TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 161, November 1971

FUNCTIONS OF FINITE λ-TYPE IN SEVERAL COMPLEX VARIABLES(1)

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Abstract. If $\lambda: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is continuous and increasing then a meromorphic function f on $\boldsymbol{C}^k$ is said to be of finite λ-type if there are positive constants s, A, B and R such that $T_f(r, s) \leqq A\lambda(Br)$ for all $r > R$ where $T_f(r, s)$ is the characteristic of f. It is shown that if $\lambda(Br)/\lambda(r)$ is bounded for r sufficiently large and $B > 1$, then every meromorphic function of finite λ-type is the quotient of two entire functions of finite λ-type.

This theorem is the result of a careful and detailed analysis of the relation between the growth of a function and the growth of its divisors. The central fact developed in this connection is: A nonnegative divisor ν on $\boldsymbol{C}^k$ with $\nu(0) = 0$ is the divisor of an entire function of finite λ-type if and only if there are positive constants A, B and R such that

$$N_{\nu|\xi}(r) \leqq A\lambda(Br),$$

$$\left| \frac{1}{p} \sum_{s < |z| \leq r} \nu|\xi(z) z^{-p} \right| \leqq A\lambda(Br) r^{-p} + A\lambda(Bs) s^{-p},$$

for all $r \geqq s > R$, all unit vectors ξ in $\boldsymbol{C}^k$, and all natural numbers p. Here $\nu|\xi$ represents the lifting of the divisor ν to the plane via the map $z \mapsto z\xi$ and $N_{\nu|\xi}$ is the valence function of that divisor.

Analogous facts for functions of zero λ-type are also presented.

Introduction. The purpose of this paper is to present a detailed description of a comprehensive theory of functions of finite λ-type in several complex variables (as announced in Bull. Amer. Math. Soc. **75** (1969), 104–107) which extends the results of Rubel and Taylor in [5] for one variable and those of Stoll in [6] for functions of exponential type in several variables. One common motivation for all of these researches is the desire to extend a theorem proved by Lindelöf in 1905 (see [4]) which gives necessary and sufficient conditions for the existence of an entire function on the plane with a prescribed set of zeros and given order and type.

Specifically, §1 contains a discussion of divisors on $\boldsymbol{C}^k$ and the concept of the restriction of a divisor to the complex line through the origin and a vector in $\boldsymbol{C}^k$ ending with Proposition 1.5 which describes the behavior of a general class of

Received by the editors December 1, 1970.

AMS 1970 *subject classifications*. Primary 32A20; Secondary 42A16, 32F05.

Key words and phrases. Several complex variables, meromorphic function, holomorphic function, characteristic of a meromorphic function, value-distribution theory, divisors of a function, plurisubharmonic functions, Fourier coefficients of a meromorphic function, growth functions, growth indices, finite λ-type, order and type of a meromorphic function.

(1) This paper is based upon the author's doctoral dissertation which was written at the University of Notre Dame under the guidance of Professor Wilhelm Stoll. The preparation of this paper was partially supported by NSF GP-13064.

growth indices for a divisor. §2 presents an application of 1.5 to the valence function of the restriction of a divisor and concludes with Proposition 2.3 which states that $N_\nu(r; \zeta)$, the valence function for the restriction of a nonnegative divisor ν on $\boldsymbol{C}^k$ to the complex line through ζ in $\boldsymbol{C}^k$, is a continuous nonnegative plurisubharmonic function of ζ on $\boldsymbol{C}^k$ provided that $\nu(\mathbf{0})=0$.

In §3 the essential facts concerning the characteristic of a meromorphic function on $\boldsymbol{C}^k$ are presented. The first of the central results of the paper is Theorem 3.9 which states that for each meromorphic function f on $\boldsymbol{C}^k$ such that $f(\mathbf{0}) \in \boldsymbol{C}\backslash\{0\}$ there are positive numbers A, B and C such that

$$T_f(r; \xi) \leqq AT_f(Br)+C$$

for all $r>0$ and all unit vectors ξ in $\boldsymbol{C}^k$. Here T_f is the characteristic of f and $T_f(\cdot\,; \xi)$ is the characteristic of f restricted to the complex line through ξ.

In §4 the basic facts concerning functions of finite λ-type are presented and the relationships between finite λ-type and the classical concepts of order and type are developed. In §5 the Rubel and Taylor Fourier coefficients of a meromorphic function are defined and their basic properties are presented.

In §6 occur the main results of this paper. Theorem 6.1 states that if f is meromorphic and of finite λ-type on $\boldsymbol{C}^k$ with $f(\mathbf{0}) \in \boldsymbol{C}\backslash\{0\}$ then there are positive constants A, B and R such that $N_\nu(r; \xi) \leqq A\lambda(Br)$ for $r>R$ and all unit vectors ξ of $\boldsymbol{C}^k$ where ν is the pole-divisor of f. This theorem is proved by an easy application of Theorem 3.9. Theorem 6.4 states that a necessary and sufficient condition for a nonnegative divisor ν on $\boldsymbol{C}^k$ with $\nu(\mathbf{0})=0$ to be the divisor of an entire function of finite λ-type is the existence of positive constants A, B and R such that

$$N_\nu(r; \xi) \leqq A\lambda(Br)$$

and

$$\left|\frac{1}{p} \sum_{s<|z|\leqq r} \nu(r; \xi)z^{-p}\right| \leqq A\lambda(Br)r^{-p}+A\lambda(Bs)s^{-p}$$

for all $r \geqq s>R$, all unit vectors ξ in $\boldsymbol{C}^k$ and all natural numbers p. The proof of this result involves a careful combination of the techniques of Rubel and Taylor with those of Stoll.

§7 contains applications of the preceding results to show the existence of a large class of functions λ for which every meromorphic function of finite λ-type can be written as the quotient of two entire functions of finite λ-type. In §8 consequences of §7 for the classical cases are derived. §9 summarizes the analogous results which may be obtained for the theory of functions of zero λ-type. Finally, §10 presents a certain strengthening of the preceding results for some of the classical cases.

1. **Divisors.** If f is a holomorphic function on an open connected neighborhood of ζ_0 in $\boldsymbol{C}^k$ and is not identically zero there, then

$$f(\zeta) = \sum_{q=\nu}^{\infty} P_q(\zeta-\zeta_0)$$

where the series converges uniformly to the function on some neighborhood of ζ_0, each P_q is either identically zero or a homogeneous polynomial of degree q, and $P_\nu \not\equiv 0$. The nonnegative integer ν, uniquely determined by f and ζ_0, is called *the zero-multiplicity of f at ζ_0* and will be denoted by $\nu_f(\zeta_0)$. A function $\nu: \boldsymbol{C}^k \to \boldsymbol{Z}$ is said to be a *divisor* if and only if for every ζ_0 in $\boldsymbol{C}^k$ there are functions $g \not\equiv 0$ and $h \not\equiv 0$ which are holomorphic on an open connected neighborhood U of ζ_0 such that $\nu(\zeta) = \nu_g(\zeta) - \nu_h(\zeta)$ for all ζ in U. The set of all divisors on $\boldsymbol{C}^k$, denoted by D_k, is a $\boldsymbol{Z}$-module of functions and is partially ordered by the usual partial ordering of real-valued functions. In particular, we define the set of *nonnegative* divisors by

$$D_k^+ := \{\nu \in D_k : \nu \geqq 0\}.$$

Then D_k^+ is closed under addition.

If ν is in D_k, then there are defining functions g and h for ν in an open connected neighborhood of ζ in $\boldsymbol{C}^k$ such that g and h are coprime at ζ (i.e., the germs of g and h are coprime in the local ring of germs of holomorphic functions at ζ). It follows that the equations $\nu^+(\zeta) := \nu_g(\zeta)$ and $\nu^-(\zeta) := \nu_h(\zeta)$ define nonnegative divisors ν^+ and ν^- which are uniquely determined by ν. Clearly, $\nu = \nu^+ - \nu^-$.

Let f be a meromorphic function on $\boldsymbol{C}^k$. If f is identically zero, let us define $\nu_f^\infty(\zeta) = 0$ for all ζ in $\boldsymbol{C}^k$. If f is not identically zero, then each ζ in $\boldsymbol{C}^k$ has an open connected neighborhood U on which there are holomorphic functions $g \not\equiv 0$ and $h \not\equiv 0$, coprime at ζ, such that $hf = g$ on U. The nonnegative integers $\nu_f^0(\zeta) := \nu_g(\zeta)$ and $\nu_f^\infty(\zeta) := \nu_h(\zeta)$ are uniquely determined by f and ζ. The functions ν_f^0 and ν_f^∞ are nonnegative divisors on $\boldsymbol{C}^k$ and are called, respectively, *the zero-divisor* and *the pole-divisor* of f. We define *the divisor of f* by

$$\nu_f := \nu_f^0 - \nu_f^\infty.$$

Then ν_f is a divisor with the following elementary properties:

$$(\nu_f)^+ = \nu_f^0 \quad \text{and} \quad (\nu_f)^- = \nu_f^\infty,$$
$$\nu_{fg} = \nu_f + \nu_g,$$
$$\nu_{1/f}^0 = \nu_f^\infty \quad \text{and} \quad \nu_{1/f}^\infty = \nu_f^0,$$
$$\nu_{1/f} = -\nu_f.$$

Moreover, $\nu_f(\zeta) \geqq 0$ for all ζ in some open subset U of $\boldsymbol{C}^k$ if and only if f is holomorphic on U; and $\nu_f(\zeta) = 0$ for all ζ in U if and only if f is holomorphic without zeros on U.

Since $\boldsymbol{C}^k$ is a contractible Stein manifold, the second Cousin problem can be solved on $\boldsymbol{C}^k$ [2, pp. 105, 181], that is, *every nonnegative divisor is the divisor of an entire function.* Thus, if ν is a divisor, then there are entire functions g and h such that $\nu_g = \nu^+$ and $\nu_h = \nu^-$ so that $\nu_{g/h} = \nu$. Therefore, *every divisor is the divisor of some meromorphic function.* We also note that *for every meromorphic function f there are entire functions g and h such that $hf = g$ on $\boldsymbol{C}^k$ and g and h are coprime at every point of $\boldsymbol{C}^k$* [2, p. 181].

The *support* of a divisor ν on $\boldsymbol{C}^k$, denoted by supp ν, is defined to be the closure in $\boldsymbol{C}^k$ of $\nu^{-1}(\boldsymbol{Z}\backslash\{0\})$. The support of the divisor of an entire function f is then $f^{-1}(0)$ so that the support of a nonnegative divisor ν is simply $\nu^{-1}(\boldsymbol{Z}\backslash\{0\})$. The support of an arbitrary divisor ν is the union of the supports of ν^+ and ν^-; therefore, the support of a divisor on $\boldsymbol{C}^k$ is either empty or an analytic set of pure dimension $k-1$ in $\boldsymbol{C}^k$. Clearly, supp ν is empty if and only if ν is the identically zero divisor. Otherwise, ν is constant on each connectivity component of the set of simple points of its support.

For each ζ in $\boldsymbol{C}^k$, define $\zeta^*: \boldsymbol{C} \to \boldsymbol{C}^k$ by $\zeta^*(z)=z\zeta$. If f is meromorphic on $\boldsymbol{C}^k$ and $\zeta^*(\boldsymbol{C}) \not\subset \operatorname{supp} \nu_f^\infty$ for some ζ in $\boldsymbol{C}^k$, then $f \circ \zeta^*$ is meromorphic on $\boldsymbol{C}$ and is called *the restriction of f to ζ*. We will also write $f|\zeta$ for $f \circ \zeta^*$. If $\zeta^*(\boldsymbol{C}) \not\subset \operatorname{supp} \nu_f$, then $f|\zeta$ is meromorphic and not identically zero on $\boldsymbol{C}$. Thus, given a divisor ν on $\boldsymbol{C}^k$ and ζ in $\boldsymbol{C}^k$ such that $\zeta^*(\boldsymbol{C}) \not\subset \operatorname{supp} \nu$, we may define *the restriction of ν to ζ*, denoted by $\nu|\zeta$, as follows: Let f be meromorphic on $\boldsymbol{C}^k$ with $\nu_f=\nu$. Then $\nu|\zeta := \nu_{f|\zeta}$. For convenience, we will write $\nu(z; \zeta)$ instead of $\nu|\zeta(z)$. Let us introduce the following notation:

$$\dot{D}_k := \{\nu \in D_k : \mathbf{0} \notin \operatorname{supp} \nu\} \quad \text{and} \quad \dot{D}_k^+ := \dot{D}_k \cap D_k^+.$$

Then $\dot{D}_k$ is a submodule of D_k and $\dot{D}_k^+ = \{\nu \in D_k^+ : \nu(\mathbf{0})=0\}$ is closed under addition. Clearly, if ν is in $\dot{D}_k$, then $\nu|\zeta$ is defined for all ζ in $\boldsymbol{C}^k$, and the mapping $\nu \mapsto \nu|\zeta$ carries D_k into D_1, $\dot{D}_k^+$ into $\dot{D}_1^+$, and is $\boldsymbol{Z}$-linear. It is easily seen that if ν is in $\dot{D}_k$ and ζ is in $\boldsymbol{C}^k$, then

$$(1.1) \qquad \nu(z; w\zeta) = \nu(zw; \zeta) \quad \text{for all } z \text{ and } w \text{ in } \boldsymbol{C}.$$

Also, for each ν in $\dot{D}_k^+$ and ζ in $\boldsymbol{C}^k$, $\nu|\zeta \geqq \nu \circ \zeta^*$, but $\nu(z\zeta)=0$ for z in $\boldsymbol{C}$ implies that $\nu(z; \zeta)=0$; therefore,

$$(1.2) \qquad z \in \operatorname{supp}(\nu|\zeta) \quad \textit{if and only if } z\zeta \in \operatorname{supp} \nu.$$

LEMMA 1.3. *For each ν in D_k and compact subset K of $\boldsymbol{C}$,*

$$\{\zeta \in \boldsymbol{C}^k : \zeta^*(K) \cap \operatorname{supp} \nu = \varnothing\}$$

is an open subset of $\boldsymbol{C}^k$.

Proof. Since K is compact there is $r>0$ such that $|z|<r$ for all z in K. If the set in question is empty, it is open. So suppose $\zeta_0^*(K) \cap \operatorname{supp} \nu = \varnothing$ for some ζ_0 in $\boldsymbol{C}^k$. Since ζ_0^* is continuous, $\zeta_0^*(K)$ is a compact subset of $\boldsymbol{C}^k$ which does not meet the closed set supp ν. Therefore, there is $s>0$ such that $s \leqq |\zeta-\zeta'|$ for all ζ in $\zeta_0^*(K)$ and all ζ' in supp ν. If ζ is in $\boldsymbol{C}^k$ with $|\zeta-\zeta_0|<s/2r$ then for each z in K, $|\zeta_0^*(z)-\zeta^*(z)| = |z\zeta_0 - z\zeta| = |z|\cdot|\zeta_0-\zeta| < r(s/2r) = s/2 < s$ so that $\zeta^*(z) \notin \operatorname{supp} \nu$. Thus, $|\zeta-\zeta_0| < s/2r$ implies $\zeta^*(K) \cap \operatorname{supp} \nu = \varnothing$. Q.E.D.

