EQUIVARIANT BORDISM AND SMITH THEORY

BY
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Abstract. The relationship between equivariant bordism and Smith homology theory on the category of pairs with involution is studied.

1. Introduction. The object of this paper is to analyze the relationship between the homology theories given by equivariant bordism $\mathfrak{N}_*^{Z_2}(X, A, T)$ and Smith homology theory $H_*^{Z_2}(X, A, T; Z_2)$ on the category of pairs with involution. These theories are related in much the same way as ordinary bordism and homology are related. (See for example [1, §8 and §17].)

In §2, basic definitions will be given, and then it will be shown that assigning to a bordism element $f: (M, \partial M, S) \rightarrow (X, A, T)$ the image of the fundamental Smith theory class of $(M, \partial M, S)$ defines a natural isomorphism

$$\bar{\mu}: \mathfrak{N}_*^{Z_2}(X, A, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2 \xrightarrow{\cong} H_*^{Z_2}(X, A, T; Z_2).$$

It should be noted, however, that one cannot have the other isomorphism relation (valid for ordinary bordism)

$$\mathfrak{N}_*^{Z_2}(X, A, T) \cong H_*^{Z_2}(X, A, T; Z_2) \otimes_{Z_2} \mathfrak{N}_*^{Z_2}$$

for all (X, A, T) since Z_2 bordism is not in general a free $\mathfrak{N}_*^{Z_2}$ module. (If $X=2$ points interchanged by T and $A=\emptyset$, $\mathfrak{N}_*^{Z_2}(X, A, T) \cong \mathfrak{N}_*$, while $H_*^{Z_2}(X, A, T) \otimes_{Z_2} \mathfrak{N}_*^{Z_2} \cong Z_2 \otimes_{Z_2} \mathfrak{N}_*^{Z_2} \cong \mathfrak{N}_*^{Z_2}$, and these are not isomorphic as graded groups.)

In §3, it will be shown that the composite

$$\mu \circ \tau_*: \mathfrak{N}_*^{Z_2}(X, A, T) \rightarrow H_*^{Z_2}(X \times BO, A \times BO, T \times S; Z_2)$$

is monic, where $\tau_*: \mathfrak{N}_*^{Z_2}(X, A, T) \rightarrow \mathfrak{N}_*^{Z_2}(X \times BO, A \times BO, T \times S)$ sends $f: (M, \partial M, Q) \rightarrow (X, A, T)$ to $f \times \tau: (M, \partial M, Q) \rightarrow (X \times BO, A \times BO, T \times S)$ with τ equivariantly classifying the stable tangent bundle of M . Thus, the dual Smith cohomology groups

$$H_{Z_2}^*(X \times BO, A \times BO, T \times S; Z_2)$$

provide characteristic numbers which determine the bordism class. There has been much recent interest in equivariant characteristic numbers (see [5], for example). Much of the effort has been devoted to more difficult cases using modern methods

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with equivariant Thom classes and such (in for example the work of T. tom Dieck or G. Hamrick). The approach taken here is much more naive.

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2. The representation theorem. Let X be a simplicial complex $T: X \rightarrow X$ a simplicial involution ($T^2 = \text{identity}$) and $A \subset X$ an invariant subcomplex ($TA \subset A$). It will be assumed that X is "finely" triangulated so that the fixed set F of T on X is a subcomplex and the orbit map $\pi: X \rightarrow X/Z_2$ is simplicial (E. E. Floyd [3] shows that this may be accomplished by taking the second barycentric subdivision).

Letting $C(X) \otimes Z_2$ denote the chains of X with mod 2 coefficients, one has an induced chain map $T_\#: C(X) \otimes Z_2 \rightarrow C(X) \otimes Z_2$ with $T_\# \circ T_\# = 1$, and one lets $C^0(X)$ denote the subgroup of chains invariant under $T_\#$ ($T_\# \sigma = \sigma$). Since the boundary operator commutes with $T_\#$, one has induced a boundary homomorphism $\partial: C^0(X) \rightarrow C^0(X)$, making $(C^0(X), \partial)$ a complex. One lets $C^0(X, A) = C^0(X)/C^0(A)$, and ∂ makes this into a chain complex. The *Smith homology groups* of (X, A, T) , $H_*^{Z_2}(X, A, T; Z_2)$ are then defined to be the homology groups of the complex $(C^0(X, A), \partial)$.

