it follows from Lemma 3.2 that there is a set function K on Σ_c with values in $L[X, Y^\omega]$ which is quasi-Gowurin with respect to the topology of uniform convergence on compact subsets such that, for $f \in C_c$, $T(f) = \int dK \cdot f$ and such that if T is (ρ, q) related, then $WK_{\rho, q''} = |T|_{\rho, q}$. This implies given a continuous seminorm q on Y , there is a compact set M_q and a continuous seminorm p on X such that

$$\begin{aligned} q''\left(\sum [K(E_i)](x_i)\right) &\leq WK_{\rho, q''} \sup_{t \in M_q} p\left(\sum \chi_{E_i}(t) \cdot x_i\right) \\ &= WK_{\rho, q''} \max_i \{p(x_i) : E_i \cap M_q \neq \emptyset\}. \end{aligned}$$

Therefore, for each x in X , $q''([K(E)](x))$ has as support M_q . Hence, K is of bounded (p, q'') variation with compact q'' -support. The uniqueness of K follows as in §7.

Theorem 6.5 now follows from Theorem 4.7 since C , the space of all continuous functions, is the completion of C_c , the space of continuous functions of compact support, both with the topology of uniform convergence on compact subsets. If Y is a normed space, then K has compact support M , which implies all functions in C are K integrable and that $\int dK \cdot f = \int dK \cdot f_M$ where f_M denotes the restriction of f to M . Also $E \int dK \cdot f = E \int dK \cdot f_M$. Lemma 4.5 implies $E \int dK \cdot f_M = \int dK \cdot f_M$ from which Theorem 6.6 follows.

10. Concluding remarks.

10.1 REMARK. In order to hypothesize the existence of the η map in Theorem 3.1, it is necessary that the dual F' of the function space F separate points of F (which is assured when F has a locally convex topology). This is sufficient to guarantee that F can be imbedded in F'' . Under this additional hypothesis we may retain the view that $F \subset F^\omega$ under the topology of uniform convergence on equicontinuous subsets of F' when X and Y are topological vector spaces. Hence, Theorem 3.1 is valid in this more general setting.

10.2 REMARK. A sufficient condition to obtain the representation theorem of Theorem 3.1 is that T has a continuous extension to $F + S(\Sigma, X)$. A seemingly weaker condition is that the η map of Theorem 3.1 exists. It is interesting to note that the existence of the η map implies the representation theorem which in turn implies that the composite map $T^\omega \eta$ (an extension of T) is continuous. The classical integral representations for functionals on real function spaces follow from Theorem 3.1 using this observation and the Hahn-Banach theorem.

10.3 REMARK. The set functions in Theorems 6.2, 6.3, 6.5, and 6.6 all can be constructed to assume values in $L[X, Y^+]$ rather than $L[X, Y^\omega]$ by using the techniques of the proof in [3] and by changing Σ_c to be the ring of sets generated

by the condition that, for each E in Σ , χ_E is the pointwise limit of a sequence of uniformly bounded continuous functions of uniform compact support.

ADDENDUM. The authors have been interested in obtaining conditions sufficient to insure that F' separates points of F (Remark 10.1). Motivated by the abstract (Abstract 672-422, Notices Amer. Math. Soc. **17** (1970), 203) of A. H. Shuchat which came to the authors' attention after the preceding manuscript was prepared, we obtain the following results.

LEMMA. Suppose F is a topological vector space of functions with topology τ from an arbitrary space H into a topological vector space X and T is a continuous linear operator from F into the topological vector space Y . Suppose further that Y' separates $T[f]$. Let $F_T = F/\text{Ker}(T)$ under the inherited topology. Then F'_T separates points in F_T .

REMARK. If Y' separates points in Y , then F'_T separates points in F_T for all T .

Hence, if $F \subset \Gamma$ and the topology τ on F can be extended to the simple functions, then it is possible to hypothesize the existence of an η map analogous to the one in Theorem 3.1 from $F_T + S_T(\Sigma, X)$ into F_T^ω . Here again, Γ is the closure under the uniform topology of the simple functions $S(\Sigma, X)$ over a ring Σ of subsets of H and τ is a topology not stronger than the topology of uniform convergence, and $S_T(\Sigma, X) = S(\Sigma, X)/(\text{Ker}(T) \cap S(\Sigma, X))$.

THEOREM. Suppose $H, \Sigma, X, Y, F, \tau, S(\Sigma, X), \Gamma$, and T are as above. Suppose further that there is a subspace $S'(\Sigma, X)$ in $S(\Sigma, X)$ such that

- (i) F is contained in the τ closure of $S'(\Sigma, X)$, and
- (ii) there is a linear map η_T from $F_T + S_T(\Sigma, X)$ into F_T^ω such that η_T restricted to each of S_T and $F_T + S'_T(\Sigma, X)$ is continuous.

Then there is a set function K on Σ with values in $L[X, Y^\omega]$ which is quasi-Gowurin with respect to τ and such that, for each $f \in F$, $T(f) = \int dK \cdot f$.

Theorems 6.1, 6.2, 6.3, 6.5, and 6.6 now follow for the case that X and Y are topological vector spaces and Y' separates points of Y .

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