INTERPOLATION THEOREMS FOR THE PAIRS OF SPACES (L^p, L^∞) AND $(L^1, L^q)^{(1)}$

BY

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Abstract. A Banach space Z has the interpolation property with respect to the pair (X, Y) if each T , which is a bounded linear operator from X to X and from Y to Y , can be extended to a bounded linear operator from Z to Z . If $X=L^p$, $Y=L^\infty$ we give a necessary and sufficient condition for a Banach function space Z on $(0, I)$, $0 < I \leq +\infty$, to have this property. The condition is that $g <^p f$ and $f \in Z$ should imply $g \in Z$; here $g <^p f$ means that $g^{**} < f^{**}$ in the Hardy-Littlewood-Pólya sense, while h^* denotes the decreasing rearrangement of the function $|h|$.

If the norms $\|T\|_X$, $\|T\|_Y$ are given, we can estimate $\|T\|_Z$. However, there is a gap between the necessary and the sufficient conditions, consisting of an unknown factor not exceeding λ_p , $\lambda_p \leq 2^{1/q}$, $1/p + 1/q = 1$.

Similar results hold if $X=L^1$, $Y=L^q$. For all these theorems, the complete continuity of T on Z is assured if T has this property on X or on Y , and if Z satisfies a certain additional necessary and sufficient condition, expressed in terms of $\|\sigma_a\|_Z$, $a > 0$, where σ_a is the compression operator $\sigma_a f(t) = f(at)$, $0 \leq t < I$.

1. Introduction. Let X , Y and Z be Banach spaces, and let $\mathcal{B}(X)$ denote the totality of bounded linear operators acting on X , let $\mathcal{B}(X, Y) = \mathcal{B}(X) \cap \mathcal{B}(Y)$. Also, let $\mathcal{B}(X, Y; K_1, K_2)$ denote the set of all operators in $\mathcal{B}(X, Y)$ satisfying $\|T\|_X \leq K_1$ and $\|T\|_Y \leq K_2$. The space Z is said to have the *interpolation property for the pair* (X, Y) , if for every $T \in \mathcal{B}(X, Y)$, T (or its unique extension $\hat{T}$ to Z) belongs to $\mathcal{B}(Z)$. The space Z has the *interpolation property for the pair* (X, Y) *in the strong sense*, if T has the interpolation property for (X, Y) and if $\|T\|_Z$ (or $\|\hat{T}\|_Z$) is majorized by a positive constant depending only on $\|T\|_X$ and $\|T\|_Y$. In the sequel, $I = (0, I)$ will be a (finite or infinite) interval of the real line, and $(X, \|\cdot\|_X)$ will be a Banach function space of locally Lebesgue integrable functions on I satisfying the following conditions:

(1.1) $|g| \leq |f|$, $f \in X$ implies $g \in X$ and $\|g\|_X \leq \|f\|_X$;

(1.2) The norm $\|\cdot\|_X$ is semicontinuous:

$$0 \leq f_n \uparrow f, \alpha = \sup_{n \geq 1} \|f_n\|_X < \infty \text{ imply } f = \bigcup_{n=1}^{\infty} f_n \in X \text{ and } \|f\|_X = \alpha.$$

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For a positive measurable function f , $d_f(y) = m[t : f(t) > y]$, $y \geq 0$, is the *distribution function* of f . Two positive functions f, g are *equimeasurable*, $f \sim g$, if they have the same distribution function. The space X is called *weakly rearrangement invariant* (*rearrangement invariant*), if $0 \leq f \in X$ and $f \sim g$ imply $g \in X$ (resp. $\|g\|_X \leq \gamma \|f\|_X$, where γ is a fixed constant independent upon f and g). We write L^p for $L^p(I)$, $1 \leq p \leq \infty$, and $\|\cdot\|_p$ for the L^p -norm on I . In his paper [2] A. P. Calderón showed that X has the interpolation property for the pair (L^1, L^∞) if and only if X is rearrangement invariant. In §3 and §4 we shall study the interpolation property for the pairs (L^p, L^∞) , $1 \leq p < \infty$, and (L^1, L^q) , $1 < q < \infty$, respectively. We characterize the Banach function spaces having the interpolation property for these pairs (Theorems 2 and 3), extending the results of [2], [11]. In §5 the complete continuity of operators acting on interpolated spaces will be dealt with. Results similar to those of [14] will be obtained, and a special case when X is an Orlicz space will be discussed in the last section.

Let X and Y be Banach function spaces consisting of locally integrable functions. By $X + Y$ we denote the set of all functions f of the form $f = f_1 + f_2$, where $f_1 \in X$ and $f_2 \in Y$. If $Z \subset X + Y$, then each operator $T \in \mathcal{B}(X, Y)$ has a natural extension onto Z . For $f \in Z$, we write $f = f_1 + f_2$, and define $Tf = Tf_1 + Tf_2$. Since T is linear, the value of Tf does not depend on the choice of f_1 and f_2 . An extension of T in this sense will be again denoted by T .

2. Quasi-orders. For a measurable function f on $I(0, l)$, f^* will denote the decreasing rearrangement of $|f|$, that is, the inverse function of $d_{|f|}(y)$, whenever it is finite. By S we denote the set of all positive simple functions, vanishing outside of a set of finite measure. It is easy to see that f^* is defined if f is locally integrable.

The main tool of this paper is different quasi-order relations between measurable functions f, g . One of them is the Hardy-Littlewood-Pólya relation $g < f$ for locally integrable f, g , which means that

$$(2.1) \quad \int_0^x g^*(t) dt \leq \int_0^x f^*(t) dt, \quad x \geq 0.$$

Although this relation is classical, some new properties of it were found in [10]. Here is a further property:

THEOREM 1. *Let $g_1 + g_2 < f$, all these functions being locally integrable and positive. Then there exist positive f_1, f_2 for which $f = f_1 + f_2$, $g_i < f_i$, $i = 1, 2$.*

LEMMA 1. *Let $g < f$, where g, f are positive and g a decreasing function in S :*

$$g = \sum_{v=1}^n \alpha_v \chi_{(c_{v-1}, c_v)}, \quad 0 = c_0 < \dots < c_n \leq l, \quad \alpha_1 \geq \dots \geq \alpha_n \geq 0.$$

Then there exist mutually disjoint sets e_v , $v = 1, \dots, n$, with the following properties:

$$(2.2) \quad m e_v = c_v - c_{v-1},$$

$$(2.3) \quad \alpha_v m e_v \leq \int f \chi_{e_v} dt.$$

Proof. First we assume that f is decreasing. In this case, we shall also have the following:

(2.4) Each set e_ν is a finite union of intervals.