Now consider a chain $\sigma = \sum a_i \Delta^i$, where Δ^i are n -simplices and $a_i \in Z_2$. The chain σ is invariant under $T_\#$ if and only if, for each n -simplex Δ , the coefficients of Δ and $T_\# \Delta$ in σ are the same. Thus σ is a sum of terms $\Delta + T_\# \Delta$ (if $T_\# \Delta \neq \Delta$) and terms Δ (with $T_\# \Delta = \Delta$), so that $C^0(X)$ decomposes into a direct sum $U \oplus V$, where U is spanned by the $\Delta + T_\# \Delta$ ($T_\# \Delta \neq \Delta$) and V is spanned by the Δ with $T_\# \Delta = \Delta$. Clearly $\partial(\Delta + T_\# \Delta)$ is a sum of terms of the same form for if a face Δ' of Δ is fixed by $T_\#$, Δ' occurs with coefficient $2 \equiv 0$ in $\partial(\Delta + T_\# \Delta)$, while if $T_\# \Delta = \Delta$, then by the "finesseness" of the triangulation, each vertex of Δ must be fixed so each face of Δ is fixed by $T_\#$. Thus $C^0(X)$ is the direct sum of the subcomplexes U and V , inducing a decomposition of $C^0(X, A)$ and $H_*^{Z_2}(X, A, T; Z_2)$.

Now let $\rho: C(X/Z_2) \otimes Z_2 \rightarrow C(X) \otimes Z_2$ be defined as follows. If Δ' is a simplex of X/Z_2 , let Δ be a simplex of X with $\pi \Delta = \Delta'$, and let $\rho(\Delta') = \Delta + T_\# \Delta$, which is well defined since Δ and $T_\# \Delta$ are the two possible lifts of Δ' . Clearly ρ is a chain map with image $U \subset C^0(X)$ and kernel $C(F) \subset C(X/Z_2)$. Thus ρ induces an isomorphism of $C(X/Z_2)/C(F) + C(A/Z_2) = C(X/Z_2, A/Z_2 \cup F)$ with the U summand of $C^0(X, A)$, and the U summand of $H_*^{Z_2}(X, A, T; Z_2)$ is identified with $H_*(X/Z_2, A/Z_2 \cup F; Z_2)$. Clearly the inclusion $i: F \rightarrow X$ identifies $C(F)$ with V , and so the V summand of $H_*^{Z_2}(X, A, T; Z_2)$ is identified with $H_*(F, F \cap A; Z_2)$. This gives the standard result

THEOREM 2.1. $H_*^{Z_2}(X, A, T; Z_2) \cong H_*(X/Z_2, A/Z_2 \cup F; Z_2) \oplus H_*(F, F \cap A; Z_2)$.

By using Čech [6], [4] or singular [2] methods to obtain a complex, this may be extended to topological pairs with involution (X, A, T) . Similarly using

$\text{Hom}(C(X); Z_2)$ one may form Smith cohomology groups dual to the homology groups.

Being given a compact differentiable manifold with boundary M^n with differentiable involution S , one may triangulate M "finely" so that S is simplicial (by triangulating M/Z_2 and lifting the triangulation). Clearly the fundamental cycle $\mu = \sum \Delta^i$, the sum of all n -simplices, is then an invariant chain, defining a fundamental class $[M, \partial M, S] \in H_n^{Z_2}(M, \partial M, S; Z_2)$. This lifts $[M, \partial M] \in H_n(M, \partial M; Z_2)$ back along the forgetful homomorphism from Smith theory to ordinary homology.

One then has a natural homomorphism

$$\mu: \mathfrak{N}_*^{Z_2}(X, A, T) \rightarrow H_*^{Z_2}(X, A, T; Z_2)$$

assigning to the equivariant bordism element $f: (M, \partial M, S) \rightarrow (X, A, T)$ the class $f_*[M, \partial M, S]$.

If $b = (B, U) \in \mathfrak{N}_*^{Z_2}$ and $\alpha = (M, \partial M, S, f) \in \mathfrak{N}_*^{Z_2}(X, A, T)$, their product $b\alpha = (B \times M, B \times \partial M, U \times S, f \circ \pi_M) \in \mathfrak{N}_*^{Z_2}(X, A, T)$ is sent by μ to

$$f_* \circ \pi_{M*}[B \times M, B \times \partial M, U \times S].$$

Since M/Z_2 and its fixed set are complexes of dimension at most that of M , Theorem 1 shows that $\pi_{M*}[B \times M, B \times \partial M, U \times S] = 0$ if the dimension of B is positive. If the dimension of B is zero, with B consisting of k points, then $\pi_M: B \times M \rightarrow M$ is a k -fold cover and π_{M*} sends the fundamental cycle of $B \times M$ to k times that of M . Thus $\mu(b\alpha) = k\mu(\alpha) = \varepsilon(b) \cdot \mu(\alpha)$, where $\varepsilon: \mathfrak{N}_*^{Z_2} \rightarrow Z_2$ is the augmentation to $\mathfrak{N}_*^{Z_2} \cong \mathfrak{N}_0 \cong Z_2$.

Thus μ induces a natural homomorphism

$$\bar{\mu}: \mathfrak{N}_*^{Z_2}(X, A, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2 \rightarrow H_*^{Z_2}(X, A, T; Z_2).$$

PROPOSITION 2.1. $\bar{\mu}$ is epic.

Proof. It suffices to show that $\bar{\mu}$ maps onto each of the summands of $H_*^{Z_2}(X, A, T; Z_2)$.