LEMMA 1.4. *Let ν in $\dot{D}_k^+$, a in $\boldsymbol{C}$ and $r>0$ be given. Suppose U is an open connected neighborhood of ζ_0 in $\boldsymbol{C}^k$ such that $(a+re^{it})\zeta \notin \operatorname{supp} \nu$ for all ζ in U and all t in $[-\pi, \pi]$. Then*

$$\sum_{|z-a|<r} \nu(z; \zeta) = \sum_{|z-a|<r} \nu(z; \zeta_0) \quad \textit{for all } \zeta \textit{ in } U.$$

Proof. Choose an entire function f with $\nu_f=\nu$. By (1.2), the hypothesis implies that for each ζ in U and all t in $[-\pi, \pi]$, $a+re^{it} \notin \operatorname{supp} \nu|\zeta$ so that $[f|\zeta](a+re^{it})\neq 0$. Thus, by the argument principle,

$$\sum_{|z-a|<r} \nu(z;\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{[f|\zeta]'(a+re^{it})}{[f|\zeta](a+re^{it})}\, re^{it}\, dt$$

for all ζ in U, where the integral can be considered to be the Lebesgue integral. But, by the chain rule,

$$[f|\zeta]'(w) = \frac{d}{dz} f(z\zeta)|_{z=w} = \sum_{p=1}^{k} u_p \frac{\partial f}{\partial z_p}(w\zeta)$$

where $\zeta=(u_1,\ldots,u_k)$ is in U and w is in $\boldsymbol{C}$. It follows that the integrand above is a continuous function of (t,ζ) on $[-\pi,\pi]\times U$ so that the integral is continuous in ζ on U. But a continuous integer-valued function on a connected set is constant; therefore, the result follows. Q.E.D.

PROPOSITION 1.5. *Suppose that $g\colon \boldsymbol{R}^+\times[\boldsymbol{C}\backslash\{0\}]\to\boldsymbol{C}$ is a continuous function such that $g(r,z)=0$ if $|z|=r$ in $\boldsymbol{R}^+$. For ν in $\dot{D}_k$ define $G_\nu\colon \boldsymbol{R}^+\times\boldsymbol{C}^k\to\boldsymbol{C}$ by*

$$G_\nu(r,\zeta) := \sum_{0<|z|\leqq r} \nu(z;\zeta)g(r,z).$$

Then G_ν is continuous on $\boldsymbol{R}^+\times\boldsymbol{C}^k$.

Proof. Since $G_\nu=G_{\nu^+}-G_{\nu^-}$ it suffices to prove the result for a given ν in $\dot{D}_k^+$. Let (r_0,ζ_0) in $\boldsymbol{R}^+\times\boldsymbol{C}^k$ be given. There are two cases to be considered: First, suppose that $z\notin \operatorname{supp}\nu|\zeta_0$ whenever $|z|\leqq r_0$. Then, since $\operatorname{supp}\nu|\zeta_0$ is a closed set of isolated points in $\boldsymbol{C}$, there is $\delta_0>0$ such that $z\notin\operatorname{supp}\nu|\zeta_0$ whenever $z\in K_0$ and $K_0:=\{z\in\boldsymbol{C} : |z|\leqq r_0+\delta_0\}$. Thus, by (1.2), $\zeta_0^*(K_0)\cap\operatorname{supp}\nu=\varnothing$, so that by (1.3) there is a neighborhood of U of ζ_0 in $\boldsymbol{C}^k$ such that $\zeta^*(K_0)\cap\operatorname{supp}\nu=\varnothing$ for all ζ in U. It follows that $G_\nu(r,\zeta)=0$ whenever $0<r<r_0+\delta_0$ and ζ is in U; therefore, G_ν is continuous on a neighborhood of (r_0,ζ_0) in $\boldsymbol{R}^+\times\boldsymbol{C}^k$. On the other hand, suppose that

$$\{z\in\operatorname{supp}\nu|\zeta_0 : |z| \leqq r_0\} = \{a_1,a_2,\ldots,a_q\}$$

where $a_\mu\neq a_\lambda$ for $\mu\neq\lambda$. Since $\mathbf{0}\notin\operatorname{supp}\nu$ it follows from (1.2) that $0\notin\operatorname{supp}\nu|\zeta_0$ so that there is δ_1 such that

(i) $0<\delta_1<|a_\mu|$ for $\mu=1,\ldots,q$.

(ii) $2\delta_1<|a_\mu-a_\lambda|$ if $\mu\neq\lambda$.

(iii) $\delta_1<r_0-|a_\mu|$ if $|a_\mu|<r_0$.

(iv) $\nu(z;\zeta_0)=0$ if $r_0<|z|\leqq r_0+\delta_1$.

Let $\varepsilon>0$ be given. By the continuity of g on $\boldsymbol{R}^+\times[\boldsymbol{C}\backslash\{0\}]$, there is $\delta_2>0$ such that $\delta_2<\delta_1$ and, for each $\mu=1,2,\ldots,q$, $|g(r,z)-g(r_0,a_\mu)|<\varepsilon/(3A)$ if $|r-r_0|<\delta_2$ and $|z-a_\mu|\leqq\delta_2$ where

$$A := \sum_{0<|z|\leqq 2r_0} \nu(z;\zeta_0) \geqq 1.$$

Moreover, by the uniform continuity of g on compact subsets of $\boldsymbol{R}^+ \times [\boldsymbol{C}\backslash\{0\}]$ we may choose the above δ_2 so small that $|g(r, z)-g(r', z')| < \varepsilon/(3A)$ whenever $|r-r'| < \delta_2$ and $|z-z'| < \delta_2$ provided that both (r, z) and (r', z') are in K where

$$K := \{(t, w) \in \boldsymbol{R}^+ \times [\boldsymbol{C}\backslash\{0\}] : |t-r_0| \leqq \delta_1, |r_0-|w|\,| \leqq \delta_1\}.$$

It follows that

$$|g(r, z)| = |g(r, z)-g(r_0, r_0 z/|z|)| < \varepsilon/(3A)$$

if $|r-r_0| < \delta_2$ and $|r_0-|z|\,| < \delta_2$ since $g(r_0, w)=0$ if $|w|=r_0$ by hypothesis. Define

$$K' := \{z \in \boldsymbol{C} : |z| \leqq r_0+\delta_1 \text{ and } |z-a_\mu| \geqq \delta_2 \text{ for } \mu = 1, \ldots, q\}.$$

Then K' is compact and $\zeta_0^*(K') \cap \operatorname{supp} \nu = \varnothing$ by (iv). By (1.3) there is $\delta_3 > 0$ such that $\zeta^*(K') \cap \operatorname{supp} \nu = \varnothing$ whenever $|\zeta-\zeta_0| < \delta_3$. It then follows from (1.4) that

$$(*) \qquad \sum_{|z-a_\mu|<\delta_2} \nu(z; \zeta) = \nu(a_\mu; \zeta_0)$$

whenever $|\zeta-\zeta_0| < \delta_3$ and $\mu = 1, 2, \ldots, q$. But then from (ii) we get

$$(**) \qquad \sum_{0<|z|\leqq r_0+\delta_1} \nu(z; \zeta) = \sum_{\mu=1}^{q} \sum_{|z-a_\mu|<\delta_2} \nu(z; \zeta) = \sum_{\mu=1}^{q} \nu(a_\mu; \zeta_0) \leqq A$$

whenever $|\zeta-\zeta_0| < \delta_3$ since $r_0+\delta_1 < 2r_0$ by (i) and $\nu(z; \zeta_0) \geqq 0$ for all z in $\boldsymbol{C}$. Suppose (r, ζ) is in $\boldsymbol{R}^+ \times \boldsymbol{C}^k$ with $|r-r_0| < \delta_2$ and $|\zeta-\zeta_0| < \delta_3$; and let $r' = \min(r, r_0)$ and $r'' := \max(r, r_0)$. Then

$$G_\nu(r, \zeta) = \sum_{0<|z|\leqq r} \nu(z; \zeta) g(r, z) = \left[\sum_{0<|z|\leqq r_0} \pm \sum_{r'<|z|\leqq r''}\right] \nu(z; \zeta) g(r, z)$$

so that

$$(***) \qquad \begin{aligned} &G_\nu(r, \zeta) - G_\nu(r_0, \zeta_0) \\ &\qquad = \sum_{0<|z|\leqq r_0} [\nu(z; \zeta) g(r, z) - \nu(z; \zeta_0) g(r_0, z)] \pm \sum_{r'<|z|\leqq r''} \nu(z; \zeta) g(r, z). \end{aligned}$$

But the absolute value of the second of these terms is less than or equal to

$$\begin{aligned} \sum_{r'<|z|\leqq r''} \nu(z; \zeta) |g(r, z)| &< \sum_{r'<|z|\leqq r''} \nu(z; \zeta)(\varepsilon/3A) \\ &\leqq (\varepsilon/3A) \sum_{0<|z|\leqq r_0+\delta_2} \nu(z; \zeta) \leqq \varepsilon/3 \end{aligned}$$

by the choice of δ_2 and (**). Moreover, the first term in (***) may be written as

$$\sum_{\mu=1}^{q} \sum_{|z-a_\mu|<\delta_2} [\nu(z; \zeta) g(r, z) - \nu(z; \zeta_0) g(r_0, z)] - \sum_{\mu=1;|a_\mu|=r_0}^{q} \sum_{|z-a_\mu|<\delta_2;|z|>r_0} \nu(z; \zeta) g(r, z)$$

by choice of δ_3 and (iii). Now, the latter term is also less than $\varepsilon/3$ in absolute value by an argument similar to the above and we observe that the former term can be written as

$$\sum_{\mu=1}^{q}\left[\sum_{|z-a_\mu|<\delta_2} \nu(z;\zeta)g(r,z)-\nu(a_\mu;\zeta_0)g(r_0,a_\mu)\right]$$
$$= \sum_{\mu=1}^{q}\left[\sum_{|z-a_\mu|<\delta_2} \nu(z;\zeta)g(r,z)-\sum_{|z-a_\mu|<\delta_2} \nu(z;\zeta)g(r_0,a_\mu)\right]$$
$$= \sum_{\mu=1}^{q}\sum_{|z-a_\mu|<\delta_2} \nu(z;\zeta)[g(r,z)-g(r_0,a_\mu)]$$

by (*), so that its absolute value is also less than $\varepsilon/3$ by the initial choice of δ_2 and (***) again.

Therefore, we have shown that G_ν is continuous at (r_0, ζ_0). Q.E.D.

2. **The growth of a divisor.** The classical indices of growth for a divisor ν in $\dot{D}_k$ are *the counting function* n_ν and *the valence function* N_ν, defined by

$$n_\nu(r;\zeta) := \sum_{|z|\leq r} \nu(z;\zeta), \qquad N_\nu(r;\zeta) := \int_0^r n_\nu(r;\zeta)\,\frac{dt}{t}$$

for $r>0$ and ζ in C^k. Clearly, for each $r>0$ and ζ in C^k, $n_\nu(r;\zeta)$ and $N_\nu(r;\zeta)$ are Z-linear functions of ν on $\dot{D}_k$; and for each ν in $\dot{D}_k^+$ and ζ in C^k, $n_\nu(r;\zeta)$ and $N_\nu(r;\zeta)$ are nonnegative increasing functions of r on R^+. By [5, 1.3, p. 57],

$$N_\nu(r;\zeta) = \sum_{0<|z|\leq r} \nu(z;\zeta)\log\frac{r}{|z|} \tag{2.1}$$

Therefore, as an immediate consequence of (1.5) we have

Proposition 2.2. *For each ν in $\dot{D}_k$, $N_\nu(r;\zeta)$ is a continuous function of (r,ζ) on $R^+\times C^k$.*

Moreover, from (1.1) and (2.1) we get

$$N_\nu(r;z\zeta) = N_\nu(r|z|;\zeta) \tag{2.3}$$

for ν in $\dot{D}_k$, $r>0$, z in C and ζ in $C^k\backslash\{0\}$.

The behavior of N_ν is further characterized by the following result:

Proposition 2.3. *For each ν in $\dot{D}_k^+$ and $r>0$, $N_\nu(r;\zeta)$ is a continuous nonnegative plurisubharmonic function of ζ on C^k.*

Given ν in $\dot{D}_k^+$, we can choose an entire function f on C^k such that $\nu_f=\nu$ which means $f(0)\neq 0$. But, by Jensen's Formula and (2.1),

$$N_\nu(r;\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{it}\zeta)|\,dt-\log|f(0)|$$

for $r>0$; and $\log|f|$ is plurisubharmonic [2, p. 44] so that the proof of Proposition 2.3 is contained in the following lemma:

Lemma 2.4. *If u is plurisubharmonic on C^k, then $(1/2\pi)\int_{-\pi}^{\pi} u(re^{it}\zeta)\,dt$ is plurisubharmonic in ζ on C^k for each $r>0$.*

Proof. By definition [2, p. 44] u plurisubharmonic on C^k implies that u is upper semicontinuous on C^k so that $u(re^{it}\zeta)$ is upper semicontinuous in t on R for each

$r>0$ and ζ in C^k. Thus, if $v(\zeta):=(1/2\pi)\int_{-\pi}^{\pi} u(re^{it}\zeta)\,dt$ for $r>0$ and ζ in C^k, then $v(\zeta)$ is in $R\cup\{-\infty\}$.

As a special case, suppose that u is a C^∞ plurisubharmonic function on C^k. Then

$$\sum_{p,q=1}^{k} w_p\bar{w}_q \frac{\partial^2 u}{\partial z_p\,\partial\bar{z}_q}(\zeta) \geqq 0$$

for all ζ and $(w_1,\ldots,w_k)$ in C^k (loc. cit.). But by the chain rule

$$\frac{\partial^2}{\partial z_p\,\partial\bar{z}_q} u(re^{it}\zeta) = re^{-it}re^{it}\frac{\partial^2 u}{\partial z_p\,\partial\bar{z}_q}(re^{it}\zeta).$$

Now, v is a C^∞ function on C^k and

$$\begin{aligned}\sum_{p,q=1}^{k} w_p\bar{w}_q \frac{\partial^2 v}{\partial z_p\,\partial\bar{z}_q}(\zeta) &= \sum_{p,q=1}^{k} w_p\bar{w}_q \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial^2}{\partial z_p\,\partial\bar{z}_q}u(re^{it}\zeta)\,dt\\ &= \frac{r^2}{2\pi}\int_{-\pi}^{\pi}\sum_{p,q=1}^{k} w_p\bar{w}_q\frac{\partial^2 u}{\partial z_p\,\partial\bar{z}_q}(re^{it}\zeta)\,dt \geqq 0\end{aligned}$$

for each ζ and $(w_1,\ldots,w_q)$ in C^k. Therefore, v is a C^∞ plurisubharmonic function on C^k (loc. cit.).