For $n=1$ the assertion holds trivially. Suppose that it holds for $n=k$. Let $n=k+1$. Putting

$$a = \sup \left\{ c: \int_c^{c+c_1} f dt \geq \alpha_1 c_1, c \leq c_n - c_1 \right\},$$

we have $\int_a^{a+c_1} f dt = \alpha_1 c_1$, unless $a = c_n - c_1$. Let $\tau_c h$ denote the translation operator, defined by

$$\begin{aligned} \tau_c h(t) &= h(t+c) \quad \text{if } t+c \in I, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

We put

$$\begin{aligned} f_1 &= (f\chi_{(0,a)} + f\chi_{(a+c_1,l)})^* = f\chi_{(0,a)} + \tau_{c_1}(f\chi_{(a+c_1,l)}), \\ g_1 &= \tau_{c_1} \left(\sum_{\nu=2}^n \alpha_\nu \chi_{(c_{\nu-1}, c_\nu)} \right) = \sum_{\nu=2}^n \alpha_\nu \chi_{(c_{\nu-1}-c_1, c_\nu-c_1)}. \end{aligned}$$

We can exclude the possibility that $a = c_n - c_1$, for then $g_1(t) \leq f_1(t)$ for all t . Since

$$\begin{aligned} \int_0^x g_1 dt &\leq \int_0^x g dt \leq \int_0^x f dt = \int_0^x f_1 dt \quad \text{if } 0 < x \leq a, \\ \int_0^x g_1 dt &= \int_0^{c_1+x} g dt - \alpha_1 c_1 \leq \int_a^{c_1+x} f dt - \int_a^{a+c_1} f dt \\ &= \int_0^x f_1 dt \quad \text{if } a < x \leq l, \end{aligned}$$

we see that $g_1 < f_1$. By the assumption, there exist mutually disjoint sets $\tilde{e}_\nu$, $2 \leq \nu \leq k+1$, such that (2.2)–(2.4) hold for f_1 and g_1 . Setting $e_1 = (a, a+c_1)$ and $e_\nu = \{\tilde{e}_\nu \cap (0, a)\} \cup \{t : t-c_1 \in \tilde{e}_\nu \cap (a, l)\}$, $2 \leq \nu \leq k+1$, we obtain mutually disjoint sets e_ν , $1 \leq \nu \leq k+1$, for which all the required conditions hold for f and g .

If f is positive but not decreasing, then, since $g < f^*$, we can find mutually disjoint measurable sets e_ν , $1 \leq \nu \leq n$, such that (2.2)–(2.4) hold for g and f^* . As each e_ν is a finite sum of intervals, we can easily find mutually disjoint sets $\tilde{e}_\nu$, $1 \leq \nu \leq n$, such that $m\tilde{e}_\nu = me_\nu$ and $\int f\chi_{\tilde{e}_\nu} dt = \alpha_\nu me_\nu$. Measurable sets $\tilde{e}_\nu$, $1 \leq \nu \leq n$, thus obtained, satisfy the requirements of Lemma 1.

We can now prove Theorem 1 when g_1 and g_2 , and consequently $g = g_1 + g_2$ belong to S . Let e_ν , $\nu = 1, \dots, n$, be sets of constancy of each of the three functions, with $g_1 = \alpha_{\nu 1}$, $g_2 = \alpha_{\nu 2}$ on e_ν . By means of the decreasing rearrangement of g and Lemma 1, we find disjoint sets $\tilde{e}_\nu$ with $m\tilde{e}_\nu = me_\nu$, $\int_{\tilde{e}_\nu} f dt \geq (\alpha_{\nu 1} + \alpha_{\nu 2})me_\nu$. Then it is possible to decompose each $\tilde{e}_\nu$ into disjoint $\tilde{e}_{\nu 1}$, $\tilde{e}_{\nu 2}$ such that $\int_{\tilde{e}_{\nu i}} f dt \geq \alpha_{\nu i} me_\nu$, $i = 1, 2$. We shall have $f\chi_{\tilde{e}_{\nu i}} > g_i\chi_{e_\nu}$, $i = 1, 2$, $\nu = 1, \dots, n$. Adding these relations, we obtain $g_i < f_i = \sum_{\nu=1}^n f\chi_{\tilde{e}_{\nu i}}$, $f_1 + f_2 \leq f$. It is now sufficient to replace f_2 by $f - f_1$ to obtain the result.

If g_1, g_2 are arbitrary positive functions, one finds increasing sequences $g_{1n} \uparrow g_1$, $g_{2n} \uparrow g_2$ from S . For the corresponding f_{1n}, f_{2n} one can use weak *-compactness on each set A where f is bounded, and the absolute continuity of the integrals $\int_e f dt$ to complete the proof.

REMARK. It is not difficult to show that the functions f_1, f_2 of Theorem 1 can be always assumed to be orthogonal (that is, with disjoint supports). However, one cannot, in general, assume that they are decreasing, even if g_1, g_2 and f are decreasing step functions with just one step.

In [10] another quasi-order $g \prec f$ has been used. With respect to two Banach function spaces this relation means the following. One must have $g, f \in X_1 + X_2$, and for each decomposition $f = f_1 + f_2$, $f_i \in X_i$, $i = 1, 2$ of f there should exist a decomposition $g = g_1 + g_2$ of g , $g_i \in X_i$, $i = 1, 2$, with the property that $\|g_i\|_{X_i} \leq \|f_i\|_{X_i}$, $i = 1, 2$. We are interested here in the case $X_1 = L^p$, $X_2 = L^\infty$. Then it is easy to see (compare also [10, p. 38]) that $g \prec f$ holds if and only if

$$(2.5) \quad \|(g^* - y)_+\|_p \leq \|(f^* - y)_+\|_p, \quad y \geq 0.$$

The quasi-order used in this paper, $g \prec^p f$, where $p \geq 1$, is defined, for two locally p -integrable functions, by the inequality

$$(2.6) \quad \int_0^x g^{*p} dt \leq \int_0^x f^{*p} dt, \quad x \geq 0,$$

that is, by $g^{*p} \prec f^{*p}$. If one writes (2.6) as $\|g^* \chi_{(0,x)}\|_p \leq \|f^* \chi_{(0,x)}\|_p$, there is an obvious similarity to (2.5).