If $\alpha \in H_*(F, F \cap A; Z_2)$, there is an ordinary bordism class $f: (N, \partial N) \rightarrow (F, F \cap A)$ with $f_*[N, \partial N] = \alpha$. Letting 1 denote the trivial involution on N , $f: (N, \partial N, 1) \rightarrow (X, A, T)$ is an equivariant bordism element with $f_*[N, \partial N, 1] = \alpha$.

If $\alpha \in H_*(X/Z_2, A/Z_2 \cup F; Z_2)$, there is an ordinary bordism element $f: (N, \partial N) \rightarrow (X/Z_2, A/Z_2 \cup F)$ with $f_*[N, \partial N] = \alpha$. Using excision and a neighborhood of F which retracts by a deformation to F , one may find a double cover $\tilde{N} \xrightarrow{\pi'} N$, with covering involution S and an equivariant map $\tilde{f}: \tilde{N} \rightarrow X$ with the diagram

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{f}} & X \\ \pi' \downarrow & & \downarrow \pi \\ N & \xrightarrow{f} & X/Z_2 \end{array}$$

commutative. Then $\tilde{f}(\partial N) \subset A \cup F$ and by applying excision methods to a neighborhood of A , one may after a small equivariant deformation if needed find a

decomposition of ∂N into two invariant submanifolds $\partial N = N_0 \cup N_1$ with $N_0 \cap N_1 = \partial N_0 = \partial N_1$ with $\tilde{f}(N_0) \subset A$, $\tilde{f}(N_1) \subset F$. Let $\hat{N}$ be the quotient of $\tilde{N}$ obtained by identifying x with Sx for $x \in N_1$, with $\hat{f}: \hat{N} \rightarrow X$ induced by $\tilde{f}$, $\hat{\pi}: \hat{N} \rightarrow N$ induced by π' and with involution $\hat{S}$ induced by S . Then $\hat{f}: (\hat{N}, \partial \hat{N}, S) \rightarrow (X, A, T)$ is an equivariant bordism element and $\hat{f}_\#$ sends the fundamental cycle of $\hat{N}$ to the lift by ρ of $f_\# \mu$ where μ is the fundamental cycle of N . Then $\hat{f}_\# [\hat{N}, \partial \hat{N}, S]$ is α in $H_*^{Z_2}(X, A, T)$.

Applying an excision to identify $(A \cup F, A)$ with $(F, F \cap A)$, the bordism exact sequence of the triple $(X, A \cup F, A)$ gives an exact sequence

$$\begin{array}{ccccc} \mathfrak{N}_*^{Z_2}(F, F \cap A, 1) & \xrightarrow{i} & \mathfrak{N}_*^{Z_2}(X, A, T) & \xrightarrow{j} & \mathfrak{N}_*^{Z_2}(X, A \cup F, T) \\ & & \partial & & \uparrow \\ & & & & \mathfrak{N}_*^{Z_2}(F, F \cap A, 1) \end{array}$$

and one has

LEMMA 2.1. *The homomorphism ∂ is zero.*

Proof. (1) Applying the fixed point homomorphism

$$F: \mathfrak{N}_*^{Z_2}(X, A, T) \rightarrow \bigoplus_{k=0}^* \mathfrak{N}_{*-k}(F \times BO_k, (F \cap A) \times BO_k)$$

of [7, §3], $F \circ i$ is a monomorphism onto a direct summand complementary to the $k=1$ term, so i is monic, or ∂ is zero.

(2) Any class $\alpha \in \mathfrak{N}_*^{Z_2}(X, A \cup F, T)$ may be represented by a map $f: (N, \partial N, S) \rightarrow (X, A \cup F, T)$, and by excising a neighborhood of the fixed set of N , one may suppose S is free. By excision and a small deformation of f one may suppose $\partial N = N_0 \cup N_1$, $\partial N_0 = \partial N_1 = N_0 \cap N_1$ with $f(N_0) \subset A$, $f(N_1) \subset F$. Let $\bar{N}$ be the quotient of N by identifying x with Sx for $x \in N_1$, with $\bar{f}: \bar{N} \rightarrow X$ and $\bar{S}: \bar{N} \rightarrow \bar{N}$ induced by f and S . Then $\bar{f}: (\bar{N}, \partial \bar{N}, \bar{S}) \rightarrow (X, A, T)$ has image α under j .

Note. This defines a splitting, identifying $\mathfrak{N}_*^{Z_2}(X, A \cup F, T)$ with the bordism classes for which the fixed point set has codimension 1.