In the general case, if u is a plurisubharmonic function on C^k, then u is the limit of a decreasing sequence $u_1, u_2, \ldots$ of C^∞ plurisubharmonic functions on C^k [2, p. 45]. Thus, $\{u_1(re^{it}\zeta)-u_p(re^{it}\zeta)\}_{p\in N}$ is an increasing sequence of nonnegative functions of t on R converging to $u_1(re^{it}\zeta)-u(re^{it}\zeta)$ as p approaches ∞. Therefore, by the monotone convergence theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[u_1(re^{it}\zeta)-u_p(re^{it}\zeta)]\,dt \to \frac{1}{2\pi}\int_{-\pi}^{\pi}[u_1(re^{it}\zeta)-u(re^{it}\zeta)]\,dt.$$

Since $v_p(\zeta)=(1/2\pi)\int_{-\pi}^{\pi} u_p(re^{it}\zeta)\,dt$ is in R and $v(\zeta)$ is in $R\cup\{-\infty\}$, it follows that $v_p(\zeta)$ converges to $v(\zeta)$ as p approaches ∞ for ζ in C^k. But $v_1, v_2, \ldots$ is a decreasing sequence of C^∞ plurisubharmonic functions by the above special case. It follows that v is plurisubharmonic on C^k (loc. cit.). Q.E.D.

3. **The growth of a meromorphic function.** In the following let σ_r be the positive element of volume on the sphere $S_k(r):=\{\zeta\in C^k : |\zeta|=r\}$, considered as a real $(2k-1)$-dimensional C^∞ manifold, oriented to the exterior of the ball $B_k(r):=\{\zeta\in C^k : |\zeta|<r\}$; and let $V_k(r)$ denote $(2\pi^k r^{2k-1})/(k-1)!$, the volume of $S_k(r)$. In order to define the characteristic of a meromorphic function, we introduce ω_k, a positive C^∞ complex exterior differential form of bidegree $(k-1,k-1)$ on C^k defined by

$$\omega_1(z) :\equiv 1,$$

$$\begin{aligned}\omega_k(z_1,\ldots,z_k) &:= \frac{1}{(k-1)!}\left[\sum_{p=1}^{k}\frac{i}{2}dz_p \wedge d\bar{z}_p\right]^{k-1}\\ &= \sum_{p=1}^{k}\left(\frac{i}{2}\right)^{k-1} dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_{p-1}\wedge d\bar{z}_{p-1}\\ &\qquad\wedge dz_{p+1}\wedge d\bar{z}_{p+1}\wedge\cdots\wedge dz_k\wedge d\bar{z}_k.\end{aligned}$$

If f is meromorphic on C^k and $r>0$, let

$$A_f(r) := \frac{1}{W_k(r)} \int_{B_k(r)} (1+|f|^2)^{-2} \frac{i}{2} df \wedge d\bar{f} \wedge \omega_k$$

where $W_k(r)$ is a normalizing factor having the value $\pi^k r^{2k-2}/(k-1)!$. Then A_f is a nonnegative, increasing, continuous function on $\boldsymbol{R}^+$ [6, p. 406]. The *characteristic* of f is given by

$$T_f(r,s) := \int_s^r A_f(t) \frac{dt}{t} \quad \text{for } r \geqq s > 0.$$

Clearly, $T_f(\cdot, s)$ is a nonnegative increasing function of class C^1 on $[s, \infty)$. Moreover, $T_f(e^x, s)$ is a convex function of x on $[\log s, \infty)$. We now list some of the other basic properties of the characteristic (see [6, pp. 406–407, 409–410]).

PROPOSITION 3.1. *If f is meromorphic on C^k and a is a complex number, then for $r \geqq s > 0$*

$$T_{f+a}(r,s) = T_f(r,s)$$

and

$$T_{1/f}(r,s) = T_f(r,s) \quad \textit{if } f \not\equiv 0.$$

PROPOSITION 3.2. *For each pair of functions f and g meromorphic on C^k and each $s>0$, there are constants A and B in $\boldsymbol{R}^+$ such that*

$$T_{f+g}(r,s) \leqq T_f(r,s) + T_g(r,s) + A$$

and

$$T_{fg}(r,s) \leqq T_f(r,s) + T_g(r,s) + B \quad \textit{for } r \geqq s.$$

PROPOSITION 3.3. *If f is meromorphic on C^k, ζ_0 is in C^k, and $g(\zeta) = f(\zeta+\zeta_0)$ for ζ in C^k, then*

$$T_f(r,s) \leqq (1+|\zeta_0|/s)^{2k-1} T_g(r+|\zeta_0|, s)$$

for $r \geqq s > 0$.

PROPOSITION 3.4. *If f is an entire function on C^k and $M_f(r) := \max\{|f(\zeta)| : |\zeta| = r\}$ for $r>0$, then for each $s>0$ there are constants A and B in $\boldsymbol{R}^+$ such that*

$$\log^+ M_f(r) \leqq A T_f(8er, s) + B$$

and

$$T_f(r,s) \leqq \log^+ M_f(r) + \tfrac{1}{2}\log 2 \quad \textit{for } r \geqq s.$$

PROPOSITION 3.5. *If f is meromorphic on C^k and ν_f is in $\dot{D}_k$, then $T_f(r) := \int_0^r A_f(t)\, dt/t$ and $T_f(r;\zeta) := T_{f|\zeta}(r)$ exist for $r \geqq 0$ and $\zeta \in C^k$.*

Moreover,

$$T_f(r) = \frac{1}{V_k(1)} \int_{S_k(1)} T_f(r;\xi)\sigma_1(\xi) \quad \textit{for } r \geqq 0.$$

PROPOSITION 3.6. *If f is meromorphic on the plane and $\nu_f(0)=0$, then*

$$T_f(r) = N_\nu(r)+\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\sqrt{(1+|f(re^{it})|^2)}\, dt-\log\sqrt{(1+|f(0)|^2)}$$

for $r\geqq 0$ where $\nu=\nu_f^\infty$ and $N_\nu(r)=N_\nu(r;1)$.

The contents of Propositions 3.1 and 3.6 are usually referred to as the *First Main Theorem* (of Value Distribution Theory). Rubel and Taylor use *the characteristic of Nevanlinna,* defined by

$$T_f^*(r) := N_\nu(r)+\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |f(re^{it})|\, dt$$

for f meromorphic on the plane, $r\geqq 0$ and $\nu=\nu_f^\infty$ with $\nu_f(0)=0$ [5, p. 75]. Since $\log^+ x\leqq\log\sqrt{(1+x^2)}$ and $\log(1+x)\leqq\log 2+\log^+ x$ for $x\geqq 0$, the First Main Theorem implies the following relations between T_f and T_f^*:

PROPOSITION 3.7. *If f is meromorphic on $\mathbf{C}$ with $\nu_f(0)=0$, then*

$$T_f^*(r) \leqq T_f(r)+\log\sqrt{(1+|f(0)|^2)}$$

and

$$T_f(r) \leqq T_f^*(r)+\log\sqrt{2} \quad \textit{for } r \geqq 0.$$

The following is an analogue of (2.3):

PROPOSITION 3.8. *If f is meromorphic on $\mathbf{C}^k$ with ν_f in $\dot{D}_k$, then $T_f(r;z\zeta)=T_f(r|z|;\zeta)$ for $r\geqq 0$, z in $\mathbf{C}$ and ζ in $\mathbf{C}^k$.*

Proof. Given f meromorphic on $\mathbf{C}^k$ with ν_f in $\dot{D}_k$, $a\in\mathbf{C}$ and $\zeta\in\mathbf{C}^k$, let $g_a=f|a\zeta$. Then by the chain rule $g_a'(z)=ag_1'(az)$. But by definition

$$\begin{aligned} A_{g_a}(r) &= \frac{1}{\pi}\int_{|z|<r}\frac{|g_a'(z)|^2}{(1+|g_a(z)|^2)^2}\frac{i}{2}\,dz\wedge d\bar z \\ &= \frac{1}{\pi}\int_{|z|<r}\frac{|g_1'(az)|^2}{(1+|g_1(az)|^2)^2}\frac{|a|^2 i}{2}\,dz\wedge d\bar z \\ &= \frac{1}{\pi}\int_{|w|<|a|r}\frac{|g_1'(w)|^2}{(1+|g_1(w)|^2)^2}\frac{i}{2}\,dw\wedge d\bar w = A_{g_1}(|a|r)\end{aligned}$$

so that

$$\begin{aligned} T_f(r;a\zeta) &= T_{g_a}(r) = \int_0^r A_{g_a}(t)\frac{dt}{t} = \int_0^r A_{g_1}(|a|t)\frac{dt}{t} \\ &= \int_0^{|a|r} A_{g_1}(s)\frac{ds}{s} = T_{g_1}(|a|r) = T_f(|a|r;\zeta). \qquad \text{Q.E.D.}\end{aligned}$$

Propositions 3.2, 3.3, 3.4 and 3.8 show that the characteristic of a meromorphic function is an analogue of the maximum modulus of an entire function. But the maximum modulus has the useful property that $M_{f|\xi}(r)\leqq M_f(r)$ for ξ in $S_k(1)$ and f entire on $\mathbf{C}^k$; we now wish to establish a similar result for the characteristic.

THEOREM 3.9. *If f is meromorphic on $\boldsymbol{C}^k$ with ν_f in $\dot{D}_k$, then there are constants A, B and C in $\boldsymbol{R}^+$ such that $T_f(r;\xi) \leqq AT_f(Br)+C$ for $r>0$ and ξ in $S_k(1)$.*

The demonstration of this theorem depends upon the following three lemmas, the first of which is a direct consequence of Proposition 3.5 of [6, pp. 406–407].

LEMMA 3.10. *If g and h are entire functions such that ν_g and ν_h are in $\dot{D}_k$, then*

$$T_{g/h}(r) \leqq \frac{1}{V_k(r)}\int_{S_k(r)} \log\sqrt{(|g|^2+|h|^2)}\sigma_r - \log\sqrt{(|g(\mathbf{0})|^2+|h(\mathbf{0})|^2)}$$

with equality holding whenever g and h are coprime at every point of $\boldsymbol{C}^k$.

LEMMA 3.11. *If g and h are entire on $\boldsymbol{C}^k$, then $\log\sqrt{(|g|^2+|h|^2)}$ is plurisubharmonic on $\boldsymbol{C}^k$.*

Proof. Since $\sqrt{(|g|^2+|h|^2)}$ is continuous and nonnegative on $\boldsymbol{C}^k$,

$$\log\sqrt{(|g|^2+|h|^2)}$$

is upper semicontinuous on $\boldsymbol{C}^k$ to $[-\infty,\infty)$. Given ξ and η in $\boldsymbol{C}^k$, $g^2(\zeta+z\eta)$ and $h^2(\zeta+z\eta)$ are entire functions of z on $\boldsymbol{C}$. Therefore, $\log|g(\zeta+z\eta)|^2$ and $\log|h(\zeta+z\eta)|^2$ are subharmonic functions of z on $\boldsymbol{C}$; and it follows that $\log(|g(\zeta+z\eta)|^2+|h(\zeta+z\eta)|^2)$ is subharmonic in z on $\boldsymbol{C}$ [2, 1.6.6 and 1.6.8, p. 18]. Clearly, $\frac{1}{2}\log(|g|^2+|h|^2)$ is plurisubharmonic on $\boldsymbol{C}^k$ by definition. Q.E.D.

LEMMA 3.12. *If u is a nonnegative plurisubharmonic function on $\boldsymbol{C}^k$ and $0<t<1$, then*

$$u(\zeta) \leqq \frac{1+t}{(1-t)^{2k-1}}\cdot\frac{1}{V_k(r)}\int_{S_k(r)} u\sigma_r$$

whenever $|\zeta| \leqq tr$.

Proof. If u is plurisubharmonic on $\boldsymbol{C}^k$, then

$$u(\zeta) \leqq \frac{r^{2k-2}}{V_k(r)}\int_{S_k(r)} u(\eta)\frac{r^2-|\zeta|^2}{|\eta-\zeta|^{2k}}\sigma_r(\eta)$$

whenever $|\zeta|<r$ [3, pp. 26–28]. But if $|\zeta| \leqq tr$ and $|\eta|=r$, then

$$\frac{r^2-|\zeta|^2}{|\eta-\zeta|^{2k}} \leqq \frac{(r+|\zeta|)(r-|\zeta|)}{(r-|\zeta|)^{2k}} \leqq \frac{(1+t)r}{(1-t)^{2k-1}r^{2k-1}}$$

and the result follows since u is nonnegative. Q.E.D.

Proof of Theorem 3.9. Given f meromorphic on $\boldsymbol{C}^k$ with ν_f in $\dot{D}_k$, we can choose entire functions g and h, coprime at every point of $\boldsymbol{C}^k$, such that $hf=g$ and $h(\mathbf{0})=1$. Then $u:=\log\sqrt{(|g|^2+|h|^2)}$ is plurisubharmonic on $\boldsymbol{C}^k$ by (3.11) so that $v(\zeta):=(1/2\pi)\int_{-\pi}^{\pi} u(e^{it}\zeta)\,dt$ is a plurisubharmonic function of ζ on $\boldsymbol{C}^k$ by (2.4). Moreover, since $u(z\zeta)$ is subharmonic in z on $\boldsymbol{C}$ the mean-value property implies that

$$v(\zeta) \geqq u(0\zeta) = u(\mathbf{0}) = \log\sqrt{(1+|g(\mathbf{0})|^2)}$$

which is nonnegative. Thus, we may apply (3.12) with $t=\frac{1}{2}$ to get

$$v(r\xi) \leqq \frac{3\cdot 2^{2k-2}}{V_k(2r)}\int_{S_k(2r)} v\sigma_{2r}$$

for ξ in $S_k(1)$. But by reversing the order of integration and using the rotational invariance of σ_{2r} we get

$$\int_{S_k(2r)} v\sigma_{2r} = \int_{S_k(2r)} u\sigma_{2r}$$

(also see [6, 1.2, p. 395]). But, for each ξ in $S_k(1)$, $\nu_{g|\xi}$ and $\nu_{h|\xi}$ are in $\dot{D}_1$ since $\nu_g=\nu_f^0$ and $\nu_h=\nu_f^\infty$. Thus, we may apply (3.10) to $f|\xi=(g|\xi)/(h|\xi)$ and to $f=g/h$ to obtain

$$T_f(r;\xi) \leqq v(r\xi) \leqq \frac{3\cdot 2^{2k-2}}{V_k(2r)}\int_{S_k(2r)} u\sigma_{2r} = 3\cdot 2^{2k-2}[T_f(2r)+u(\mathbf{0})]$$

for ξ in $S_k(1)$. Therefore, (3.9) is true when $A=3\cdot 2^{2k-2}$, $B=2$, and $C=Au(\mathbf{0})$. Q.E.D.