From the definition we see that

$$(2.7) \quad g \prec^p f \text{ is equivalent to } g^* \prec^p f^*.$$

By a theorem of Hardy, Littlewood and Pólya, [3], $g \prec^p f$ implies $\Phi(|g|) \prec \Phi(|f|)$, where $\Phi(u)$, $u \geq 0$, is convex and increasing. In particular,

$$(2.8) \quad g \prec f \text{ implies } g \prec^p f.$$

We also have

$$(2.9) \quad g_i \prec^p f, \quad i = 1, 2, \quad \alpha_1, \alpha_2 \geq 0, \quad \alpha_1 + \alpha_2 = 1 \text{ imply } \alpha_1 g_1 + \alpha_2 g_2 \prec^p f.$$

In fact, for $x \in I$ we have, because of the inequality $(f_1 + f_2)^* \prec f_1^* + f_2^*$ and (2.8),

$$\begin{aligned} \int_0^x (\alpha_1 g_1 + \alpha_2 g_2)^{*p} dt &\leq \int_0^x (\alpha_1 g_1^* + \alpha_2 g_2^*)^p dt \\ &\leq \alpha_1 \int_0^x g_1^{*p} dt + \alpha_2 \int_0^x g_2^{*p} dt \leq \int_0^x f^{*p} dt. \end{aligned}$$

LEMMA 2. (i) Relation $g \prec^p f$ implies $g \prec f$; (ii) for each $p > 1$, there is a smallest constant λ_p , $1 < \lambda_p \leq 2^{1/q}$ ($1/p + 1/q = 1$), for which $g \prec f$ implies $g \prec^p \lambda_p f$.

Proof. (i) For a given $y \geq 0$, we consider the function $\Phi(u) = (u^{1/p} - y)_+^p$, which is increasing and convex. Thus, by the theorem of Hardy, Littlewood and Pólya mentioned above $g \prec^p f$ implies $\Phi(g^{*p}) \prec \Phi(f^{*p})$; relation (2.5) follows from this.

(ii) Assume $g \prec f$. If $e_0 \subset I$ is a given set, with $me_0 = a > 0$, let $\alpha = f^*(a)$, and let $f_2 = f^{(\alpha)} \in L^\infty$ be the α -truncation of f , let $f_1 = f - f^{(\alpha)} \in L^p$. There exist g_i , $i = 1, 2$, with $g = g_1 + g_2$, $\|g_1\|_p \leq \|f - f^{(\alpha)}\|_p$, $\|g_2\|_\infty \leq \alpha$. Let $e \subset I$, $me \leq a$. Then

$$\|g\chi_e\|_p \leq \|g_1\chi_e\|_p + \|g_2\chi_e\|_p \leq \|g_1\|_p + \alpha a^{1/p} \leq \|f - f^{(\alpha)}\|_p + \|f^{(\alpha)}\chi_{e_0}\|_p \leq 2^{1/q} \|f\chi_{e_0}\|_p.$$

We have used here the fact that if $f_1, f_2 \geq 0$, then $\|f_1\|_p + \|f_2\|_p \leq 2^{1/q} \|f_1 + f_2\|_p$. From (2.10) it follows that $g \prec^{p2^{1/q}} f$. We leave to the reader the proof that $\lambda_p > 1$.

3. An interpolation theorem for the pair L^p, L^∞ . In this section we assume that X is a Banach function space satisfying $X \subset L^p + L^\infty$ for some p , $1 \leq p < +\infty$. We shall say that X is *monotone with respect to the relation $\prec^p$* , or that X *belongs to the class $\mathcal{M}^p$* if $g \prec^p f$ and $f \in X$ imply $g \in X$. For $A > 0$, we shall say that $X \in \mathcal{M}^p(A)$ if $g \prec^p f$ and $f \in X$ imply $g \in X$ and $\|g\|_X \leq A\|f\|_X$.

LEMMA 3. *If $X \in \mathcal{M}^p$, then $X \in \mathcal{M}^p(A)$ for some $A > 0$; moreover, X is rearrangement invariant.*

Proof. By (2.8) it is clear that X is weakly rearrangement invariant if $X \in \mathcal{M}^p$. Suppose that $\mathcal{M}^p(A)$ is violated for each $A > 0$. Then there exist positive functions f_n, g_n , $n = 1, 2, \dots$, such that $g_n \prec^p f_n$, $\|g_n\|_X \geq n$ and $\|f_n\|_X \leq 2^{-2n}$. Putting $f = \sum_{n=1}^\infty 2^n f_n$, we have $f \in X$ and $2^n g_n \prec^p f$, $n \geq 1$. By (2.9) we get

$$g = \sum_{n=1}^\infty g_n = \sum_{n=1}^\infty 2^{-n} (2^n g_n) \prec^p f,$$

hence $g \in X$. This, however, contradicts the fact that $\|g\|_X \geq \|g_n\|_X \geq n$ for all $n \geq 1$. Thus, the condition $\mathcal{M}^p(A)$ holds for some $A > 0$, and X is necessarily rearrangement invariant.

(For $p = 1$, Lemma 3 was given in [12], [16], but the present proof is simpler.)

A space $X \subset L^p$ is *normally imbedded* in L^p if X is dense in L^p and if $\|f\|_p \leq \|f\|_X$ for all $f \in X$. Each of the Lorentz spaces $\Lambda(C, p)$ [9] (where C is a class of decreasing positive functions c with $\int c \, dt = 1$) is normally imbedded in L^p and satisfies $\mathcal{M}^p(1)$. Here is an example in the opposite direction:

EXAMPLE 1. The space Λ_α , $\alpha = p^{-1}$, $p > 1$, with the norm $\|f\|_\Lambda = \alpha \int_0^1 t^{\alpha-1} f^*(t) \, dt$, is normally imbedded in L^p . On the other hand, it does not satisfy $\mathcal{M}^p$. Indeed, let $\phi(t) = t^{-\alpha} \log^{-1}(1/t)$, $0 < t \leq e^{-1}$; $= 0$, $e^{-1} < t < 1$. Then $\phi \in L^p$, but $\|\phi\|_\Lambda = +\infty$. We put, for $0 < a < e^{-1}$,

$$\begin{aligned} g_a(t) &= \phi(a), & 0 \leq t \leq a, & & f_a(t) &= \phi(a), & 0 \leq t \leq b, \\ &= \phi(t), & a \leq t \leq 1, & & &= 0, & b \leq t \leq 1, \end{aligned}$$

selecting b in such a way that $\|f_a\|_p = \|g_a\|_p$.

Then for each a , $g_a \prec^p f_a$, but $\|g_a\|_\Lambda \rightarrow \infty$, $\|f_a\|_\Lambda = \|f_a\|_p = \|g_a\|_p \rightarrow \|\phi\|_p$ for $a \rightarrow 0$. Lemma 3 shows that $\mathcal{M}^p$ is violated.

LEMMA 4. Assume that $f_0, g_0, g_0 \in S$ are positive and that $g_0 \prec^p f_0$. Then there exists a positive operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ with the property that $g_0 \leq Tf_0$.