Special Note. The splitting $\varphi: \mathfrak{N}_*^{Z_2}(X, A \cup F, T) \rightarrow \mathfrak{N}_*^{Z_2}(X, A, T)$ just constructed is only a splitting as $\mathfrak{N}_*$ modules and not as $\mathfrak{N}_*^{Z_2}$ modules (see [7, p. 57, note (4)]). However, being given $f: (N, \partial N, S) \rightarrow (X, A \cup F, T)$ representing α with S free and $\partial N = N_0 \cup N_1$, $f(N_0) \subset A$, $f(N_1) \subset F$, as above, and $m = [M, U] \in \mathfrak{N}_*^{Z_2}$, $\varphi(m\alpha) - m\varphi(\alpha)$ is the class of

$$g: M \times I \times N_1 / \{(m, 0, n) \sim (Um, 0, Sn), (m, 1, n) \sim (m, 1, Sn)\} \rightarrow F$$

by $g(m, t, n) = f(n)$ and with involution induced by $U \times 1 \times S$. In particular, $f: (N_1, \partial N_1, S) \rightarrow (F, F \cap A, 1)$ is a free Z_2 bordism element and is a sum of products $u \otimes v$ with $u \in \mathfrak{N}_*^{Z_2}(\text{Free})(\text{point})$ and $v \in \mathfrak{N}_*(F, F \cap A)$. Crossing with $M \times I$ and acting as above sends this to a term $u' \otimes v$ with $u' \in \mathfrak{N}_*^{Z_2}$, and since $\dim I > 0$, $\dim u' > 0$. Thus $\varphi(m\alpha) - m\varphi(\alpha)$ is decomposable in the $\mathfrak{N}_*^{Z_2}$ module structure, so that φ induces a splitting

$$\varphi': \mathfrak{N}_*^{Z_2}(X, A \cup F, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2 \rightarrow \mathfrak{N}_*^{Z_2}(X, A, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2$$

for

$$j': \mathfrak{N}_*^{Z_2}(X, A, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2 \rightarrow \mathfrak{N}_*^{Z_2}(X, A \cup F, T) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2.$$

PROPOSITION 2.2. $\bar{\mu}$ is an isomorphism.

Proof. Tensoring the short exact sequence of the lemma with Z_2 and applying $\bar{\mu}$ gives a commutative diagram

$$\begin{array}{ccccccc}
 Q & \longrightarrow & \mathfrak{N}_*^{Z_2}(F, F \cap A) \otimes Z_2 & \xrightarrow{i'} & \mathfrak{N}_*^{Z_2}(X, A) \otimes Z_2 & \xrightarrow{j'} & \mathfrak{N}_*^{Z_2}(X, A \cup F) \otimes Z_2 \longrightarrow 0 \\
 & & \downarrow a \downarrow \bar{\mu} & & \downarrow b \downarrow \bar{\mu} & & \downarrow c \downarrow \bar{\mu} \\
 0 & \longrightarrow & H_*^{Z_2}(F, F \cap A) & \xrightarrow{i_*} & H_*^{Z_2}(X, A) & \xrightarrow{j_*} & H_*^{Z_2}(X, A \cup F) \longrightarrow 0 \\
 & & \wr \parallel & & & & \wr \parallel \\
 & & H_*(F, F \cap A) & & & & H_*(X/Z_2, A/Z_2 \cup F)
 \end{array}$$

in which the bottom exact sequence is split and gives the direct sum decomposition of Theorem 2.1, with extraneous data being dropped from the notation, and with Q being a Tor-term.

First, a is an isomorphism. Since $\mathfrak{N}_*^{Z_2}$ is a free $\mathfrak{N}_*$ module, the multiplication $\mathfrak{N}_*^{Z_2} \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(F, F \cap A) \rightarrow \mathfrak{N}_*^{Z_2}(F, F \cap A)$ is an isomorphism, being an isomorphism of homology theories on the category of pairs (with trivial action). Thus $\mathfrak{N}_*^{Z_2}(F, F \cap A) \otimes Z_2$ is isomorphic to

$$Z_2 \otimes_{\mathfrak{N}_*^{Z_2}} \mathfrak{N}_*^{Z_2} \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(F, F \cap A) \cong Z_2 \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(F, F \cap A) \cong H_*(F, F \cap A),$$

and these isomorphisms are just $\bar{\mu} = a$. In particular, $i_*a = bi'$ is monic, so i' is monic, and the top sequence is short exact.

Next, c is an isomorphism. First, the pair $(X, A \cup F)$ is relatively free as a Z_2 pair, so $\mathfrak{N}_*^{Z_2}(X, A \cup F)$ is isomorphic to the free bordism group $\mathfrak{N}_*^{Z_2}(\text{Free})(X, A \cup F)$. Assigning to a free bordism element $f: (N, \partial N, S) \rightarrow (X, A \cup F, T)$ the map $\bar{f}: (N/Z_2, \partial N/Z_2) \rightarrow (X/Z_2, A/Z_2 \cup F)$ gives an $\mathfrak{N}_*$ module isomorphism

$$\rho: \mathfrak{N}_*^{Z_2}(X, A \cup F) \rightarrow \mathfrak{N}_*(X/Z_2, A/Z_2 \cup F).$$

Clearly $\bar{f}_*[N/Z_2, \partial N/Z_2]$ in $H_*(X/Z_2, A/Z_2 \cup F)$ represents $\mu(N, \partial N, S, f)$ under the identification with the Smith group, and hence

$$\hat{\mu}: \mathfrak{N}_*^{Z_2}(X, A \cup F) \otimes_{\mathfrak{N}_*} Z_2 \rightarrow H_*^{Z_2}(X, A \cup F)$$

induced by μ is an isomorphism. Letting

$$\theta: \mathfrak{N}_*^{Z_2}(X, A \cup F) \otimes_{\mathfrak{N}_*} Z_2 \rightarrow \mathfrak{N}_*^{Z_2}(X, A \cup F) \otimes_{\mathfrak{N}_*^{Z_2}} Z_2$$

be the quotient, θ and $\bar{\mu}$ are epic while $\hat{\mu} = \bar{\mu} \circ \theta$ is an isomorphism. Thus θ and $c = \bar{\mu}$ are isomorphisms.