We can now demonstrate the following analogue of Liouville's Theorem:

COROLLARY 3.13. *If f is meromorphic on $\boldsymbol{C}^k$, then the following are equivalent:*

(i) *f is constant.*

(ii) *$A_f\equiv 0$ on $\boldsymbol{R}^+$.*

(iii) *$T_f\equiv 0$ on $\boldsymbol{R}^+$.*

(iv) *For each $s>0$, $T_f(\cdot, s)$ is bounded on $[s,\infty)$.*

(v) *There is some $s>0$ such that $T_f(\cdot, s)$ is bounded on $[s,\infty)$.*

Proof. Clearly, (i) implies (ii), (ii) implies (iii), (iii) implies (iv), and (iv) implies (v). Thus, it suffices to show that (v) implies (i). As a special case suppose that f is meromorphic and nonconstant on the plane with $\nu_f(0)=0$. Then there is z_0 in $\boldsymbol{C}\backslash\{0\}$ such that $b=f(z_0)$ is in $\boldsymbol{C}\backslash\{f(0)\}$. Let $g(z)=1/(f(z)-b)$ for z in $\boldsymbol{C}$. Then g is meromorphic on the plane with $\nu_g(0)=0$. Now, if $r>|z_0|$ and $\nu=\nu_g^\infty$, then by (2.1)

$$N_\nu(r) = \sum_{0<|z|\leqq r} \nu(z)\log\frac{r}{|z|} \geqq \nu(z_0)\log\frac{r}{|z_0|} > 0.$$

But, by the First Main Theorem,

$$T_g(r) \geqq N_\nu(r)-\log\sqrt{(1+|g(\mathbf{0})|^2)} \quad \text{for } r\geqq 0$$

so that T_g is unbounded. But from (3.1) for each $s>0$ and $r\geqq s$,

$$T_f(r,s) = T_g(r,s) = T_g(r)-T_g(s)$$

so that $T_f(\cdot, s)$ is unbounded on $[s,\infty)$.

In the general case, suppose f is meromorphic on $\boldsymbol{C}^k$ with $T_f(\cdot, s)$ bounded for some $s>0$. If supp $\nu_f=\boldsymbol{C}^k$, then $f\equiv 0$ on $\boldsymbol{C}^k$. If supp $\nu_f\neq\boldsymbol{C}^k$, choose ζ_0 in $\boldsymbol{C}^k\backslash$supp ν_f. Let $g(\zeta)=f(\zeta-\zeta_0)$. Then ν_g is in $\dot{D}_k$ and $T_g(\cdot, s)$ is bounded on $[s,\infty)$ by (3.3).

Therefore, T_g is bounded on $\mathbf{R}^+$ and $T_{g|\xi}$ is bounded on $\mathbf{R}^+$ for each ξ in $S_k(1)$ by Theorem 3.9. But, from the above special case it follows that $g|\xi$ is constant for each ξ in $S_k(1)$. Thus, for each ζ in $\mathbf{C}^k\backslash\{\mathbf{0}\}$,

$$g(\zeta) = [g|(1/|\zeta|)\zeta](|\zeta|) = [g|(1/|\zeta|)\zeta](0) = g(\mathbf{0})$$

and, therefore, $f(\zeta)=g(\zeta+\zeta_0)=g(\mathbf{0})=f(-\zeta_0)$ for all ζ in $\mathbf{C}^k$. Q.E.D.

The above proof illustrates how the theorem may be used to reduce problems concerning several variables to questions involving only one variable. Of course, this example is elementary. A more powerful application will be made in the study of functions of finite λ-type which follows. Meanwhile, we will explore a few more immediate consequences of this theorem. First we have the following modification:

COROLLARY 3.14. *If f is meromorphic on $\mathbf{C}^k$ with ν_f in $\dot{D}_k$, then there are constants A, B and R in $\mathbf{R}^+$ such that $T_f(r;\xi)\leqq AT_f(Br)$ for all $r>R$ and ξ in $S_k(1)$.*

Proof. If f is constant, the result is trivial. In the other case, T_f is unbounded by the preceding corollary so that there is $R>0$ such that $C\leqq T_f(Br)$ for all $r>R$ where A, B and C are the constants of (3.9). Thus, $T_f(r;\xi)\leqq (A+1)T_f(Br)$ for $r>R$ and ξ in $S_k(1)$ by (3.9). Q.E.D.

COROLLARY 3.15. *If f is meromorphic and nonconstant on $\mathbf{C}^k$ with ν_f in $\dot{D}_k$, then*

$$\limsup_{r\to\infty} \frac{\log r}{T_f(r)} < \infty.$$

Proof. If f is not constant, then $A_f\not\equiv 0$ by (3.13) so that $A_f(R)>0$ for some $R>0$. But $(d/dx)T_f(e^x)=A_f(e^x)$ for x in R by definition; and $T_f(e^x)-T_f(e^{x'}) \geqq A_f(e^{x'})(x-x')$ for $x\geqq x'$ by the mean-value theorem since A_f is increasing. Replacing x' by $\log R$ and x by $\log r$, we get

$$T_f(r) \geqq T_f(R)+A_f(R)\log(r/R) > 0 \quad \text{for } r > R$$

and the result follows. Q.E.D.

COROLLARY 3.16. *If f is meromorphic on $\mathbf{C}^k$ with ν_f in $\dot{D}_k$, then there are constants A, B and R in $\mathbf{R}^+$ such that $T_{f\circ\tau}(r)\leqq AT_f(BM_\tau r)$ for all $r>R$ and each $\mathbf{C}$-linear mapping $\tau\colon \mathbf{C}^j\to\mathbf{C}^k$ provided $M_\tau := \max\{|\tau(\xi)| : \xi\in S_j(1)\}$.*

Proof. If f is meromorphic on $\mathbf{C}^k$ with ν_f in $\dot{D}_k$ and $\tau\colon \mathbf{C}^j\to\mathbf{C}^k$ is a $\mathbf{C}$-linear mapping then $f\circ\tau$ is meromorphic on $\mathbf{C}^j$ with $\nu_{f\circ\tau}$ in $\dot{D}_j$. Let ξ in $S_j(1)$ be given. If $\tau(\xi)=\mathbf{0}$, then $f\circ\tau|\xi\equiv f(\mathbf{0})$ so that $T_{f\circ\tau}(r;\xi)=0$ for all $r>0$. On the other hand, if $\tau(\xi)\neq\mathbf{0}$, then $f\circ\tau|\xi=f|\tau(\xi)$ so that

$$\begin{aligned} T_{f\circ\tau}(r;\xi) &= T_f(r;\tau(\xi)) = T_f(r|\tau(\xi)|;(1/|\tau(\xi)|)\tau(\xi)) \\ &\leqq T_f(rM_\tau;(1/|\tau(\xi)|)\tau(\xi)) \leqq AT_f(BM_\tau r) \end{aligned}$$

for all $r>R$ by (3.8), the definition of M_τ, and (3.9). Thus, for all ξ in $S_j(1)$, $T_{f\circ\tau}(r;\xi)\leqq AT_f(BM_\tau r)$ for each $r>R$ so that $T_{f\circ\tau}(r)\leqq AT_f(BM_\tau r)$ for each $r>R$ by (3.5). Q.E.D.

COROLLARY 3.17. *If f is meromorphic on $\mathbf{C}^k$ with ν_f in $\dot{D}_k$, then there are constants A, B and R in $\mathbf{R}^+$ such that $T_{f|V}(r) \leqq AT_f(Br)$ for all $r > R$ and each $\mathbf{C}$-linear subspace V of $\mathbf{C}^k$.*

Proof. If $V = \{\mathbf{0}\}$, the result is clear. If $V \neq \{\mathbf{0}\}$, then choose a unitary basis $\xi_1, \ldots, \xi_j$ of V and extend it to a unitary basis $\xi_1, \ldots, \xi_k$ of $\mathbf{C}^k$. Define $\tau_V \colon \mathbf{C}^j \to \mathbf{C}^k$ by

$$\tau_V(z_1, \ldots, z_j) = z_1\xi_1 + \cdots + z_j\xi_j.$$

Then τ_V is a $\mathbf{C}$-linear mapping and $M_{\tau_V} = 1$. If we identify $f|V$ with $f \circ \tau_V$ as usual, then the result follows from (3.15). Q.E.D.

4. **Functions of finite λ-type.** We say that λ is a *growth function* if and only if $\lambda \colon \mathbf{R}^+ \to \mathbf{R}^+$ is continuous and increasing. *In the remainder of this paper the symbols "λ" and "μ" will always represent growth functions.* Clearly, $\lambda + \mu$, $\lambda\mu$, $\max(\lambda, \mu)$ and $\min(\lambda, \mu)$ are growth functions. Moreover, $\max(1, T_f(\cdot, s))$ is a growth function for each function f meromorphic on $\mathbf{C}^k$ and each $s > 0$ (here we agree that $T_f(r, s) = 0$ if $0 < r < s$).

We will say that a meromorphic function f on $\mathbf{C}^k$ is *of finite λ-type* whenever there are constants $s > 0$ and $B > 0$ such that

$$\limsup_{r \to \infty} \frac{T_f(r, s)}{\lambda(Br)} < \infty.$$

Let $M_k(\lambda)$ denote the set of functions of finite λ-type on $\mathbf{C}^k$, $E_k(\lambda)$ denote the set of entire functions of finite λ-type on $\mathbf{C}^k$, $\dot{M}_k(\lambda)$ denote the set of f in $M_k(\lambda)$ with ν_f in $\dot{D}_k$, and $\dot{E}_k(\lambda)$ denote $E_k(\lambda) \cap \dot{M}_k(\lambda)$.

THEOREM 4.1. (i) *$M_k(\lambda)$ is an extension field of $\mathbf{C}$ and is invariant under affine transformations of the variable in $\mathbf{C}^k$.*

(ii) *$E_k(\lambda)$ is an integral domain containing $\mathbf{C}$ and is invariant under affine transformations of the variable in $\mathbf{C}^k$.*

(iii) *Either $M_k(\lambda) = E_k(\lambda) = \mathbf{C}$ or $E_k(\lambda)$ contains the ring of complex polynomial functions on $\mathbf{C}^k$ and $M_k(\lambda)$ contains the field of complex rational functions on $\mathbf{C}^k$.*

Proof. $E_k(\lambda)$ is an integral domain by (3.2) and contains $\mathbf{C}$, the constant functions, by (3.13). $M_k(\lambda)$ is a field by (3.1) and (3.2). The invariance under affine transformations of the variable is a consequence of (3.3) and (3.16). If $M_k(\lambda)$ contains a nonconstant function, then (3.15) implies that $\limsup (\log r)/\lambda(Br) < \infty$ for some $B > 0$ so that, in particular, λ is unbounded. On the other hand, if $f_p(z_1, \ldots, z_k) := z_p$ for $p = 1, 2, \ldots, k$, then each f_p is entire on $\mathbf{C}^k$ and, clearly, $M_{f_p}(r) = r$ so that $\log^+ M_{f_p}(r) = \log r$ for $r > 1$. Hence, $\limsup (\log^+ M_{f_p}(r))/\lambda(Br) < \infty$ and f_p is in $E_k(\lambda)$ for each $p = 1, 2, \ldots, k$, by (3.4). Since $E_k(\lambda)$ is an integral domain containing $\mathbf{C}$, it follows that $E_k(\lambda)$ contains each polynomial function; and since $M_k(\lambda)$ is a field containing $E_k(\lambda)$, it follows that $M_k(\lambda)$ contains each rational function. Q.E.D.

It follows that $M_k(\lambda)$ contains $Q(E_k(\lambda))$, the field of quotients of $E_k(\lambda)$. The primary objective of this paper is to determine conditions on λ which will imply that $M_k(\lambda)$ is exactly $Q(E_k(\lambda))$. We have an immediate reduction of the problem as follows:

PROPOSITION 4.2. *If* $\dot{M}_k(\lambda)\subset Q(E_k(\lambda))$, *then* $M_k(\lambda)=Q(E_k(\lambda))$.

Proof. Let f in $M_k(\lambda)$ be given. If $f\equiv 0$, then f is in $E_k(\lambda)$ and there is nothing to prove. If $f\not\equiv 0$, then there is η in $\boldsymbol{C}^k\backslash\operatorname{supp}\nu_f$. Let $F(\zeta):=f(\zeta+\eta)$. Then F is in $\dot{M}_k(\lambda)$ by (4.1)(i). If $\dot{M}_k(\lambda)\subset Q(E_k(\lambda))$, then there are functions G and H in $E_k(\lambda)$ such that $HF=G$ and $H\not\equiv 0$. Define $g(\zeta):=G(\zeta-\eta)$ and $h(\zeta):=H(\zeta-\eta)$. Then g and h are in $E_k(\lambda)$ by (4.1)(ii). But clearly, $hf=g$ and $h\not\equiv 0$ so that f is in $Q(E_k(\lambda))$. Q.E.D.

Combining (3.5), (3.14) and the definition of $M_k(\lambda)$ we get

PROPOSITION 4.3. *Suppose f is meromorphic on $\boldsymbol{C}^k$ with ν_f in $\dot{D}_k$. Then f is in $\dot{M}_k(\lambda)$ if and only if there are constants A, B and R in $\boldsymbol{R}^+$ such that $T_f(r;\xi)\leqq A\lambda(Br)$ for all $r>R$ and all ξ in $S_k(1)$.*

The constants A, B and R as in the above proposition will be called a *uniform defining system for f in $M_k(\lambda)$*. The above result could be paraphrased as: "*A function f meromorphic on $\boldsymbol{C}^k$ with ν_f in $\dot{D}_k$ is in $\dot{M}_k(\lambda)$ if and only if $f|\xi$ is in $\dot{M}_1(\lambda)$ uniformly in ξ on $S_k(1)$.*"

Let us examine the relation between finite λ-type and some other traditional concepts. Recall that *the zero type of f*, meromorphic on $\boldsymbol{C}^k$, is defined by

$$\tau_0(f) := \limsup_{r\to\infty} \frac{T_f(r,s)}{\log r} \quad \text{for some } s>0,$$

and *the ρ-type of f* for $\rho>0$ by

$$\tau_\rho(f) := \limsup_{r\to\infty} \frac{T_f(r,s)}{r^\rho} \quad \text{for some } s>0.$$

Also, *the order of f* is defined by

$$\operatorname{Ord} f := \limsup_{r\to\infty} \frac{\log^+ T_f(r,s)}{\log r} \quad \text{for some } s>0.$$

Clearly, these definitions are independent of s; and $\operatorname{Ord} f$ is in $[0,\infty]$. If $\operatorname{Ord} f=\rho$ in $[0,\infty)$, then $\tau_\rho(f)$ is called *the type of f* and is denoted by $\tau(f)$.

As easy consequences of the definitions involved we have

PROPOSITION 4.4. *Let* $\lambda(r):=\max(1,\log r)$ *for* $r>0$. *Then the following are equivalent for f meromorphic on $\boldsymbol{C}^k$*:

(i) *f is of finite λ-type.*

(ii) $\tau_0(f)<\infty$.

(iii) $\operatorname{Ord} f=0$ *and* $\tau(f)<\infty$.

PROPOSITION 4.5. *For $\rho>0$, let $\lambda(r):=r^\rho$ for $r>0$. Then the following are equivalent for f meromorphic on $\boldsymbol{C}^k$*:

(i) *f is of finite λ-type.*

(ii) $\tau_\rho(f)<\infty$.