Proof. Let g_0 be given by

$$(3.1) \quad g_0 = \sum_{\nu=1}^n \alpha_\nu \chi_{\tilde{e}_\nu}, \quad \alpha_1 \geq \dots \geq \alpha_n \geq 0, \quad \tilde{e}_\nu \cap \tilde{e}_\mu = \emptyset, \nu \neq \mu.$$

By Lemma 1 (applied to g_0^{*p}) there exist disjoint subsets e_ν , $\nu=1, \dots, n$, of I for which $me_\nu = m\tilde{e}_\nu$ and

$$(3.2) \quad \int_0^t f_0^p \chi_{e_\nu} dt \geq \alpha_\nu^p me_\nu, \quad \nu = 1, \dots, n.$$

We denote by h_ν positive functions with the properties that $\|h_\nu\|_q = 1$ ($1/p + 1/q = 1$) and

$$\langle f_0 \chi_{e_\nu}, h_\nu \rangle = \int_0^t f_0 \chi_{e_\nu}(t) h_\nu(t) dt = \|f_0 \chi_{e_\nu}\|_p.$$

We define an operator T on the set of all locally p -integrable functions by

$$Tf = \sum_{\nu=1}^n \frac{\langle f \chi_{e_\nu}, h_\nu \rangle}{\|\chi_{e_\nu}\|_p} \chi_{\tilde{e}_\nu}.$$

Clearly T is positive and linear and

$$\|Tf\|_p^p \leq \sum_{\nu=1}^n \|f \chi_{e_\nu}\|_p^p \leq \|f\|_p^p$$

for all $f \in L^p$. On the other hand, for any $f \in L^\infty$,

$$|\langle f \chi_{e_\nu}, h_\nu \rangle| \leq \|f \chi_{e_\nu}\|_p \leq \|f\|_\infty \|\chi_{e_\nu}\|_p, \quad 1 \leq \nu \leq n.$$

Consequently, $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$. Furthermore by (3.2),

$$Tf_0 = \sum_{\nu=1}^n \frac{\|f_0 \chi_{e_\nu}\|_p}{\|\chi_{e_\nu}\|_p} \chi_{\tilde{e}_\nu} \geq \sum_{\nu=1}^n \alpha_\nu \chi_{\tilde{e}_\nu} = g_0.$$

Now we can prove

THEOREM 2. Let X be a Banach function space over I with $X \subset L^p + L^\infty$. The necessary and sufficient condition for X to have the interpolation property for the pair (L^p, L^∞) or, equivalently, this property in the strong sense for (L^p, L^∞) is that $X \in \mathcal{M}^p$.

Proof. First let $X \in \mathcal{M}^p$. By Lemma 3, $X \in \mathcal{M}^p(A)$ for some $A > 0$. Let $g, f \in X$ and $g \prec f$ (with respect to L^p, L^∞). Then by Lemma 2, $g \prec^p \lambda_p f$, and so $\|g\|_X \leq$

$A\lambda_p\|f\|_X$. Now if $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, then for each $f \in X$, $Tf \prec f$. It follows that T maps X into itself and that $\|T\|_X \leq A\lambda_p$. And if $0 \neq T \in \mathcal{B}(L^p, L^\infty; K_p, K_\infty)$, then $\alpha T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, where $\alpha^{-1} = \text{Max}(K_p, K_\infty)$. This shows that X has the interpolation property in the strong sense.

Conversely, suppose that X has the interpolation property for the pair (L^p, L^∞) , but fails to satisfy $\mathcal{M}^p$. Then there exist positive functions f and g such that $f \in X$, $\|f\|_X = 1$, $g \prec^p f$, but $g \notin X$. Let $0 \leq g_n \in S$ and $g_n \uparrow g$. As $g_n \prec^p f$, there exist, by Lemma 4, positive operators $T_n \in \mathcal{B}(L^p, L^\infty; 1, 1)$ such that $g_n \leq T_n f$ for each $n \geq 1$. This implies that $g_n \in X$ for each $n \geq 1$. Since $\|\cdot\|_X$ satisfies (1.2), $\|g_n\|_X \uparrow \infty$ holds. We may therefore assume without loss of generality that $\|g_n\|_X > n \cdot 2^n$, $n \geq 1$. It follows that $\|T_n\|_X \geq \|T_n f\|_X > n \cdot 2^n$, $n \geq 1$. Putting $T = \sum_{n=1}^{\infty} 2^{-n} T_n$, we obtain a positive operator belonging to $\mathcal{B}(L^p, L^\infty; 1, 1)$. On the other hand, $\|Tf\|_X \geq \|2^{-n} T_n f\|_X > n$ holds for each n , since T_n is a positive operator. This contradicts the fact that $Tf \in X$, and shows that the condition is necessary.

From the proof above, we have immediately

COROLLARY 1. *If X satisfies the condition $\mathcal{M}^p(A)$ and $T \in \mathcal{B}(L^p, L^\infty; K_p, K_\infty)$, then*

$$(3.3) \quad Tf \prec^p \lambda_p \text{Max}(K_p, K_\infty)f \quad \text{for each } f \in L^p;$$

$$(3.4) \quad \|T\|_X \leq A\lambda_p \text{Max}(K_p, K_\infty).$$

In the last inequalities, $\lambda_p \leq 2^{1/q} \leq 2$. We shall show that λ_p cannot be here replaced by 1.

EXAMPLE 2. For each p , $1 < p < +\infty$, there exists an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, for which $Tf \prec^p f$ is not true for some f .

Let $\alpha > 1$ be chosen so that $c = \frac{1}{2}(\alpha^{p-1} + 1)/(\alpha^p + 1)^{p-1} < 1$ (actually, this is true for any $\alpha > 1$). We define

$$\begin{aligned} f_0(x) &= \alpha & \text{on } (0, \tfrac{1}{2}), & & g_0(x) &= \beta & \text{on } (0, c), \\ &= 1 & \text{on } (\tfrac{1}{2}, 1), & & &= 0 & \text{on } (c, 1), \end{aligned}$$

where $\beta = (\alpha^p + 1)/(\alpha^{p-1} + 1)$. An easy calculation shows that

$$(3.5) \quad \|g_0\|_p = \|f_0\|_p,$$

$$(3.6) \quad \|f_0^{p-1}\|_1 \|g_0\|_\infty = \|f_0\|_p^p.$$

We define the positive operator

$$(3.7) \quad Tf = \frac{1}{\|f_0\|_p^p} \langle f, f_0^{p-1} \rangle g_0.$$

Since $\|h^{p-1}\|_q = \|h\|_p^{p-1}$ for $h \geq 0$, it follows from (3.5) that, if $f \in L^p$,

$$\|Tf\|_p \leq \frac{1}{\|f_0\|_p^p} \|f\|_p \|f_0^{p-1}\|_q \|g_0\|_p = \|f\|_p,$$

and from (3.6) that, if $f \in L^\infty$,

$$\|Tf\|_\infty \leq \frac{1}{\|f_0\|_p^p} \|f\|_\infty \|f_0^{p-1}\|_1 \|g_0\|_\infty = \|f\|_\infty.$$

Also, $Tf_0 = g_0$. However, $g_0 \prec^p f_0$ is incorrect, since

$$(3.8) \quad \int_0^c g_0^p dt = \|f_0\|_p^p > \int_0^c f_0^p dt.$$

EXAMPLE 3. There exists a space $X \in \mathcal{M}^p(1)$, and an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, for which $\|T\|_X > 1$.