Note. This indicates that free bordism as $\mathfrak{N}_*^{Z_2}$ module essentially is given by the $\mathfrak{N}_*$ module structure: This is not precise, however.

Finally, since a and c are isomorphisms, so is b by the five lemma, which gives the proposition.

3. The characteristic number theorem. Let $R^\infty \oplus R^\infty$ denote the countable direct sum of copies of R , with involution $s: R^\infty \oplus R^\infty \rightarrow R^\infty \oplus R^\infty: (x, y) \rightarrow (x, -y)$, and let BO_n be the space of n -dimensional subspaces of $R^\infty \oplus R^\infty$ with

involution S induced by s . Then (BO_n, S) is a classifying space for n -plane bundles with involution over reasonably decent spaces with involution. The inclusion $R^\infty \oplus R^\infty \rightarrow R \oplus R^\infty \oplus R^\infty: (x, y) \rightarrow (0, x, y)$ and identification of $R \oplus R^\infty$ with R^∞ (with trivial involutions) gives an equivariant inclusion $i: BO_n \rightarrow BO_{n+1}$, so that if γ_n denotes the universal n -plane bundle (with involution) $i^*(\gamma_{n+1}) \cong \gamma_n \oplus 1_+$, with 1_+ being the trivial bundle with trivial involution. Let (BO, S) be the limit of the sequence of BO_n 's with these maps.

If (M, Q) is a compact n -manifold with boundary, the tangent bundle of M , as bundle with involution, is classified by a map $\tau: (M, Q) \rightarrow (BO_n, S)$. If (M, Q) is a regularly imbedded invariant submanifold of ∂V with (V, Q') an involution, then the tangent bundle of V restricts to $\tau_M \oplus 1_+$ on M , which is classified by $i \circ \tau: (M, Q) \rightarrow (BO_{n+1}, S)$. Thus one has a well-defined homotopy class of maps $\tau_M: (M, Q) \rightarrow (BO, S)$ classifying the stable tangent bundle of M , and if (M, Q) is regularly imbedded in the boundary of (V, Q') by $f: M \rightarrow \partial V$, then $\tau_V \circ f = \tau_M$.

Being given a pair (X, A, T) with involution, one then defines a natural transformation

$$\tau_*: \mathfrak{N}_*^{Z_2}(X, A, T) \rightarrow \mathfrak{N}_*^{Z_2}(X \times BO, A \times BO, T \times S)$$

by sending the class of $f: (M, \partial M, Q) \rightarrow (X, A, T)$ to the class of $f \times \tau_M: (M, \partial M, Q) \rightarrow (X \times BO, A \times BO, T \times S)$, where τ_M classifies the stable tangent bundle of M equivariantly.

The remainder of this section will be devoted to a proof of

PROPOSITION 3.1. *The composite*

$$\mathfrak{N}_*^{Z_2}(X, A, T) \xrightarrow{\tau_*} \mathfrak{N}_*^{Z_2}(X \times BO, A \times BO, T \times S) \xrightarrow{\mu} H_*^{Z_2}(X \times BO, A \times BO, T \times S; Z_2)$$

is monic.

Proof. $\mu \circ \tau_*$ is a natural transformation of equivariant homology theories, and so one has a commutative diagram

$$\begin{array}{ccc}
 0 & & \\
 \downarrow & & \\
 \mathfrak{N}_*^{Z_2}(F, F \cap A) & \xrightarrow[\alpha]{\mu \circ \tau_*} & H_*^{Z_2}(F \times BO, (F \cap A) \times BO) \\
 i \downarrow & & i_* \downarrow \\
 \mathfrak{N}_*^{Z_2}(X, A) & \xrightarrow[\beta]{\mu \circ \tau_*} & H_*^{Z_2}(X \times BO, A \times BO) \\
 j \downarrow & & j_* \downarrow \\
 \mathfrak{N}_*^{Z_2}(X, A \cup F) & \xrightarrow[\gamma]{\mu \circ \tau_*} & H_*^{Z_2}(X \times BO, (A \cup F) \times BO) \\
 \downarrow & & \\
 0 & &
 \end{array}
 \quad \left. \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right\} \partial$$

Since $F \times BO$ contains the fixed set of $X \times BO$, the pair $(X \times BO, (A \cup F) \times BO)$ is relatively free, and hence

$$H_*^{Z_2}(X \times BO, (A \cup F) \times BO) \cong H_*((X \times BO)/Z_2, ((A \cup F) \times BO)/Z_2).$$