(iii) $\operatorname{Ord} f<\rho$ *or both* $\operatorname{Ord} f=\rho$ *and* $\tau(f)<\infty$.

Moreover, by (3.16) we have

PROPOSITION 4.6. *For each f meromorphic on $\boldsymbol{C}^k$ with ν_f in $\dot{D}_k$ there are constants A and B in $\boldsymbol{R}^+$ such that $\tau_\rho(f\circ\varphi)\leqq A(BM_\varphi)^\rho\tau_\rho(f)$ for $\rho\geqq 0$ and* $\operatorname{Ord}(f\circ\varphi)\leqq\operatorname{Ord} f$ *for each $\boldsymbol{C}$-linear mapping $\varphi\colon \boldsymbol{C}^j\to\boldsymbol{C}^k$.*

And, from (3.16) we obtain

PROPOSITION 4.7. *If f is in $M_k(\lambda)$, then $f\circ\varphi$ is in $M_j(\lambda)$ for each $\boldsymbol{C}$-linear mapping $\varphi\colon\boldsymbol{C}^j\to\boldsymbol{C}^k$ provided that $\varphi(\boldsymbol{C}^j)$ is not a subset of* $\operatorname{supp}\nu_f^\infty$.

For the sake of completeness we mention the following result:

PROPOSITION 4.8. (i) *If* $\limsup_{r\to\infty}\lambda(r)/\mu(Br)<\infty$ *for some $B>0$, then $M_k(\lambda)\subset M_k(\mu)$.*

(ii) $M_k(\lambda)\cup M_k(\mu)\subset M_k(\lambda+\mu)=M_k(\max(\lambda,\mu))\subset M_k(\lambda\mu)$.

(iii) $M_k(\lambda)\cap M_k(\mu)=M_k(\min(\lambda,\mu))$.

5. **The Fourier coefficients of a meromorphic function.** In [5] Rubel and Taylor introduce the *Fourier coefficients* of a function f meromorphic on the plane with $\nu_f(0)=0$ as follows:

$$c_p(r;f):=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{it})|e^{-ipt}\,dt$$

for p in $\boldsymbol{Z}$ and $r>0$ (where the integral is the Lebesgue integral). If f is meromorphic on $\boldsymbol{C}^k$ with ν_f in $\dot{D}_k$ and ζ is in $\boldsymbol{C}^k$, let us define

$$c_p(r;f,\zeta):=c_p(r;f|\zeta).$$

PROPOSITION 5.1. *If f and g are meromorphic on $\boldsymbol{C}^k$ with ν_f and ν_g in $\dot{D}_k$, then for each p in $\boldsymbol{Z}$, $r>0$, and ζ in $\boldsymbol{C}^k$,*

(i) $c_{-p}(r;f,\zeta)=\overline{c_p(r;f,\zeta)}$,

(ii) $c_p(r;1/f,\zeta)=-c_p(r;f,\zeta)$,

(iii) $c_p(r;fg,\zeta)=c_p(r;f,\zeta)+c_p(r;g,\zeta)$,

(iv) $c_p(r;f,se^{it}\zeta)=e^{ipt}c_p(rs;f,\zeta)$ *for $s>0$ and t in $\boldsymbol{R}$,*

(v) $\log|f(re^{it}\zeta)|=\sum_{p=-\infty}^{\infty}c_p(r;f,\zeta)e^{ipt}$ *where we mean that the symmetric partial sums of the series converge in the L^2-norm (with respect to t on $[-\pi,\pi]$) to the function on the left.*

Proof. For (v) see [5, 4.2, p. 76]; the rest is proved by trivial verifications. Q.E.D.

If f is meromorphic on the plane with $\nu_f(0)=0$, then we can choose $s>0$ so that f is holomorphic and never zero on $B_1(2s)$. Let us define

$$\alpha_p(f) := \frac{1}{2\pi i p}\int_{|z|=s} \frac{f'(z)}{f(z)} z^{-p}\, dz$$

for each p in N. Clearly, the definition is independent of the choice of s. For f meromorphic on C^k with ν_f in $\dot{D}_k$ and ζ in C^k, let $\alpha_p(f;\zeta) := \alpha_p(f|\zeta)$. In this case we can choose $s>0$ so that f is holomorphic and never zero on $B_k(s)$. By Taylor's Theorem

$$\frac{[f|\zeta]'}{[f|\zeta]}(z) = \sum_{p=1}^{\infty} p\alpha_p(f;\zeta)z^{p-1}$$

for z in $B_1(s/|\zeta|)$ and ζ in $C^k\backslash\{\mathbf{0}\}$. However, there is a function g holomorphic on $B_k(s)$ such that $\exp(g)=f$ there. Let

$$g(\zeta) = \sum_{p=0}^{\infty} P_p(\zeta)$$

be the expansion of g in terms of homogeneous polynomials converging uniformly on $B_k(t)$ for some t with $0<t\leqq s$. Clearly, $\exp[g|\zeta]=f|\zeta$ on $B_1(t/|\zeta|)$ for each ζ in $C^k\backslash\{\mathbf{0}\}$ so that $[g|\zeta]'=[f|\zeta]'/[f|\zeta]$ on $B_1(t/|\zeta|)$. Moreover, $[g|\zeta](z)=g(z\zeta)=\sum_{p=0}^{\infty} P_p(\zeta)z^p$ so that $[g|\zeta]'(z)=\sum_{p=1}^{\infty} pP_p(\zeta)z^{p-1}$ for z in $B_1(t/|\zeta|)$. Therefore, by the uniqueness of the Taylor coefficients we have $\alpha_p(f;\zeta)=P_p(\zeta)$ for all ζ in $C^k\backslash\{\mathbf{0}\}$. Since $\alpha_p(f;\mathbf{0})=0$ for all p in N, we have proved

PROPOSITION 5.2. *For each function f meromorphic on C^k with ν_f in $\dot{D}_k$ and each p in N, $\alpha_p(f;\zeta)$ is either identically zero or a homogeneous polynomial of degree p in ζ on C^k.*

For ν in $\dot{D}_k$, ζ in C^k, p in N and $r>0$, let

$$N_p'(r;\nu,\zeta) := \frac{1}{p}\sum_{0<|z|\leqq r} \nu(z;\zeta)(r/z)^p,$$

$$N_p''(r;\nu,\zeta) := \frac{1}{p}\sum_{|z|\leqq r} \nu(z;\zeta)(\bar{z}/r)^p,$$

$$N_p := N_p' - N_p''.$$

Then, since $g(r,z)=(r/z)^p-(\bar{z}/r)^p$ is continuous on $R^+\times[C\backslash\{0\}]$ and $g(r,z)=0$ if $|z|=r$, an immediate consequence of (1.5) is

PROPOSITION 5.3. *For each ν in $\dot{D}_k$ and p in N, $N_p(r;\nu,\zeta)$ is a continuous function of (r,ζ) on $R^+\times C^k$.*

By Lemma 4.2 of [5, pp. 76–77] we have

PROPOSITION 5.4. *If f is meromorphic on C^k with ν_f in $\dot{D}_k$, then for each $r>0$ and ζ in C^k*

$$c_0(r;f,\zeta) = \log|f(\mathbf{0})| + N_{\nu_f}(r;\zeta)$$

and

$$2c_p(r; f, \zeta) = r^p \alpha_p(f; \zeta) + N_p(r; \nu_f, \zeta)$$

for each p in $\boldsymbol{N}$.

Combining (5.1)(i), (5.2), (5.3) and (5.4), we obtain

PROPOSITION 5.5. *If f is meromorphic on* $\boldsymbol{C}^k$ *with* ν_f *in* $\dot{D}_k$, *then for each p in* $\boldsymbol{Z}$, $c_p(r; f, \zeta)$ *is a continuous function of* (r, ζ) *on* $\boldsymbol{R}^+ \times \boldsymbol{C}^k$.

6. **Admissible divisors.** We will say that a divisor ν in $\dot{D}_k^+$ is *of finite* λ*-density* if and only if there are constants A, B and R in $\boldsymbol{R}^+$ such that $N_\nu(r; \xi) \leqq A\lambda(Br)$ for all $r > R$ and all ξ in $S_k(1)$.

THEOREM 6.1. *If f is in* $\dot{M}_k(\lambda)$, *then* ν_f^∞ *is of finite* λ*-density.*

Proof. Let A, B and R be a uniform defining system for f in $\dot{M}_k(\lambda)$. By the First Main Theorem

$$N_{\nu_f^\infty}(r; \xi) \leqq T_f(r; \xi) + \log \sqrt{(1 + |f(\mathbf{0})|^2)}$$

since $\log \sqrt{(1 + |f(re^{it}\xi)|^2)} \geqq 0$ for $r > 0$, t in $[-\pi, \pi]$ and ξ in $S_k(1)$. Therefore, if $r > R$ and ξ is in $S_k(1)$,

$$\begin{aligned} N_{\nu_f^\infty}(r; \xi) &\leqq A\lambda(Br) + \log \sqrt{(1 + |f(\mathbf{0})|^2)} \\ &\leqq \left[A + \frac{\log \sqrt{(1 + |f(\mathbf{0})|^2)}}{\lambda(BR)}\right] \lambda(Br) \end{aligned}$$

so that ν_f^∞ is of finite λ-density. Q.E.D.

REMARK. When $k = 1$, the above result is a consequence of Theorem 5.3 of [5, p. 88]. In fact, the above proof is essentially the same as that given for the corresponding result in [5] if we make the observation that the defining constants for $\nu | \xi$ to be of finite λ-density depend upon λ, $f(\mathbf{0})$ and the defining constants for $f | \xi$ to be of finite λ-type. In particular, and this is the crucial point, the constants derived in [5] *do not* depend upon ξ. In this case, we have repeated the details of the proof in order to emphasize this fact. In the sequel we will omit proofs that are totally analogous to those given in [5] for corresponding results. (See also [5, remark following 1.11, p. 58].)

For ν in $\dot{D}_k$, p in $\boldsymbol{N}$, $r > s > 0$ and ξ in $S_k(1)$ let

$$\begin{aligned} C_p(r, s; \nu, \xi) &:= r^{-p} N_p'(r; \nu, \xi) - s^{-p} N_p'(s; \nu, \xi) \\ &= \frac{1}{p} \sum_{s < |z| \leqq r} \nu(z; \xi) z^{-p} \end{aligned}$$

and let $C_p(s, r; \nu, \xi) := C_p(r, s; \nu, \xi)$. We will say that ν in $\dot{D}_k^+$ is λ*-balanced* provided that there are constants A, B and R in $\boldsymbol{R}^+$ such that

$$|C_p(r, s; \nu, \xi)| \leqq A\lambda(Br)/r^p + A\lambda(Bs)/s^p$$

for all p in $\boldsymbol{N}$ and all ξ in $S_k(1)$ whenever $r > s > R$. And ν in $\dot{D}_k^+$ will be called *λ-admissible* whenever ν is both λ-balanced and of finite λ-density. Note that, by taking appropriate maxima, we can assume that for each λ-admissible divisor there is *one* set of constants A, B and R which satisfy the conditions for finite λ-density and for λ-balancing.

LEMMA 6.2. *If ν in $\dot{D}_k^+$ is of finite λ-density and there is a sequence $\{\alpha_p\}_{p\in N}$ of complex-valued functions on $S_k(1)$ and constants A, B and R in $\boldsymbol{R}^+$ such that $|r^p\alpha_p(\xi)+N_p(r;\nu,\xi)|\leqq A\lambda(Br)$ for all p in N and all ξ in $S_k(1)$ whenever $r>R$, then ν is λ-admissible.*

Proof. See [5, 2.5, pp. 65–66]. The balancing constants can be taken to be $3A$, eB and R if we also assume, without loss of generality, that A, B and R are also the defining constants for the finite λ-density of ν. Q.E.D.

LEMMA 6.3. *If ν in $\dot{D}_k^+$ is λ-admissible, then there is a sequence $\{\alpha_p\}_{p\in N}$ of continuous complex-valued functions on $S_k(1)$ and constants A, B and R in $\boldsymbol{R}^+$ such that $|r^p\alpha_p(\xi)+N_p(r;\nu,\xi)|\leqq A\lambda(Br)/(1+p)$ for all p in N and all ξ in $S_k(1)$ whenever $r>R$.*

Proof. (This is a generalization of [5, 2.5, p. 65]; but since the proof given there requires some modification in order to achieve the continuity of the α_p, we give a complete proof here.) Let ν be λ-admissible with defining constants A, B and R. First of all we observe from [5, 1.11, p. 58] that if $A':=2A$ and $B':=2Be$, then

$$(*)\qquad |C_p(r,s;\nu,\xi)| \leqq A'\lambda(B'r)/pr^p+A'\lambda(B's)/ps^p$$

for all p in N and all ξ in $S_k(1)$ whenever $r>R$ and $s>R$.

Now, let $m:=\min\{p\in \boldsymbol{N} : \liminf_{r\to\infty} r^{-p}\lambda(B'r)=0\}$ where we agree that $\min\varnothing=\infty$. Thus, m is in $\boldsymbol{N}\cup\{\infty\}$.

Definition of α_p for p in N and $p<m$. Since $r^{-p}\lambda(B'r)>0$ and $\liminf_{r\to\infty} r^{-p}\lambda(Br)\neq 0$, it follows that $\liminf_{r\to\infty} r^{-p}\lambda(B'r)>0$. But $r^{-p}\lambda(B'r)$ is continuous and strictly positive on $[R+1,\infty)$ so that $I_p:=\inf\{r^{-p}\lambda(B'r) : r\geqq R+1\}>0$ and there is $r_p\geqq R+1$ such that $(r_p)^{-p}\lambda(B'r_p)\leqq 2I_p\leqq 2r^{-p}\lambda(B'r)$ for all $r>R+1$. For ξ in $S_k(1)$ define

$$\alpha_p(\xi) := -(r_p)^{-p}N_p(r_p;\nu,\xi).$$

Then α_p is continuous on $S_k(1)$ by (5.3). Moreover,

$$\begin{aligned}|\alpha_p(\xi)+r^{-p}N_p'(r;\nu,\xi)| &\leqq |C_p(r_p,r;\nu,\xi)|+(r_p)^{-p}|N_p''(r_p;\nu,\xi)|\\ &\leqq \frac{A'\lambda(B'r_p)}{p(r_p)^p}+\frac{A'\lambda(B'r)}{pr^p}+\frac{N_\nu(er_p;\xi)}{p(r_p)^p}\end{aligned}$$

for $r>R$ and ξ in $S_k(1)$ by (*) and [5, 2.2, p. 65]. Therefore, since ν is of finite λ-density, we have

$$(**)\qquad |\alpha_p(\xi)+r^{-p}N_p'(r;\nu,\xi)| \leqq \frac{2A'\lambda(B'r)}{pr^p}+\frac{A'\lambda(B'r)}{pr^p}+\frac{A\lambda(Ber_p)}{p(r_p)^p} \leqq \frac{5A'\lambda(B'r)}{pr^p}$$

for $r>R+1$ and ξ in $S_k(1)$ by the choice of r_p.