In the notations of the last example, we take $X = L^p$ with the norm $\|f\|_X = \|f^* \chi_{(0,c)}\|_p$. It is immediately clear that $g \prec^p f$ implies $\|g\|_X \leq \|f\|_X$, so that $X \in \mathcal{M}^p(1)$. For the operator (3.7) we have $g_0 = Tf_0$, but $\|g_0\|_X > \|f_0\|_X > 0$ by (3.8).

If $g = Tf$ and $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, then we have $g \prec f$. We shall show that the converse is not true, in general. This will also show that one cannot replace the relation $\prec^p$ by $\prec$ in Lemma 4.

EXAMPLE 4. Let $p > 1$ be an integer, and let f_0 and g_0 be the functions of the Example 2. We put $f_1 = f_0 + 1$ and $g_1 = g_0 + 1$, where 1 denotes the characteristic function of $(0, 1)$. Let

$$G(t) = \|f_0 + t1\|_p^p - \|g_0 + t1\|_p^p, \quad t \geq 0.$$

Using (3.5), (3.6) and elementary calculations (for instance, with induction in k) we can show that

$$G(0) = G'(0) = 0, \quad G^{(k)}(0) \geq 0, \quad 2 \leq k \leq p.$$

It follows that $G(t) \geq 0$ for all $0 \leq t \leq 1$, hence we have $g_1 \prec f_1$ on account of (2.5).

Now suppose that there exists an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ such that $Tf_1 = g_1$. Since $\|T1\|_\infty \leq 1$, we have $0 \leq g_1 - 1 \leq Tf_1 - T1$, hence

$$\|g_0\|_p = \|g_1 - 1\|_p \leq \|Tf_1 - T1\|_p \leq \|f_1 - 1\|_p = \|f_0\|_p.$$

From (3.5) it follows that $T1 = 1$ and $Tf_0 = g_0$. Since $\chi_{(1/2,1)} = (\alpha 1 - f_0)/(\alpha - 1)$, we have

$$T\chi_{(1/2,1)} = (\alpha - 1)^{-1}(\alpha 1 - g_0).$$

The last function has values $\alpha/(\alpha - 1) > 1$ on the interval $(c, 1)$, hence $\|T\chi_{(1/2,1)}\|_\infty > 1$, a contradiction. Consequently, there does not exist an operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ with the property $Tf_1 = g_1$.

4. Interpolation theorems for the pair L^1, L^q . In this section we assume that X is a Banach function space for which $X \subset L^1 + L^q$, $1 < q < +\infty$, and that p is the conjugate exponent, $1/p + 1/q = 1$. We define a quasi-order relation $\prec_q$. We write $f_1 \prec_q f_2$, $f_1, f_2 \in L^q$, if there exists, for every $g_1 \in L^p$, a $g_2 \in L^p$ such that both $g_2 \prec^p g_1$ and $\langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle$. For example, $0 \leq f_1 \leq f_2$ implies $f_1 \prec_q f_2$, for here we can take $g_2 = |g_1|$. We begin with some properties of the relation $\prec_q$. For given $f, g \geq 0$, there exists a $\tilde{g} \geq 0$ with the properties $g \sim \tilde{g}$ and $\langle f^*, g \rangle = \langle f, \tilde{g} \rangle$, [8, p. 61]. From this, using (2.7), it is not difficult to derive

$$(4.1) \quad f_1 \prec_q f_2 \text{ if and only if } f_1^* \prec_q f_2^*.$$

If $f_1 <_q f_2$, then for each g_1 we have $\langle f_1^*, g_1 \rangle \leq \langle f_1^*, g_1^* \rangle \leq \langle f_2^*, g_1^* \rangle$. Hence, by (4.1),

$$(4.2) \quad f_1 <_q f_2 \text{ implies } f_1 <_q f_2.$$

Similar to (2.9) is the property

$$(4.3) \quad f_i <_q f, i = 1, 2, \text{ and } \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1 \text{ imply } \alpha_1 f_1 + \alpha_2 f_2 <_q f.$$

In fact, for each $g \in L^p$, we can find g_1 and g_2 such that $g_i <^p g$ and $\langle f_i, g \rangle \leq \langle f, g_i \rangle$, $i=1, 2$. Hence,

$$\langle \alpha_1 f_1 + \alpha_2 f_2, g \rangle \leq \langle f, \alpha_1 g_1 \rangle + \langle f, \alpha_2 g_2 \rangle = \langle f, \alpha_1 g_1 + \alpha_2 g_2 \rangle,$$

where $\alpha_1 g_1 + \alpha_2 g_2 <^p g$ by (2.9). Since g is arbitrary, we get (4.3).

For a Banach function space X , X' will denote the conjugate space of X , that is, the space of all measurable functions g such that

$$\|g\|_{X'} = \sup \{ |\langle f, g \rangle|; f \in X, \|f\|_X \leq 1 \} < \infty.$$

For any operator T acting on X , T' will denote the conjugate operator of T acting on the conjugate space X' . Note that $T \in \mathcal{B}(L^1, L^q; K_1, K_q)$ implies $T' \in \mathcal{B}(L^p, L^\infty; K_q, K_1)$.

LEMMA 5. If $T \in \mathcal{B}(L^1, L^q; 1, 1)$, then

$$(4.4) \quad Tf <_q \lambda_p f \text{ for each } f \in L^q.$$

Proof. We have $T' \in \mathcal{B}(L^p, L^\infty; 1, 1)$, hence $T'g <^p \lambda_p g$ holds for every $g \in L^p$, by (3.3). If $f \in L^q$ and $g_1 \in L^p$ are given, we select $g_2 = (1/\lambda_p)T'g_1$. Then $g_2 <^p g_1$ and $\langle Tf, g_1 \rangle = \langle f, T'g_1 \rangle = \langle \lambda_p f, g_2 \rangle$, and we have proven (4.4).

We shall use the following monotony conditions for a Banach function space X :

$$X \in \mathcal{M}_q, \text{ if } g <_q f, f \in X \text{ imply } g \in X;$$

$$X \in \mathcal{M}_q(A), \text{ if } g <_q f, f \in X \text{ imply } g \in X \text{ and } \|g\|_X \leq A\|f\|_X.$$

With the same proof as for Lemma 3 we have

LEMMA 6. If $X \in \mathcal{M}_q$, then $X \in \mathcal{M}_q(A)$ for some $A > 0$; moreover, X is rearrangement invariant.