If Z is a free Z_2 space, so is $Z \times BO$ and if $\gamma_r \rightarrow Z \times BO_r$ is the pull back of the universal bundle the quotient $E(\gamma_r)/Z_2 \rightarrow Z \times BO_r/Z_2$ is an r -plane bundle over $Z \times BO_r/Z_2$ and is classified by a map $\tilde{\gamma}_r: Z \times BO_r/Z_2 \rightarrow BO_r$. The bundle 1_+ over $Z \times BO_r$ gives a trivial bundle over $Z \times BO_r/Z_2$ and one obtains a map $\tilde{\gamma}: Z \times BO/Z_2 \rightarrow BO$. Letting $\tilde{\pi}: Z \times BO/Z_2 \rightarrow Z/Z_2$ be induced by projection on Z , one has

$$\tilde{\pi} \times \tilde{\gamma}: Z \times BO/Z_2 \rightarrow (Z/Z_2) \times BO$$

and this induces an isomorphism on Z_2 cohomology. (It is clearly an isomorphism on bordism for the free bordism of $Z \times BO$ may be identified with the bordism of $(Z/Z_2) \times BO$ by sending $f \times \xi: M \rightarrow Z \times BO$ to the class of $\tilde{f} \times \tilde{\xi}: M/Z_2 \rightarrow Z/Z_2 \times BO$ with $\tilde{f}$ induced by f and $\tilde{\xi}$ classifying $E(\xi)/Z_2 \rightarrow M/Z_2$.) By excision, this also holds in the relatively free case, so

$$H_*^{Z_2}(X \times BO, (A \cup F) \times BO) \cong H_*((X/Z_2) \times BO, (A/Z_2 \cup F) \times BO).$$

Now $\mathfrak{N}_*^{Z_2}(X, A \cup F)$ is isomorphic to the free bordism and so to $\mathfrak{N}_*(X/Z_2, A/Z_2 \cup F)$, sending a free map $f: (M, \partial M) \rightarrow (X, A \cup F)$ to the class of $\tilde{f}: (M/Z_2, \partial M/Z_2) \rightarrow (X/Z_2, A/Z_2 \cup F)$. Applying τ_* and $\tilde{\pi} \times \tilde{\gamma}$ sends $(M, \partial M, f)$ to the image of the fundamental class of

$$\tilde{f} \times \tilde{\tau}: (M/Z_2, \partial M/Z_2) \rightarrow (X/Z_2 \times BO, (A/Z_2 \cup F) \times BO)$$

but $\tilde{\tau}$ classifies the stable tangent bundle of M/Z_2 . Thus, the diagram

$$\begin{array}{ccc} \mathfrak{N}_*^{Z_2}(X, A \cup F) & \xrightarrow{\gamma} & H_*^{Z_2}(X \times BO, (A \cup F) \times BO) \\ \downarrow \cong & & \downarrow \cong \\ \mathfrak{N}_*(X/Z_2, A/Z_2 \cup F) & \xrightarrow{c} & H_*((X/Z_2) \times BO, (A/Z_2 \cup F) \times BO) \end{array}$$

commutes, with $c = \mu \circ \tau_*$ in ordinary bordism, which is monic. Thus γ is monic.

Next, consider the fixed set of (BO, S) . Clearly, $FBO_n = \bigcup_{j+k=n} BO_j \times BO_k$ with $BO_j \times BO_k$ given by the planes in $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$ of the form $U \oplus V$, $\dim U = j$, $\dim V = k$. Since $i: BO_n \rightarrow BO_{n+1}$ sends $BO_j \times BO_k$ into $BO_{j+1} \times BO_k$, one has $FBO = \bigcup_{k=0}^\infty BO \times BO_k$. Let $BO' \subset BO$ denote the fixed component $BO \times BO_0 \cong BO$. One then has an equivariant inclusion

$$q: (X \times BO', A \times BO') \rightarrow (X \times BO, A \times BO)$$

with $F(X \times BO') = F \times BO'$. In Smith theory, one then has

$$\begin{array}{ccc}
 H_*^{Z_2}(X \times BO', A \times BO') & \xrightarrow{q_*} & H_*^{Z_2}(X \times BO, A \times BO) \\
 \wr \parallel & & \downarrow pj_* \\
 H_*\left(\frac{X}{Z_2} \times BO', \left(\frac{A}{Z_2} \cup F\right) \times BO'\right) & & H_*\left(\frac{X}{Z_2} \times BO, \left(\frac{A}{Z_2} \cup F\right) \times BO\right) \\
 \oplus & & \\
 H_*(F \times BO', (F \cap A) \times BO') & &
 \end{array}$$

and it is immediate that the composite pj_*q_* on the first summand is just the homomorphism induced by the map

$$\frac{Z}{Z_2} \times BO' \xrightarrow{\alpha} \frac{Z \times BO}{Z_2} \xrightarrow{\tilde{\pi} \times \tilde{\gamma}} \frac{Z}{Z_2} \times BO$$

(on free spaces after excision). Now BO' is just a copy of BO and clearly $\tilde{\pi} \circ \alpha$ is the projection on Z/Z_2 while $\tilde{\gamma} \circ \alpha$ classifies the universal bundle over BO' , so $(\tilde{\pi} \times \tilde{\gamma}) \circ \alpha$ is an equivalence. Thus pj_*q_* is epic, but p being an isomorphism, j_* is then epic.