Definition of α_p for p in N and $p \geqq m$. By the definition of m there is an increasing unbounded sequence of real numbers $\{s_q\}_{q\in N}$ with $s_1 > R+1$ such that $\lim_{q\to\infty}(s_q)^{-m}\lambda(B's_q)=0$. Consider the sequence $\{(s_q)^{-p}N_p(s_q;\nu,\xi)\}_{q\in N}$. As in (**) (for $q'\geqq q$)

$$
\begin{aligned}
&|(s_q)^{-p}N_p(s_q;\nu,\xi)-(s_{q'})^{-p}N_p(s_{q'};\nu,\xi)| \\
&\qquad \leqq |C_p(s_q,s_{q'};\nu,\xi)|+(s_q)^{-p}|N''_p(s_q;\nu,\xi)|+(s_{q'})^{-p}|N''_p(s_{q'};\nu,\xi)| \\
&\qquad \leqq \frac{A'\lambda(B's_q)}{p(s_q)^p}+\frac{A'\lambda(B's_{q'})}{p(s_{q'})^p}+\frac{A\lambda(Bes_q)}{p(s_q)^p}+\frac{A\lambda(Bes_{q'})}{p(s_{q'})^p} \\
&\qquad \leqq \frac{2A'\lambda(B's_q)}{(s_q)^m}+\frac{2A'\lambda(B's_{q'})}{(s_{q'})^m}
\end{aligned}
$$

which converges to zero as q approaches ∞. Thus, the sequence is uniformly Cauchy on $S_k(1)$ so that

$$\alpha_p(\xi) := -\lim_{q\to\infty}(s_q)^{-p}N_p(s_q;\nu,\xi)$$

is continuous on $S_k(1)$ by (5.3). Moreover, as in (**)

$$
\begin{aligned}
&|-(s_q)^{-p}N_p(s_q;\nu,\xi)+r^{-p}N'_p(r;\nu,\xi)| \\
&\qquad \leqq |C_p(s_q,r;\nu,\xi)|+(s_q)^{-p}|N''_p(s_q;\nu,\xi)| \leqq \frac{A'\lambda(B'r)}{pr^p}+\frac{2A'\lambda(B's_q)}{(s_q)^m}
\end{aligned}
$$

for ξ in $S_k(1)$ and $r>R+1$. But $\lim_{q\to\infty}(s_q)^{-m}\lambda(B's_q)=0$ by choice; therefore,

$$
(***)\quad
\begin{aligned}
|\alpha_p(\xi)+r^{-p}N'_p(r;\nu,\xi)| &= \lim_{q\to\infty}|-(s_q)^{-p}N_p(s_q;\nu,\xi)+r^{-p}N'_p(r;\nu,\xi)| \\
&\leqq \frac{A'\lambda(B'r)}{pr^p}\leqq\frac{5A'\lambda(B'r)}{pr^p}
\end{aligned}
$$

for all ξ in $S_k(1)$ whenever $r>R+1$.

Consequently, for each p in N,

$$
\begin{aligned}
|r^p\alpha_p(\xi)+N_p(r;\nu,\xi)| &\leqq r^p|\alpha_p(\xi)+r^{-p}N'_p(r;\nu,\xi)|+|N''_p(r;\nu,\xi)| \\
&\leqq \frac{5A'\lambda(B'r)}{p}+\frac{A\lambda(Ber)}{p} \\
&\leqq \frac{11A\lambda(2Ber)}{p}\leqq\frac{22A\lambda(2Ber)}{p+1}
\end{aligned}
$$

for all ξ in $S_k(1)$ whenever $r>R+1$ by (**) and (***). Q.E.D.

The major result of this paper is the following:

THEOREM 6.4. *A divisor ν in $\dot{D}^+_k$ is the divisor of a function f in $\dot{E}_k(\lambda)$ if and only if ν is λ-admissible.*

The proof will depend upon the properties of a pair of operators, the first of which is the integral operator δ^k defined by

$$\delta^k[f](\zeta) := (k-1)\int_0^1 (1-t)^{k-2} f(t\zeta)\,dt$$

for ζ in $B_k(r)$ and $f\colon B_k(r) \to \boldsymbol{C}$.

LEMMA 6.5. *If f is holomorphic on $B_k(r)$ for some $r>0$ and $k \geqq 2$, and ζ is in $B_k(r)$, then*

(i) *$\delta^k[f]$ is holomorphic on $B_k(r)$.*

(ii) *$\delta^k[f](\zeta) = (1/V_k(1)) \int_{S_k(1)} f((\zeta|\xi)\xi)\sigma_1(\xi)$ where $(\zeta|\xi)$ denotes the standard Hermitian product of ζ and ξ.*

(iii) *If $f(\mathbf{0}) \neq 0$, then*

$$\delta^k[\log|f|](\zeta) \leqq \frac{1}{V_k(1)}\int_{S_k(1)} \log|f((\zeta|\xi)\xi)|\sigma_1(\xi).$$

(iv) *If f is entire with $f(\mathbf{0}) = 1$, then*

$$\log^+ M_f(r) \leqq 8^{3k-1}[B_f(4r) + 2N_f(4er)]$$

for all $r>0$ where

$$B_f(r) := \max\{\delta^k[\log|f|](\zeta) : |\zeta| \leqq r\}$$

and

$$N_f(r) := \frac{1}{V_k(1)}\int_{S_k(1)} N_{\nu_f}(r;\ \xi)\sigma_1(\xi).$$

Proof. (i) is clear. (ii) is Lemma 5.3 of [6, p. 414]. (iii) is Lemma 5.4 of [6, p. 414]. (iv) is a consequence of Lemma 5.6 of [6, p. 415] and the proof of Proposition 2.2 of [5, p. 65]. Q.E.D.

For f holomorphic on $B_k(r)$ we define the $\boldsymbol{C}$-linear differential operator δ_k by

$$\delta_k[f](\zeta) := \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}[z^{k-1}f(z\zeta)]_{z=1}$$

for ζ in $B_k(r)$.

LEMMA 6.6. *Let f be holomorphic on $B_k(r)$ for some $r>0$ and $k \geqq 2$. Then*

(i) *If $f(\mathbf{0}) = 0$, then $\delta_k[f](\mathbf{0}) = 0$.*

(ii) *$\delta_k[f]$ is holomorphic on $B_k(r)$.*

(iii) *$\delta_k \circ \delta^k[f] = \delta^k \circ \delta_k[f] = f$.*

Proof. (i) is obvious. (ii) is an immediate consequence of the Leibnitz formula and the chain rule. That $\delta_k \circ \delta^k = \delta^k \circ \delta_k$ is proved by a standard interchange of integral and differential operators. If f is a homogeneous polynomial then $\delta^k \circ \delta_k[f] = f$ by (6.5)(ii) and Lemma 5.8 of [6, p. 415]. If f is holomorphic on $B_k(r)$, then f is the uniform limit of a sequence of homogeneous polynomials on a neighborhood of $\mathbf{0}$ so that $\delta^k \circ \delta_k[f] = f$ by the preceding case, (ii), (6.5)(i) and the identity principle. Q.E.D.

Proof of Theorem 6.4. If f is in $\dot{E}_k(\lambda)$ then ν_f is λ-admissible by an analogue of the proof of Theorem 4.6 of [5, p. 78] followed by (5.4) and (6.2). Conversely, if ν in

$\dot{D}_1^+$ is λ-admissible then $\nu=\nu_f$ for some f in $\dot{E}_1(\lambda)$ by Theorem 5.2 of [5, p. 87]. Therefore, suppose that ν in $\dot{D}_k^+$ is λ-admissible where $k\geqq 2$. Choose an entire function F on $\boldsymbol{C}^k$ with $\nu_F=\nu$ and $F(\boldsymbol{0})=1$. By (6.3) there is a sequence $\{\alpha_p\}_{p\in \boldsymbol{N}}$ of continuous complex-valued functions on $S=S_k(1)$ and constants A, B and R in $\boldsymbol{R}^+$ such that $|r^p\alpha_p(\xi)+N_p(r;\nu,\xi)|\leqq A\lambda(Br)/(1+p)$ for all ξ in S and p in $\boldsymbol{N}$ whenever $r>R$. Let $c_0(r;\xi):=N_\nu(r;\xi)$, $c_p(r;\xi):=\frac{1}{2}[r^p\alpha_p(\xi)+N_p(r;\nu,\xi)]$, and

$$c_{-p}(r;\xi) := \overline{c_p(r;\xi)} \quad \text{for } p \text{ in } \boldsymbol{N}.$$

Then for each ξ in S and $r>R$,

$$\sum_{p=-\infty}^{\infty} |c_p(r;\xi)|^2 \leqq [A\lambda(Br)]^2 C < \infty \quad \text{where } C := \sum_{p=1}^{\infty} 1/p^2. \tag{1}$$

[Here we have assumed that A, B and R are also the defining constants for the finite λ-density of ν.] By Theorem 5.1 of [5, p. 84] there is a unique entire function f_ξ on C for each ξ in S such that $f_\xi(0)=1$, $\nu_{f_\xi}=\nu|\xi$ and $c_p(r;f_\xi)=c_p(r;\xi)$ for each p in $\boldsymbol{Z}$ and $r>0$. It follows immediately from (5.4) that for each p in $\boldsymbol{N}$ and ξ in S

$$\alpha_p(f_\xi) = \alpha_p(\xi) \tag{2}$$

so that $\alpha_p(f_\xi)$ is continuous in ξ on S. Moreover, for each ξ in S and $r>R$

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} |\log|f_\xi(re^{it})||\,dt \\ &\leqq \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |\log|f_\xi(re^{it})||^2\,dt\right]^{1/2} && \text{[by Holder's Theorem]} \\ &= \left[\sum_{p=-\infty}^{\infty} |c_p(r;f_\xi)|^2\right]^{1/2} && \text{[by Parseval's Theorem]} \\ &\leqq AC^{1/2}\lambda(Br) && \text{[by (1)].} \end{aligned} \tag{3}$$

But $F|\xi$ is entire on $\boldsymbol{C}$ for each ξ in S; and $F|\xi(0)=1$ and $\nu_{F|\xi}=\nu|\xi$ by construction. Therefore, there is a unique entire function g_ξ on $\boldsymbol{C}$ for each ξ in S such that

$$F|\xi = f_\xi \exp(g_\xi) \quad \text{and} \quad g_\xi(0) = 0. \tag{4}$$

Since Re g_ξ is harmonic on $\boldsymbol{C}$ for each ξ in S, it follows that for z in $\boldsymbol{C}$ with $|z|\leqq r$ and $r>R$

$$\begin{aligned} \operatorname{Re} g_\xi(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re} g(2re^{it})\frac{4r^2-|z|^2}{|2re^{it}-z|^2}\,dt && \text{[Poisson Formula]} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(2re^{it}\xi)|\frac{4r^2-|z|^2}{|2re^{it}-z|^2}\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f_\xi(2re^{it})|\frac{4r^2-|z|^2}{|2re^{it}-z|^2}\,dt \\ & && \text{[by (4)]} \\ &\leqq 3\log^+ M_F(2r) + \frac{1}{2\pi}\int_{-\pi}^{\pi} |\log|f_\xi(2re^{it})||\frac{4r^2-|z|^2}{|2re^{it}-z|^2}\,dt \\ &\leqq 3\log^+ M_F(2r) + 3AC^{1/2}\lambda(Br) && \text{[by (3)]} \\ &=: 3K(2r). \end{aligned}$$

Therefore, by the Borel-Carathéodory Lemma [7, p. 175], since $g_\xi(0)=0$, $M_{g_\xi}(r) \leqq 12K(4r)$ for each ξ in S and $r>R$. Let $a_p(\xi)$ be the pth Taylor coefficient of g_ξ at 0. Then by the Cauchy estimates, $|a_p(\xi)| \leqq 12K(4r)r^{-p}$ for each p in N, ξ in S and $r>R$. Thus, for all ζ in $B_k(r)$ with $r>R$, ξ in S, and p in N,

$$|a_p(\xi)(\zeta|\xi)^p| \leqq \frac{12K(8r)}{(2r)^p}|(\zeta|\xi)|^p \leqq \frac{12K(8r)}{2^p}$$

so that

$$g_\xi((\zeta|\xi)) = \sum_{p=1}^{\infty} a_p(\xi)(\zeta|\xi)^p$$

converges uniformly in (ζ, ξ) on $B_k(r)\times S$ for $r>R$. However, from (4) we obtain

$$a_p(\xi) = \frac{1}{2\pi i p}\int_{|z|=s} g'_\xi(z)z^{-p}\,dz = \alpha_p(F;\xi)-\alpha_p(f_\xi)$$

when $B_k(s)\cap \operatorname{supp}\nu=\varnothing$. Now $\alpha_p(F;\xi)$ is continuous in ξ on S by (5.2) and $\alpha_p(f_\xi)$ is continuous in ξ on S by (2). Therefore, $g_\xi((\zeta|\xi))$ is continuous in ξ on S for each ζ in C^k and is holomorphic in ζ on C^k for each ξ in S.

Let $V:=V_k(1)$ and $G(\zeta):=(1/V)\int_S g_\xi((\zeta|\xi))\sigma_1(\xi)$. From the preceding it is clear that G is entire on C^k with $G(\mathbf{0})=0$. Let $H:=\delta_k[G]$. Then by (6.6)(i) and (ii) H is entire on C^k with $H(\mathbf{0})=0$.

Let $f:=F\exp(-H)$. Then f is entire on C^k with $\nu_f=\nu_F=\nu$ and $f(\mathbf{0})=1$. To show that f is of finite λ-type we proceed as follows: Since $f|\xi=F|\xi\exp(-H|\xi)$ $=f_\xi\exp(g_\xi-H|\xi)$ by (4), we have for each ζ in C^k

$$\begin{aligned}(5)\quad &\frac{1}{V}\int_S \log|f((\zeta|\xi)\xi)|\sigma_1(\xi)\\ &= \frac{1}{V}\int_S \log|f_\xi((\zeta|\xi))|\sigma_1(\xi)+\frac{1}{V}\int_S \operatorname{Re}\,[g_\xi((\zeta|\xi))-H((\zeta|\xi)\xi)]\sigma_1(\xi).\end{aligned}$$

But by (6.5)(ii) and (6.6)(iii) this second integral is equal to the real part of

$$(6)\qquad \frac{1}{V}\int_S g_\xi((\zeta|\xi))\sigma_1(\xi)-\delta^k[H](\zeta) = G(\zeta)-\delta^k\circ\delta_k[G](\zeta) = 0.$$

And since $\log|f_\xi|$ is subharmonic on C for each ξ in S we have for ζ in $B_k(r)$ with $r>R$ by the Poisson estimate

$$\begin{aligned}\log|f_\xi((\zeta|\xi))| &\leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f_\xi(2re^{it})|\frac{4r^2-|(\zeta|\xi)|^2}{|2re^{it}-(\zeta|\xi)|^2}\,dt\\ (7)\qquad &\leqq \frac{3}{2\pi}\int_{-\pi}^{\pi}\log|f_\xi(2re^{it})|\,dt\\ &\leqq 3AC^{1/2}\lambda(2Br)\quad\text{by (3).}\end{aligned}$$

Consequently, combining (6.5)(iv), (6.5)(iii), (5), (6) and (7) we get

$$\begin{aligned}\log^+ M_f(r) &\leqq 8^{3k-1}[3AC^{1/2}\lambda(8Br)+2A\lambda(4Ber)]\\ &\leqq 8^{3k-1}5AC^{1/2}\lambda(4Ber)\end{aligned}$$

for $r>R$. Thus, f is of finite λ-type by (3.4). Q.E.D.