LEMMA 7. If the space X does not satisfy the condition $\mathcal{M}_q(A)$, then there exists a positive operator $T \in \mathcal{B}(L^1, L^q; 1, 1)$ and a function $0 \leq f \in X$ for which $\|Tf\|_X > A\|f\|_X$.

Proof. We shall first show that under the assumptions of Lemma 7, the conjugate space X' of X does not satisfy $\mathcal{M}^p(A)$. There exist functions $f_1, f_2 \in X$ such that $f_1 <_q f_2$ and $\|f_1\|_X > A\|f_2\|_X$. For any $\varepsilon > 0$ satisfying $(1-\varepsilon)\|f_1\|_X > A\|f_2\|_X$, we can find, by virtue of the reflexivity of the semicontinuous norm $\|\cdot\|_X$, a function $g_1 \in X' \cap L^p$ such that $\|g_1\|_{X'} = 1$ and $(1-\varepsilon)\|f_1\|_X \leq \langle f_1, g_1 \rangle$. Since $f_1 <_q f_2$, there exists a function $g_2 \in L^p$ for which $g_2 <^p g_1$ and $\langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle$. This implies

$$A\|f_2\|_X < (1-\varepsilon)\|f_1\|_X \leq \|f_2\|_X \|g_2\|_{X'}.$$

Thus, we have obtained two functions $g_1, g_2 \in X'$, for which $g_2 \prec^p g_1$, but $\|g_2\|_{X'} > A\|g_1\|_{X'}$, contradicting the condition $\mathcal{M}^p(A)$.

For g_1 and g_2 , obtained above, we may assume $g_1, g_2 \geq 0$. Since $\|\cdot\|_{X'}$ is also semicontinuous, we can select an $h \in S \cap X'$ such that $0 \leq h \leq g_2$ and $\|h\|_{X'} > A\|g_1\|_{X'}$. By Lemma 4 there exists a positive operator $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$ for which $Tg_1 \geq h$. Choose an $\varepsilon > 0$ such that $(1-\varepsilon)\|h\|_{X'} > A\|g_1\|_{X'}$. There exists a function $0 \leq \tilde{f} \in X$, $\|\tilde{f}\|_X = 1$ with the property $\langle \tilde{f}, h \rangle \geq (1-\varepsilon)\|h\|_{X'}$. It follows that $(1-\varepsilon)\|h\|_{X'} \leq \langle \tilde{f}, h \rangle \leq \langle \tilde{f}, Tg_1 \rangle \leq \|T'\tilde{f}\|_X \|g_1\|_{X'}$. Consequently, we get $\|T'\tilde{f}\|_X > A$, for the positive operator $T' \in \mathcal{B}(L^1, L^q; 1, 1)$.

Now we can state our interpolation theorem for the pair (L^1, L^q) .

THEOREM 3. *Let X be a Banach function space over I with $X \subset L^1 + L^q$. The necessary and sufficient condition for X to have the interpolation property for the pair (L^1, L^q) , or, equivalently, this property for (L^1, L^q) in the strong sense, is that $X \in \mathcal{M}_q$.*

Proof. First let $X \in \mathcal{M}_q$. By Lemma 6, $X \in \mathcal{M}_q(A)$ for some $A > 0$. Let $0 \neq T \in \mathcal{B}(L^1, L^q; K_1, K_q)$, we put $\alpha^{-1} = \text{Max}(K_1, K_q)$. Then $\alpha T \in \mathcal{B}(L^1, L^q; 1, 1)$ and so $\alpha Tf \prec_q \lambda_p f$ holds for all $f \in L^q$ by Lemma 5. Thus, $f \in L^q \cap X$ implies $Tf \in X$ and $\|Tf\|_X \leq \lambda_p A \alpha^{-1} \|f\|_X$. We extend this relation to all $f \in X$. Since $f \in L^1 + L^q$, all truncations $f^{(n)}$ belong to L^q , and all differences $f - f^{(n)}$ belong to L^1 for $n = 1, 2, \dots$. Since $|f - f^{(n)}| \rightarrow 0$ a.e. and $\|T(f - f^{(n)})\|_1 \leq K_1 \|f - f^{(n)}\|_1$, we have $\|Tf - Tf^{(n)}\|_1 \rightarrow 0$. Taking, if necessary, a subsequence, we can assume that the sequence $Tf^{(n)}$, $n = 1, 2, \dots$, converges a.e. to Tf . By (1.2) and the semicontinuity of $\|\cdot\|_X$ we have

$$\begin{aligned} \|Tf\|_X &\leq \liminf_{n \rightarrow \infty} \|Tf^{(n)}\|_X \\ &\leq \liminf_{n \rightarrow \infty} \lambda_p A \alpha^{-1} \|f^{(n)}\|_X \leq \lambda_p A \alpha^{-1} \|f\|_X. \end{aligned}$$

This shows that $T \in \mathcal{B}(X)$, and that $\|T\|_X \leq \lambda_p A \alpha^{-1}$.

The necessity of the condition $\mathcal{M}_q(A)$ follows exactly as in the proof of Theorem 2.

COROLLARY 2. *If $X \in \mathcal{M}_q(A)$ and $T \in \mathcal{B}(L^1, L^q; K_1, K_q)$, then $T \in \mathcal{B}(X)$ and*

$$(4.5) \quad \|T\|_X \leq \lambda_p A \text{Max}(K_1, K_q).$$

5. Complete continuity of operators in interpolation theorems. In this section we give necessary and sufficient conditions for the space X in order that every operator T in $\mathcal{B}(L^p, L^\infty)$ (or in $\mathcal{B}(L^1, L^q)$) should be completely continuous on X if T is completely continuous on one of the spaces of the pair. We assume that $X \subset L^p + L^\infty$ (or $X \subset L^1 + L^q$) for the pair (L^p, L^∞) (respectively, (L^1, L^q)). The basic idea of the arguments below is due to the paper [4], and the setting and the proofs follow the lines of [11], [14]. The conditions are given in terms of the norms of compression operators. We denote by σ_a , $a > 0$, the *compression operator*:

$$(5.1) \quad \begin{aligned} \sigma_a f(t) &= f(at) \quad \text{if } 0 < at < I, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

For any rearrangement invariant space $(X, \|\cdot\|_X)$ with $\gamma=1$, we have $\sigma_a \in \mathcal{B}(X)$ and (see [13])

$$(5.2) \quad \|\sigma_a\|_X \leq 1 \quad \text{if } a \geq 1; \quad 1 \leq \|\sigma_a\|_X \leq a^{-1} \quad \text{if } 0 < a \leq 1.$$

It is clear that $\sigma_{ab} = \sigma_a \sigma_b$ if $b \geq 1$, or if $0 < a, b \leq 1$. It follows from this and (5.2) that

$$(5.3) \quad \|\sigma_a\|_X \leq (c/a) \|\sigma_c\|_X \quad \text{if } 0 < a \leq c, c > 1.$$

The norms $\|\sigma_a\|_X$, which play an important role in the theory of function spaces, have been discussed in [1], [13], [14]. We improve the inequality (3.4) of Corollary 1.