The Smith theory sequence of the main diagram is then short exact, j_* being epic, with γ monic. By standard five lemma type arguments, β will be monic if α is monic.

In order to prove α is monic, one must consider $\mathfrak{N}_*^{Z_2}(F, F \cap A)$. One has an isomorphism

$$P: \bigoplus_{k=0; k \neq 1}^* \mathfrak{N}_{*-k}(F \times BO_k, (F \cap A) \times BO_k) \rightarrow \mathfrak{N}_*^{Z_2}(F, F \cap A)$$

which assigns to $f: (M, \partial M) \rightarrow (F \times BO_k, (F \cap A) \times BO_k)$ the class obtained as follows. Let ξ be the k -plane bundle over M induced by $\pi_2 \circ f$, $\bar{M} = D(\xi)/\{x \sim -x \mid x \in S(\xi)\}$ with involution Q induced by -1 in the fibers of ξ and with projection $\pi: \bar{M} \rightarrow M$ induced by projection in $D(\xi)$. Then $P(M, \partial M, f)$ is the class of $\pi_1 \circ f \circ \pi: (\bar{M}, \partial \bar{M}) \rightarrow (F, F \cap A)$. If $k=0$, $\bar{M}$ is just M with the trivial involution.

Then $H_*^{Z_2}(F \times BO, (F \cap A) \times BO)$ may be identified with the direct sum of $R = H_*(F \times BO/Z_2, (F \cap A) \times BO/Z_2 \cup F \times FBO)$ and

$$S = H_*(F \times FBO, (F \cap A) \times FBO).$$

First, if $\alpha = (M, \partial M, f) \in \mathfrak{N}_*(F \times BO_0, (F \cap A) \times BO_0)$, $\mu \circ \tau_* \circ P(\alpha)$ is an element of S since $\bar{M}$ is pointwise fixed by Q . Now S decomposes as a direct sum corresponding to the components of FBO , with $\tau: \bar{M} \rightarrow FBO$ mapping into the fixed component $BO \times BO_0 = BO'$. Clearly

$$\mu \circ \tau_* \circ P: \mathfrak{N}_*(F \times BO_0, (F \cap A) \times BO_0) \rightarrow H_*(F \times BO \times BO_0, (F \cap A) \times BO \times BO_0)$$

is just $\mu \circ \tau_*$ in ordinary bordism, hence is monic.

Next, if $\alpha = (M, \partial M, f) \in \mathfrak{N}_{*-k}(F \times BO_k, (F \cap A) \times BO_k)$, $k > 1$, the fixed set of $\bar{M}$ is M (image of zero section, codimension k) and $RP(\xi)$ (image of $S(\xi)$, codimension 1) so that $\mu \circ \tau_* \circ P(\alpha)$ lies in the summand R . In order to evaluate $\mu \circ \tau_* \circ P(\alpha)$ one defines a boundary homomorphism

$$\partial: R \rightarrow H_*(F \times FBO \times BZ_2, (F \cap A) \times FBO \times BZ_2) = T.$$

Specifically, one considers $\mathfrak{N}_*^{Z_2}(F \times BO, (F \cap A) \times BO \cup F \times FBO)$ as the free bordism and takes the boundary to the free bordism

$$\mathfrak{N}_*^{Z_2}(\text{Free})(F \times FBO, (F \cap A) \times FBO)$$

which is isomorphic to $\mathfrak{N}_*(F \times FBO \times BZ_2, (F \cap A) \times FBO \times BZ_2)$. Equivalently $(F \times BO \times EZ_2, ((F \cap A) \times BO \cup F \times FBO) \times EZ_2)$ projects to

$$(F \times BO, (F \cap A) \times BO \cup F \times FBO)$$

inducing an isomorphism on Smith theory, and one applies the boundary to

$$\begin{aligned} H_*^{Z_2}(((F \cap A) \times BO \cup F \times FBO) \times EZ_2, (F \cap A) \times BO \times EZ_2) \\ \cong H_*^{Z_2}(F \times FBO \times EZ_2, (F \cap A) \times FBO \times EZ_2) \\ \cong H_*(F \times FBO \times BZ_2, (F \cap A) \times FBO \times BZ_2) = T. \end{aligned}$$

T is, of course, a direct sum over the components of FBO .