REMARK. It is easily verified that the function f in the above is canonical in the sense that it is the unique entire function on $\mathbf{C}^k$ such that $f(\mathbf{0})=1$, $\nu_f=\nu$ and

$$\alpha_p(f;\zeta) = \binom{p+k-1}{p}\frac{1}{V}\int_S \alpha_p(\xi)(\zeta|\xi)^p\sigma_1(\xi).$$

7. **Regular growth functions.** We will say that a growth function λ is *regular for* k if and only if $M_k(\lambda)=Q(E_k(\lambda))$. Also, λ is called *regular* if λ is regular for all dimensions.

PROPOSITION 7.1. *If λ is regular for $k+1$, then λ is regular for k.*

Proof. Suppose that λ is regular for $k+1$. By (4.2) it suffices to show that $\dot{M}_k(\lambda)$ is a subset of $Q(E_k(\lambda))$. Let f in $\dot{M}_k(\lambda)$ be given. Define $\tau\colon \mathbf{C}^{k+1}\to\mathbf{C}^k$ by $\tau(z_1,\ldots,z_{k+1})=(z_1,\ldots,z_k)$. Then τ is $\mathbf{C}$-linear so that $F:=f\circ\tau$ is in $\dot{M}_{k+1}(\lambda)$ by (4.7). By hypothesis, there is H in $E_{k+1}(\lambda)$ such that $H\not\equiv 0$ and HF is in $E_{k+1}(\lambda)$. Choose ζ^* in $\mathbf{C}^{k+1}\backslash(\operatorname{supp}\nu_H\cup\operatorname{supp}\nu_F)$. For ζ in $\mathbf{C}^{k+1}$ define $H^*(\zeta)=H(\zeta+\zeta^*)$ and $F^*(\zeta)=F(\zeta+\zeta^*)$. By (4.1) H^* and H^*F^* are in $\dot{E}_{k+1}(\lambda)$. Define $\tau^*\colon\mathbf{C}^k\to\mathbf{C}^{k+1}$ by $\tau^*(z_1,\ldots,z_k)=(z_1,\ldots,z_k,0)$. Then τ^* is $\mathbf{C}$-linear so that $h^*=H^*\circ\tau^*$ and h^*f^* are in $\dot{E}_k(\lambda)$ when $f^*=F^*\circ\tau^*$. For η in $\mathbf{C}^k$ define $h(\eta):=h^*(\eta-\tau(\zeta^*))$ and $f^\#(\eta-\tau(\zeta^*))$. Then h is in $E^k(\lambda)$ and $hf^\#$ is in $E^k(\lambda)$. But by construction $h\not\equiv 0$ and $f^\#=f$. Q.E.D.

We will say that λ is *strictly regular for* k if and only if for each ν in $\dot{D}_k^+$ which is of finite λ-density, there is ν' in $\dot{D}_k^+$ such that $\nu'\geqq\nu$ and ν' is λ-admissible. As before, we say λ is *strictly regular* if λ is strictly regular for all dimensions.

THEOREM 7.2. *If λ is strictly regular for k, then λ is regular for k. If λ is regular for k, then λ is strictly regular for* 1.

Proof. Suppose λ is strictly regular for k. Let f in $\dot{M}_k(\lambda)$ be given. Then ν_f^∞ is of finite λ-density by (6.1) and there is a divisor ν' in $\dot{D}_k^+$ such that $\nu'\geqq\nu$ and ν' is λ-admissible. By (6.4) there is h in $\dot{E}_k(\lambda)$ such that $\nu_h=\nu'$. By (4.1)(i), hf is in $\dot{M}_k(\lambda)$; but $\nu_{hf}=\nu_h+\nu_f=\nu'+\nu_f^0-\nu_f^\infty\geqq 0$ so that hf is in $E_k(\lambda)$ by the elementary properties of divisors. Thus, λ is regular for k by (4.2). The second statement in the theorem is a consequence of (7.1) and Theorem 5.4 of [5, p. 90]. Q.E.D.

In order to show the existence of a reasonably large class of regular growth functions we present the following sequence of results:

PROPOSITION 7.3. *If λ is a growth function for which there are constants A, B and R in $\mathbf{R}^+$ and p_0 in $\mathbf{N}$ such that*

$$\int_s^r \lambda(t)t^{-p-1}\,dt \leqq A\lambda(Br)r^{-p}+A\lambda(Bs)s^{-p}$$

whenever $r\geqq s>R$ and $p_0\leqq p$ in $\mathbf{N}$, then λ is strictly regular.

Proof. (Compare [5, p. 71].) Suppose ν in $\dot{D}_k^+$ is of finite λ-density with defining constants A', B' and R', which can be assumed to be greater than 1. Choose an entire function f on C^k with $\nu_f=\nu$. Let $\omega:=\exp(2\pi i/p_0)$ and define $f_p(\zeta):=f(\omega^p\zeta)$ for p in N and ζ in C^k and $g:=f_1f_2\cdots f_{p_0}$. Clearly, g is entire on C^k and $\nu':=\nu_g=\nu_1+\cdots+\nu_{p_0}\geqq\nu_{p_0}=\nu$ where $\nu_p:=\nu_{f_p}$. Moreover, $\nu_p(z;\xi)=\nu(z;\omega^p\xi)$ by construction. But $|\omega^p|=1$ so that

$$\begin{aligned} n_{\nu'}(r;\xi) &= \sum_{|z|\leqq r}\sum_{q=1}^{p_0}\nu_q(z;\xi) = \sum_{q=1}^{p_0}\sum_{|z|\leqq r}\nu(z;\omega^q\xi) \\ &= \sum_{q=1}^{p_0}\sum_{|z|\leqq r}\nu(z\omega^q;\xi) = p_0 n_\nu(r;\xi) \leqq p_0 N_\nu(er;\xi), \end{aligned}$$

and

$$N_{\nu'}(r;\xi) = p_0N_\nu(r;\xi)$$

by (1.1) and [5, p. 65]. Thus, ν' is of finite λ-density since $N_{\nu'}(r;\xi)\leqq p_0A'\lambda(B'r)$ for ξ in $S_k(1)$ and $r>R'$. It now suffices to show that ν' is λ-balanced. Observe that if $p<p_0$ then $\sum_{q=1}^{p_0}\omega^{pq}=0$ so that

$$\begin{aligned} C_p(r,s;\nu',\xi) &= \frac{1}{p}\sum_{s<|z|\leqq r}\nu'(z;\xi)z^{-p} \\ &= \frac{1}{p}\sum_{s<|z|\leqq r}\sum_{q=1}^{p_0}\nu(z;\omega^q\xi)z^{-p} \\ &= \sum_{q=1}^{p_0}\frac{1}{p}\sum_{s<|z|\leqq r}\nu(z\omega^q;\xi)z^{-p} \quad \text{[by (1.1)]} \\ &= \sum_{q=1}^{p_0}\omega^{pq}C_p(r,s;\nu,\xi) = 0. \end{aligned}$$

On the other hand, if $p\geqq p_0$ and $r\geqq s>R'':=\max(R,R')$, then

$$\begin{aligned} |C_p(r,s;\nu',\xi)| &\leqq \frac{1}{p}\sum_{s<|z|\leqq r}\nu'(z;\xi)|z|^{-p} = \frac{1}{p}\int_s^r t^{-p}\,dn_{\nu'}(t;\xi) \\ &\leqq \frac{1}{p}n_{\nu'}(r;\xi)r^{-p}+\int_s^r n_{\nu'}(t;\xi)t^{-p-1}\,dt \quad \text{[integrating by parts]} \\ &\leqq \frac{p_0}{p}N_\nu(er;\xi)r^{-p}+p_0\int_s^r N_\nu(et;\xi)t^{-p-1}\,dt \\ &\leqq A'\lambda(B'er)r^{-p}+p_0A'\int_s^r\lambda(B'et)t^{-p-1}\,dt \\ &\leqq A'\lambda(B'er)r^{-p}+p_0A'\int_{B'es}^{B'er}\lambda(u)u^{-p-1}(B'e)^{-p-2}\,du \\ &\leqq A'\lambda(B'er)r^{-p}+p_0A'(Be)^{-2p-2}[A\lambda(BB'er)r^{-p}+A\lambda(BB'es)s^{-p}] \\ &\leqq A''\lambda(B''r)r^{-p}+A''\lambda(B''s)s^{-p} \end{aligned}$$

where $A'':=A'+p_0A'A$ and $B'':=B'e+BB'e$ since $B'>1$. Thus, ν' is λ-balanced. Q.E.D.

COROLLARY 7.4. *If λ is slowly increasing, i.e., if there is $B>1$ such that $\lambda(Br)/\lambda(r)$ is bounded for r sufficiently large, then λ is strictly regular.*

Proof. Lemma 3.7 of [5, p. 71] shows that the hypothesis of (7.3) holds when λ is slowly increasing. Q.E.D.

8. **Applications.** Since $\max(1, \log r)$ and r^ρ for $\rho>0$ are slowly increasing functions of r, they are regular by (7.4). Combining this with (4.4) and (4.5) we obtain

PROPOSITION 8.1. *Let $\rho \geqq 0$ be given. A meromorphic function on C^k has finite ρ-type if and only if it is the quotient of two entire functions having finite ρ-type.*

PROPOSITION 8.2. *The following are equivalent for f meromorphic on C^k:*

(i) *f is a rational function.*

(ii) $\tau_0(f)<\infty$.

(iii) $\operatorname{Ord} f=0$ *and* $\tau(f)<\infty$.

Proof. (ii) and (iii) are equivalent by (4.4). Since each coordinate projection has finite zero-type by (3.4), it follows that each polynomial and, therefore, each rational function has finite zero-type by (4.4) and (4.1)(i). Conversely, by (8.1) it suffices to show that each entire function having finite zero-type is a polynomial. But if f is entire and $\tau_0(f)<\infty$, then assuming without loss of generality that ν_f is in $\dot{D}_f^+$, from (3.4), (3.14) and (4.4) we obtain

$$\begin{aligned}\log^+ M_{f|\xi}(r) &\leqq AT_{f|\xi}(8er, s)+B \leqq AT_f(8er; \xi)+B \\ &\leqq A'AT_f(8B'er)+B \leqq A''A'A \log(8B'B''er)+B\end{aligned}$$

for r sufficiently large. The Cauchy estimates then show that $f|\xi$ is a polynomial of degree no larger than $A''A'A$. Consequently, f is a polynomial. Q.E.D.

We can now state the following criteria for algebraic divisors:

PROPOSITION 8.3. *Let $\lambda(r)=\max(1, \log r)$ for $r>0$. Then the following are equivalent for ν in $\dot{D}_k^+$:*

(i) *ν is λ-admissible.*

(ii) *ν is the divisor of a polynomial in C^k.*

(iii) *$n_\nu(r; \xi)$ is bounded for all $r>0$ and all ξ in $S_k(1)$.*

(iv) *ν is of finite λ-density.*

Proof. (i) and (ii) are equivalent by (8.2), (4.4) and (6.4). If ν is the divisor of a polynomial then $n_\nu(r; \xi)$ is bounded by the degree of that polynomial. Thus, (ii) implies (iii). Suppose that $n_\nu(r; \xi) \leqq A$. Then $N_\nu(r; \xi)=\int_0^r n_\nu(t; \xi)\, dt/t = \int_{s_0}^r n_\nu(t; \xi)\, dt/t \leqq A \log r/s_0$ for $r \geqq s_0$ where $\operatorname{supp} \nu \cap B_k(s_0)=\varnothing$. Thus, ν is of finite λ-density; and (iii) implies (iv). If, on the other hand, ν is of finite λ-density, then for r sufficiently large

$$N_\nu(r; \xi) = \int_{s_0}^r n_\nu(t; \xi) \frac{dt}{t} \leqq A \log Br$$

so that $\int_{s_0}^r [n_\nu(t;\xi)-A]\,dt/t \leqq A \log Bs_0$. But if $n_\nu(t_0;\xi)-A \geqq \varepsilon > 0$ for any $t_0>0$, then $n_\nu(t;\xi)-A\geqq\varepsilon>0$ for all $t>t_0$ and $\int_{s_0}^r [n_\nu(t;\xi)-A]\,dt/t$ is unbounded since $\int_{t_0}^\infty \varepsilon/t\,dt=\infty$. Thus, (iv) implies (iii). It now suffices to show that (iii) implies (i). But $|C_p(r,s;\nu,\xi)|\leqq s^{-p}n_\nu(r;\xi)$ for all $r\geqq s>0$, so that ν is λ-admissible if n_ν is bounded. Q.E.D.

Let us now examine the meaning of finite λ-density and λ-admissibility for the other classical growth functions.

PROPOSITION 8.4. *Let $\lambda(r):=r^\rho$ for $\rho>0$. Then*

(i) *A divisor ν in $\dot{D}_k^+$ is of finite λ-density if and only if $r^{-\rho}n_\nu(r;\xi)$ is bounded for all $r>0$ and all ξ in $S_k(1)$.*

(ii) *When ρ is not an integer, every divisor of finite λ-density is λ-admissible.*

(iii) *When ρ is an integer, a divisor ν in $\dot{D}_k^+$ is λ-admissible if and only if there is $R>0$ such that $r^{-\rho}N'_\rho(r;\nu,\xi)$ is bounded for all $r>R$ and all ξ in $S_k(1)$ and ν is of finite λ-density.*

The proof is an appropriate modification of that given for Proposition 3.3 of [5, p. 69].