LEMMA 8. *If X satisfies the condition $\mathcal{M}^p(A)$, $1 \leq p < \infty$, then, for every $0 \neq T \in \mathcal{B}(L^p, L^\infty; K_p, K_\infty)$,*

$$(5.4) \quad \|T\|_X \leq A \lambda_p K_\infty \|\sigma_a\|_X,$$

where $a = K_\infty^p \cdot K_p^{-p}$.

Proof. In the assumptions of the lemma, both operators $T' = K_\infty^{-1} T \sigma_a^{-1}$ and $T'' = K_\infty^{-1} \sigma_a^{-1} T$ belong to $\mathcal{B}(L^p, L^\infty; 1, 1)$, as can be easily seen. If $a \geq 1$, $\sigma_a^{-1} \sigma_a = I$ on X , hence $T = K_\infty T' \sigma_a$; if $0 < a < 1$, $\sigma_a \sigma_a^{-1} = I$ on X , hence $T = K_\infty \sigma_a T''$. In both cases, (5.4) follows from Corollary 1.

An operator A on the set of locally integrable functions is called an *averaging operator* if A is defined by

$$(5.5) \quad Af = \sum_{v=1}^n (me_v)^{-1} \langle f, \chi_{e_v} \rangle \chi_{e_v},$$

where $me_v < \infty$, $e_v \cap e_\mu = \emptyset$, if $v \neq \mu$ and $n \geq 1$. For convenience, we sometimes denote the operator (5.5) by A_g , where $g = \sum_{v=1}^n \alpha_v \chi_{e_v}$, is any function in S corresponding to the sets $e_1, \dots, e_n$. It is clear that $A_g g = g$ for all $g \in S$. An averaging operator A belongs to $\mathcal{B}(L^1, L^\infty)$. If X is rearrangement invariant, then, because of the relation $Af \prec f$, A and $I - A$ belong to $\mathcal{B}(X)$. Moreover, A is completely continuous. For each p , $1 \leq p < \infty$, there exists a sequence of averaging operators A_n , $n = 1, 2, \dots$, which converges in L^p strongly to the identity operator I_n [5, p. 21].

We have

THEOREM 4. *Let X satisfy $\mathcal{M}^p(A)$, $1 \leq p < \infty$. In order that every operator $T \in \mathcal{B}(L^p, L^\infty)$, which is completely continuous on L^p , should be also completely continuous on X , it is necessary and sufficient that*

$$(5.6) \quad \lim_{a \rightarrow \infty} \|\sigma_a\|_X = 0.$$

Proof. Assume that (5.6) is satisfied. The image TV of the unit ball V in L^p has compact closure in L^p . We select a sequence A_n , $n = 1, 2, \dots$, of averaging operators converging strongly to I in L^p . Then

$$\lim_{n \rightarrow \infty} \left\{ \sup_{f \in V} \|(I - A_n)Tf\|_p \right\} = 0,$$

hence $\lim_{n \rightarrow \infty} \|(I - A_n)T\|_p = 0$. Since $\|(I - A_n)T\|_\infty \leq \|T\|_\infty$, putting $a_n = (\|(I - A_n)T\|_\infty \|(I - A_n)T\|_p^{-1})^p$ and $c_n = (\|T\|_\infty \|(I - A_n)T\|_p^{-1})^p$, we have $a_n \leq c_n$ and $c_n \rightarrow \infty$. Using (5.4) and (5.3) we obtain

$$\begin{aligned} \|(I - A_n)T\|_X &\leq A\lambda_p \|(I - A_n)T\|_\infty \|\sigma_{a_n}\|_X \\ &\leq A\lambda_p \|T\|_\infty \|\sigma_{c_n}\|_X \rightarrow 0. \end{aligned}$$

Since T is the uniform limit of the operators $A_n T$, $n = 1, 2, \dots$, which are completely continuous on X , T also has the property.

Conversely, assume that (5.6) is not valid for X . It has been shown in [14] that there exists an operator $T_0 \in \mathcal{B}(L^1, L^\infty)$ which is completely continuous on L^1 , but fails to be so on X . Such an operator T_0 is also completely continuous on L^p , $1 \leq p < \infty$ [4], [14]. Thus the necessity is proved.

If I is a finite interval, then for each operator T which is completely continuous on L^∞ , there exists a sequence of averaging operators A_n , $n = 1, 2, \dots$, such that $\|(I - A_n)T\|_\infty \rightarrow 0$ [5, p. 22]. This fact can be used in the proof of the following theorem.

THEOREM 5. *Let I be a finite interval, and let X satisfy $\mathcal{M}^p(A)$, $1 \leq p < \infty$. In order that every $T \in \mathcal{B}(L^p, L^\infty)$ which is completely continuous on L^∞ should also be completely continuous on X , it is necessary and sufficient that*

$$(5.7) \quad \lim_{a \rightarrow 0} a^{1/p} \|\sigma_a\|_X = 0.$$

Proof. The sufficiency is derived from (5.4) in a similar manner as in the proof of Theorem 4. Without loss of generality, we prove the necessity for $l=1$. First we note that the condition $\mathcal{M}^p(A)$ implies

$$(5.8) \quad a^{1/p} \|\sigma_a\|_X \leq A \quad \text{for all } a, 0 < a \leq 1.$$

This follows from the relation $a^{1/p} \sigma_a f \prec^p f$, $0 < a \leq 1$, $f \in L^p$, which can be easily verified.

Suppose that (5.7) is not true. Then there exists a $\delta > 0$ such that for arbitrarily small $a > 0$, $a^{1/p} \|\sigma_a\|_X > \delta$. For each a of this kind there exists a function g , which we may assume positive, such that

$$(5.9) \quad g \in S; \quad \|g\|_X \leq 1; \quad a^{1/p} \|\sigma_a g\|_X > \delta.$$

We can replace g by $\chi_{(0,a)} g$, since this will not change $\sigma_a g$. Then, for $n \rightarrow \infty$ we will have $\chi_{(1/n,a)} g \uparrow g$, $\sigma_a(\chi_{(1/n,a)} g) \uparrow \sigma_a g$. From (1.2) it follows that we can assume that the functions g in (5.9) have support (c, a) , $0 < c < a$. In addition to (5.9) we have

$$(5.10) \quad a^{1/2p} \|\sigma_{\sqrt{a}} g\|_X > A^{-1} \delta,$$

since by (5.8) and (5.9)

$$\delta < a^{1/p} \|\sigma_{\sqrt{a}}\|_X \|\sigma_{\sqrt{a}} g\|_X.$$

We can select a sequence of functions g_n , with supports (c_n, a_n) , $n=1, 2, \dots$, which satisfy (5.9) and (5.10) and for which, in addition, all intervals (c_n, a_n) , $n=1, 2, \dots$, are disjoint, all intervals $(c_n/(a_n)^{1/2}, (a_n)^{1/2})$ are disjoint, and $\sum a_n^{1/2p} < +\infty$.