One now analyzes $\partial \circ \mu \circ \tau_* \circ P(\alpha)$. Specifically, $\tau: \bar{M} \rightarrow BO$ is equivariant sending M into the fixed component $BO \times BO_k$ to classify $\tau_M \oplus \xi$ (the eigenbundles) while $RP(\xi)$ maps into $BO \times BO_1$ classifying $\tau_{RP(\xi)} \oplus \lambda$ where λ is the canonical line bundle (the normal bundle in $\bar{M} = RP(\xi \oplus 1)$). Excising the fixed set in $\bar{M}$ gives a free bordism element $S(\xi) \times [0, 1]$ with involution -1×1 which maps into $F \times BO$ so that $S(\xi) \times 0 \rightarrow F \times FBO$ by projection on M composed with $\pi_1 f \times (\tau_M \times \xi)$, hence maps into $F \times BO \times BO_k$, and $S(\xi) \times 1 \rightarrow F \times FBO$ is given by projection on $RP(\xi)$ composed with $(\pi_1(f \circ \pi) \times (\tau_{RP(\xi)} \times \lambda))$, hence maps into $F \times BO \times BO_1$.

Taking the boundary, and excising $S(\xi|_{\partial M}) \times [0, 1]$ which maps into $(F \cap A) \times BO$, one has the remaining boundary given by $S(\xi) \times \{0, 1\}$ and maps as described. Thus $\partial \cdot \mu \cdot \tau_* \cdot P(\alpha)$ is the image of the fundamental classes of

$$(RP(\xi), \partial RP(\xi)) \rightarrow (F \times BO \times BO_k \times BZ_2, (F \cap A) \times BO \times BO_k \times BZ_2)$$

and

$$(RP(\xi), \partial RP(\xi)) \rightarrow (F \times BO \times BO_1 \times BZ_2, (F \cap A) \times BO \times BO_1 \times BZ_2)$$

with the maps given by $RP(\xi) \xrightarrow{\pi} M \xrightarrow{f \times \iota} F \times BO$, $RP(\xi) \xrightarrow{\pi} M \xrightarrow{\xi} BO_k$, $RP(\xi) \xrightarrow{\lambda} BO_1$ and $RP(\xi) \rightarrow BZ_2$ classifying the double cover by $S(\xi)$.

Letting

$$\pi_k: T \rightarrow H_*(F \times BO \times BO_k \times BZ_2, (F \cap A) \times BO \times BO_k \times BZ_2)$$

be the projection for the k th component, one has

$$\begin{aligned} \pi_k \circ \partial \circ \mu \circ \tau_* \circ P: \mathfrak{N}_{*-k}(F \times BO_k, (F \cap A) \times BO_k) \\ \rightarrow H_*(F \times BO \times BO_k \times BZ_2, (F \cap A) \times BO \times BO_k \times BZ_2) \end{aligned}$$

sending $(M, \partial M, f)$ into the image of the fundamental class of $RP(\xi)$ given by the maps

$$RP(\xi) \xrightarrow{\pi} M \xrightarrow{\pi_1 f \times \tau \times \xi} F \times BO \times BO_k \quad \text{and} \quad RP(\xi) \xrightarrow{\sigma} BZ_2$$

classifying the double covers $S(\xi)$. To see that $\pi_k \circ \partial \circ \mu \circ \tau_* \circ P$ is monic, let $c \in H^1(BZ_2, Z_2)$ denote the generator, so that $H^*(RP(\xi), \partial RP(\xi); Z_2)$ is the free $H^*(M, \partial M; Z_2)$ module via π^* on 1, $\sigma^*(c), \dots, \sigma^*(c)^{k-1}$. Then for any $x \in H^*((F, F \cap A) \times BO \times BO_k)$,

$$\begin{aligned} \langle x \otimes c^{k-1}, \pi_k \partial \mu \tau_* P(\alpha) \rangle &= \langle \pi^*(\pi_1 f \times \tau \times \xi)^*(x) \cup \sigma^*(c)^{k-1}, [RP(\xi), \partial RP(\xi)] \rangle \\ &= \langle (\pi_1 f \times \tau \times \xi)^*(x), [M, \partial M] \rangle \end{aligned}$$

but the homomorphism

$$\mathfrak{N}_{*-k}(F \times BO_k, (F \cap A) \times BO_k) \rightarrow H_*(F \times BO \times BO_k, (F \cap A) \times BO \times BO_k)$$

sending $(M, \partial M, f)$ to $(\pi_1 f \times \tau \times \xi)_*[M, \partial M]$ is just $\mu \circ \tau_*$ in ordinary bordism, which is monic. Since homology and cohomology are dual, the numbers $\langle x \otimes c^{k-1}, \pi_k \partial \mu \tau_* P(\alpha) \rangle$ then determine α .

Thus, the direct sum of the homomorphisms $\pi_k \circ \partial \circ \mu \circ \tau_* \circ P$ is monic, so $\partial \circ \mu \circ \tau_* \circ P$ is monic to R on the $k \neq 0$ summand. Thus $\mu \circ \tau_* \circ P$ is monic to $R \oplus S$, and since P is epic, $\mu \circ \tau_* = \alpha$ is monic.

This completes the proof of the proposition.

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