We define the operators

$$(5.11) \quad T = \sum_{n=1}^{\infty} T_n; \quad T_n = a_n^{1/2p} \sigma_{(a_n)^{1/2}} A_{g_n}, \quad n = 1, 2, \dots,$$

where A_{g_n} are the averaging operators corresponding to the functions g_n . Then $\|T_n\|_{\infty} \leq a_n^{1/2p}$; the T_n are completely continuous on L^{∞} . It follows that also T is completely continuous in L^{∞} .

For any f , $T_n f = T_n(f\chi_{(c_n, a_n)})$. Also, $T_n f$ has support $(c_n/(a_n)^{1/2}, (a_n)^{1/2})$. Thus, all $T_n f$ are disjoint. It is easy to see that $\|\sigma_a\|_p \leq a^{-1/p}$, $0 < a < 1$. From this it follows that $\|T_n\|_p \leq 1$. Therefore

$$\|Tf\|_p^p = \sum_{n=1}^{\infty} \|T_n f\|_p^p \leq \sum_{n=1}^{\infty} \|f\chi_{(c_n, a_n)}\|_p^p \leq \|f\|_p^p, \quad f \in L^p,$$

and we see that $T \in \mathcal{B}(L^p)$.

It remains to show that T is not completely continuous on X . For the sequence of functions g_n , bounded in norm in X , we have $Tg_n = T_n g_n$, and by (5.10), $\|T_n g_n\|_X \geq A^{-1}\delta > 0$, also $T_n g_n(t) \rightarrow 0$ everywhere. If Tg_n would have a convergent subsequence in X , it could converge only to 0, and this is impossible.

We turn now to the pair (L^1, L^q) , $1 < q < \infty$. Applying similar arguments (or considering the conjugate spaces) we obtain

LEMMA 9. If $X \in \mathcal{M}_q(A)$, $1 < q < \infty$, then, for every $0 \neq T \in \mathcal{B}(L^1, L^q; K_1, K_q)$, we have

$$(5.12) \quad \|T\|_X \leq \lambda_p A(K_q^q K_1^{-1})^{1/(q-1)} \|\sigma_a\|_X,$$

where $a = (K_q^q K_1^{-1})^{a/(q-1)}$.

We also have

THEOREM 6. Let $X \in \mathcal{M}_q(A)$, $1 < q < \infty$. In order that every operator $T \in \mathcal{B}(L^1, L^q)$ which is completely continuous on L^q (or L^1) should be also completely continuous on X , it is necessary and sufficient that the following condition (5.13) (resp. (5.14)) hold:

$$(5.13) \quad \lim_{a \rightarrow 0} a \|\sigma_a\|_X = 0;$$

$$(5.14) \quad \lim_{a \rightarrow \infty} a^{1/q} \|\sigma_a\|_X = 0.$$

6. Orlicz spaces. In view of Examples 2 and 3 it appears to be worthwhile to give examples of classes of function spaces $X \in \mathcal{M}^p(1)$, $1 \leq p < \infty$, for which $\|T\|_X \leq 1$ holds for every $T \in \mathcal{B}(L^p, L^{\infty}; 1, 1)$.

We consider N -functions (compare [6]) M having the expression

$$(6.1) \quad M(u) = \int_0^u (u-t)^p d\phi(t), \quad u > 0,$$

where $1 \leq p < \infty$ and ϕ is a positive nondecreasing left continuous function with $\phi(0)=0$. For example, for r with $p \leq r < \infty$, the N -function $M(u)=u^r$, $u > 0$, has an expression (6.1). For an N -function M , let $L_M=L_M(I)$ denote the Orlicz space defined by M with the norm $\|\cdot\|_M$, where,

$$\|f\|_M = \inf \{ \xi : \rho_M(\xi^{-1}f) \leq 1, \xi > 0 \}$$

and

$$\rho_M(f) = \int_I M(|f(t)|) dt, \quad f \in L_M.$$

Then we have

THEOREM 7. *Let M have the expression (6.1). The Orlicz space L_M has the interpolation property for the pair (L^p, L^∞) in the strong sense. In addition, for every $T \in \mathcal{B}(L^p, L^\infty; K_p, K_\infty)$,*

$$(6.2) \quad \|T\|_M \leq \text{Max}(K_p, K_\infty).$$

Proof. We may assume that $K_p=K_\infty=1$. Let $f \in L_M$ and $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$. We have

$$\rho_M(Tf) = \int_I M(|Tf(t)|) dt = \int_I \left\{ \int_0^{|Tf(t)|} (|Tf(t)|-s)^p d\phi(s) \right\} dt$$

by (6.1). By Fubini's theorem this implies

$$\rho_M(Tf) = \int_0^\infty d\phi(s) \int_{E_s} (|Tf(t)|-s)^p dt,$$

where E_s , $s > 0$, is the set $\{t : |Tf(t)| > s, t \in I\}$. In view of the equality $(|Tf| - s1)_{\chi_{E_s}} = |Tf| - |Tf|^{(s)} = |Tf - T(f)^{(s)}|$, the last term is equal to

$$\int_0^\infty d\phi(s) \int_I |Tf(t) - (Tf)^{(s)}|^p dt = \int_0^\infty \|Tf - (Tf)^{(s)}\|_p^p d\phi(s).$$

Since $T \in \mathcal{B}(L^p, L^\infty; 1, 1)$, we get

$$\|Tf - (Tf)^{(s)}\|_p^p \leq \|Tf - T(f)^{(s)}\|_p^p \leq \|f - f^{(s)}\|_p^p,$$

which, in turn, implies $\rho_M(Tf) \leq \rho_M(f)$. Consequently, on account of the fact that $\|f\|_M \leq 1$ if and only if $\rho_M(f) \leq 1$, we have $\|Tf\|_M \leq \|f\|_M$. As f is arbitrary, we obtain $\|T\|_M \leq 1$.

In view of the proof above, we see that this theorem is also valid for Lipschitz operators acting on both L^p and L^∞ if the norms of the operators are now interpreted as their bounds. Thus, $\|T\|_X$ is now the smallest number γ satisfying $\|Tf - Tg\|_X \leq \gamma \|f - g\|_X$ for all $f, g \in X$. Since every N -function M has the expression (6.1) for $p=1$, Theorem 6 is a generalization of a theorem by W. Orlicz [13].

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