Combining (6.4) with (4.4), (4.5), (8.3) and (8.4) we obtain the following generalization of a theorem of Lindelöf (see [4], [5, p. 88] and [6, p. 418]):

PROPOSITION 8.5. *A divisor ν in $\dot{D}_k^+$ is the divisor of an entire function having finite ρ-type (for $\rho\geqq 0$) if and only if $r^{-\rho}n_\nu(r;\xi)$ is bounded for all $r>0$ and all ξ in $S_k(1)$ and, when ρ is a natural number, $r^{-\rho}N'_\rho(r;\nu,\xi)$ is bounded for all ξ in $S_k(1)$ and all r sufficiently large.*

9. **Functions of zero λ-type.** We say that a meromorphic function f on $\boldsymbol{C}^k$ is *of zero λ-type* whenever there are constants $s>0$ and $B>0$ such that

$$\lim_{r\to\infty} T_f(r,s)/\lambda(Br) = 0.$$

Let $M_k^0(\lambda)$ and $E_k^0(\lambda)$ denote, respectively, the classes of meromorphic and entire functions of zero λ-type on $\boldsymbol{C}^k$. As an immediate consequence of the definitions involved we have the following relations between finite and zero λ-type:

PROPOSITION 9.1. (i) *$M_k^0(\lambda)\subset M_k(\lambda)$ and $E_k^0(\lambda)\subset E_k(\lambda)$.*

(ii) *If λ is bounded, then $E_k^0(\lambda)=M_k^0(\lambda)=E_k(\lambda)=M_k(\lambda)=C$.*

(iii) *If λ is swiftly increasing, i.e. if there is $B>1$ such that* $\lim_{r\to\infty}\lambda(r)/\lambda(Br)=0$, *then $M_k^0(\lambda)=M_k(\lambda)$ and $E_k^0(\lambda)=E_k(\lambda)$.*

(iv) *$M_k^0(\lambda)$ is the union of those classes $M_k(\mu)$ where* $\lim_{r\to\infty}\mu(r)/\lambda(B_\mu r)=0$ *for some $B_\mu>0$.*

(v) *If* $\lim_{r\to\infty}\mu(r)/\lambda(Br)=0$ *for some $B>0$, then $M_k(\mu)\subset M_k^0(\lambda)$.*

NOTE. In view of (ii), in the sequel we will assume that *all growth functions are unbounded.*

We say that $A_0\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is a *vanishing function* provided that A_0 is decreasing and converges to zero at ∞. The theory of functions of zero λ-type is completely analogous to the theory of functions of finite λ-type and is developed by judiciously replacing the constants "A" by vanishing functions "A_0". We now list some of the results which can be obtained in this manner:

PROPOSITION 9.2. (SEE 4.1). (i) *$M_k^0(\lambda)$ is an extension field of $\boldsymbol{C}$ and is invariant under affine transformations of the variable in $\boldsymbol{C}^k$.*

(ii) *Either $M_k^0(\lambda)=E_k^0(\lambda)=\boldsymbol{C}$ or $E_k^0(\lambda)$ contains the ring of complex polynomial functions on $\boldsymbol{C}^k$ and $M_k^0(\lambda)$ contains the field of rational functions on $\boldsymbol{C}^k$.*

PROPOSITION 9.3 (SEE 4.3). *Suppose f is meromorphic on $\boldsymbol{C}^k$ with ν_f in $\dot{D}_k$. Then f is in $M_k^0(\lambda)$ if and only if there are constants $B>0$ and $R>0$ and a vanishing function A_0 such that $T_f(r;\,\xi)\leqq A_0(r)\lambda(Br)$ for all $r>R$ and all ξ in $S_k(1)$.*

We say that a divisor ν in $\dot{D}_k^+$ is *of zero λ-density* if and only if there are constants $B>0$ and $R>0$ and a vanishing function A_0 such that $N_\nu(r;\,\xi)\leqq A_0(r)\lambda(Br)$ for all $r>R$ and all ξ in $S_k(1)$. Let $\dot{M}_k^0(\lambda)=\dot{M}_k(\lambda)\cap M_k^0(\lambda)$.

PROPOSITION 9.4 (SEE 6.1). *If f is in $\dot{M}_k^0(\lambda)$, then ν_f^∞ is of zero λ-density.*

We say that ν in $\dot{D}_k^+$ is *finely λ-balanced* if and only if there are constants $B>0$ and $R>0$ and a vanishing function A_0 such that

$$|C_p(r,s;\nu,\xi)| \leqq A_0(r)\lambda(Br)r^{-p}+A_0(s)\lambda(Bs)s^{-p}$$

for all p in $\boldsymbol{N}$ and all ξ in $S_k(1)$ whenever $r>s>R$. And we say ν is *finely λ-admissible* whenever ν is of zero λ-density and is finely λ-balanced.

PROPOSITION 9.5 (SEE 6.3). *If ν in $\dot{D}_k^+$ is finely λ-admissible, then there is a sequence $\{\alpha_p\}_{p\in N}$ of continuous complex-valued functions on $S_k(1)$ and constants $B>0$ and $R>0$ and a vanishing function A_0 such that*

$$|r^p\alpha_p(\xi)+N_p(r;\nu,\xi)| \leqq A_0(r)\lambda(Br)/(1+p)$$

for all p in $\boldsymbol{N}$ and all ξ in $S_k(1)$ whenever $r>R$.

REMARK. In modifying the proof of (6.3) it is necessary to define m as $\min\{p\in \boldsymbol{N} : \liminf A_0(r)\lambda(B'r)r^{-p}=0\}$.

PROPOSITION 9.6 (SEE 6.4). *A divisor ν in $\dot{D}_k^+$ is the divisor of an entire function of zero λ-type if and only if ν is finely λ-admissible.*

We say that λ is *zero-regular for k* if and only if every meromorphic function of zero λ-type on $\boldsymbol{C}^k$ is the quotient of two entire functions of zero λ-type on $\boldsymbol{C}^k$; and λ is *zero-regular* whenever λ is zero-regular for all dimensions. In view of (9.1)(iii) we have

PROPOSITION 9.7. *When λ is swiftly increasing, λ is zero-regular for k if and only if λ is regular for k.*

PROPOSITION 9.8 (SEE 7.1). *If λ is zero-regular for $k+1$, then λ is zero-regular for k.*

We say that λ is *strictly zero-regular for k* if and only if for each ν in $\dot{D}_k^+$ which is of zero λ-density there is a finely λ-admissible divisor ν' in $\dot{D}_k^+$ such that $\nu' \geqq \nu$; and λ is *strictly zero-regular* whenever λ is strictly zero-regular for all dimensions.

PROPOSITION 9.9 (SEE 7.2). *If λ is strictly zero-regular for k, then λ is zero-regular for k. If λ is zero-regular for k, then λ is strictly zero-regular for* 1.

PROPOSITION 9.10 (SEE 7.3). *If λ is a growth function for which there are constants A, B and R in $\boldsymbol{R}^+$, a vanishing function A_0, and p_0 in $\boldsymbol{N}$ such that*

$$\int_s^r \lambda(t)t^{-p-1}\,dt \leqq A_0(r)\lambda(Br)r^{-p} + A\lambda(Bs)s^{-p}$$

whenever $r \geqq s > R$ and $p_0 \leqq p$ in $\boldsymbol{N}$, then λ is strictly zero-regular.

PROPOSITION 9.11 (SEE 7.4). *If λ is slowly increasing, then λ is strictly zero-regular.*

Turning to the classical growth functions we have

PROPOSITION 9.12 (SEE 4.4 AND 8.2). *Let* $\lambda(r) := \max(1, \log r)$. *Then the following are equivalent for f meromorphic on $\boldsymbol{C}^k$:*

(i) *f is of zero λ-type.*
(ii) $\tau_0(f) = 0$.
(iii) $\operatorname{Ord} f = 0$ *and* $\tau(f) = 0$.
(iv) *f is constant on $\boldsymbol{C}^k$.*

PROPOSITION 9.13 (SEE 4.5 AND 8.1). *For $\rho > 0$, let $\lambda(r) := r^\rho$. Then the following are equivalent for f meromorphic on $\boldsymbol{C}^k$:*

(i) *f is of zero λ-type.*
(ii) $\tau_\rho(f) = 0$.
(iii) $\operatorname{Ord} f < \rho$ *or both* $\operatorname{Ord} f = \rho$ *and* $\tau(f) = 0$.
(iv) *f is the quotient of two entire functions of zero λ-type on $\boldsymbol{C}^k$.*

PROPOSITION 9.14 (SEE 8.3). *Let* $\lambda(r) := \max(1, \log r)$. *Then the following are equivalent for ν in $\dot{D}_k^+$:*

(i) *ν is finely λ-admissible.*
(ii) *ν is the divisor of a constant function on $\boldsymbol{C}^k$.*
(iii) *$n_\nu(r; \xi) = 0$ for all $r > 0$ and all ξ in $S_k(1)$.*
(iv) *ν is of zero λ-density.*

PROPOSITION 9.15 (SEE 8.4). *For $\rho > 0$, let $\lambda(r) := r^\rho$. Then*

(i) *A divisor ν in $\dot{D}_k^+$ is of zero λ-density if and only if $r^{-\rho}n_\nu(r; \xi)$ converges to zero at $r = \infty$ uniformly in ξ on $S_k(1)$.*

(ii) *When ρ is not an integer, every divisor of zero λ-density is finely λ-admissible.*

(iii) *When ρ is an integer, a divisor ν in $\dot{D}_k^+$ is finely λ-admissible if and only if ν is of zero λ-density and $r^{-\rho}N'_\rho(r;\nu,\xi)$ converges to a complex-valued function on $S_k(1)$ as r approaches ∞ uniformly in ξ on $S_k(1)$.*

Proof. (i) If ν is of zero λ-density, then $n_\nu(r;\xi) \leqq N_\nu(er;\xi) \leqq A_0(er)(Be)^\rho r^\rho$ for r sufficiently large, where A_0 is a vanishing function. Thus, $r^{-\rho}n_\nu(r;\xi)$ converges to zero at $r=\infty$ uniformly in ξ on $S_k(1)$. Conversely, suppose that there is $R(\varepsilon)>0$ for each $\varepsilon>0$ such that $r^{-\rho}n_\nu(r;\xi)<\varepsilon$ whenever $r>R(\varepsilon)$ and ξ is in $S_k(1)$. Let $K(\varepsilon)=\sup\{N_\nu(eR(\varepsilon);\xi) : \xi \in S_k(1)\}$ which is finite since $N_\nu(eR(\varepsilon);\xi)$ is continuous in ξ on $S_k(1)$ by (2.2). Let $\varepsilon_0=\varepsilon\rho/2$. Then whenever ξ is in $S_k(1)$ and $r>\max(R(\varepsilon_0), [2K(\varepsilon_0)/\varepsilon]^{1/\rho})$ we have

$$\begin{aligned} N_\nu(r;\xi) &= N_\nu(R(\varepsilon_0);\xi)+\int_{R(\varepsilon_0)}^r n_\nu(t;\xi)t^{-1}\,dt \\ &< K(\varepsilon_0)+(\varepsilon/2)r^\rho \end{aligned}$$

so that $r^{-\rho}N_\nu(r;\xi)<\varepsilon$. It follows that

$$A_0^*(r) = \sup\{s^{-\rho}N_\nu(s;\xi) : s \geqq r,\ \xi \in S_k(1)\}$$

is a vanishing function such that $N_\nu(r;\xi) \leqq A_0^*(r)r^\rho$ for all $r>0$ and ξ in $S_k(1)$. Thus, ν is of zero λ-density.

(ii) Suppose ν is of zero λ-density. For each natural number $p<\rho$ let $\varepsilon_p=\varepsilon(\rho-p)/2$. As in (i) we obtain that for each $\varepsilon>0$

$$r^{p-\rho}\int_0^r n_\nu(t;\xi)t^{-p-1}\,dt < \varepsilon$$

whenever $r>\max(R(\varepsilon_p), [2K(\varepsilon_p)/\varepsilon]^{1/\rho})$ so that

$$A_p(r) = \sup\left\{s^{p-\rho}\int_0^s n_\nu(t;\xi)t^{-p-1}\,dt : s \geqq r,\ \xi \in S_k(1)\right\}$$

is a vanishing function. Now, for each natural number p and $r \geqq s>0$

$$|C_p(r,s;\nu,\xi)| \leqq (1/p)\sum_{s<|z|\leqq r}\nu(z;\xi)|z|^{-p} = (1/p)\int_s^r t^{-p}\,d(n_\nu(t;\xi)).$$

Integration by parts shows that this integral is dominated by

$$r^{-p}n_\nu(r;\xi)+\int_s^r n_\nu(t;\xi)t^{-p-1}\,dt.$$

Thus, for $p<\rho$ and $r \geqq s>0$

$$|C_p(r,s;\nu,\xi)| \leqq A_0^*(r)r^{\rho-p}+A_p(r)r^{\rho-p}.$$

But, for $p>\rho$ and $r \geqq s>0$ we have

$$|C_p(r,s;\nu,\xi)| \leqq A_0^*(r)r^{\rho-p}+A_0^*(s)s^{\rho-p}/(p-\rho).$$

Hence, if ρ is not an integer we see that ν is finely λ-balanced by the vanishing function

$$A_0^{**} = \sum_{p<\rho} A_p + A_0^* \Big/ \min_{p>\rho} (p-\rho).$$

Thus, ν is finely λ-admissible.

(iii) If ν is of zero λ-density and $\lim_{r\to\infty} r^{-\rho} N'_\rho(r; \nu, \xi) = \alpha(\xi)$ where the convergence is uniform in ξ on $S_k(1)$ then $A_\rho(r)$ is a vanishing function where $A_\rho(r)=1$ for $0<r\leqq R_1$ and for $r>R_1$

$$A_\rho(r) = \sup\left\{(1/\rho)\left|\sum_{s<|z|} \nu(z;\xi)z^{-\rho}\right| : s \geqq r,\ \xi\in S_k(1)\right\}$$

where R_1 is chosen so that $|\sum_{r<|z|} \nu(z;\xi)z^{-\rho}|<\rho$ whenever $r>R_1$. Once again it follows that ν is finely λ-balanced by the vanishing function $A_0^{**}+A_\rho$. Conversely, if ν is finely λ-admissible then ν is of zero λ-density by definition and the balancing condition on C_ρ implies the Cauchy criterion for the uniform convergence of $r^{-\rho}N'_\rho(r;\nu,\xi)$ at $r=\infty$ on $S_k(1)$. Q.E.D.

REMARK. An examination of the proof of (9.5) will show that in (iii) $\lim_{r\to\infty} r^{-\rho}N'_\rho(r;\nu,\xi)=\alpha_\rho(\xi)$ which is continuous on $S_k(1)$.

PROPOSITION 9.16 (SEE 8.5). *A divisor ν in $\dot{D}_k^+$ is the divisor of an entire function having zero ρ-type (for $\rho\geqq 0$) if and only if $r^{-\rho}n_\nu(r;\xi)$ converges to zero at $r=\infty$ uniformly in ξ on $S_k(1)$ and, when ρ is a natural number, $r^{-\rho}N'_\rho(r;\nu,\xi)$ converges to a complex-valued function as r approaches ∞ uniformly in ξ on $S_k(1)$.*

10. **Extra-regular growth functions.** We say that a growth function λ is *extra-regular* if and only if every meromorphic function of finite λ-type is the quotient of two entire functions of finite λ-type which are everywhere locally relatively prime. And we can make an analogous definition for *extra-zero-regularity*. The following is then an easy consequence of the arguments used in (7.2) and (9.9), of Proposition 3.5 of [6, p. 406], and of (8.3), (8.4)(ii), (9.14) and (9.15)(ii):

PROPOSITION 10.1. *If $\lambda(r):=\max(1,\log r)$ for $r>0$, then λ is extra-regular and extra-zero-regular. If $\lambda(r)=r^\rho$ for nonintegral positive ρ, then λ is extra-regular and extra-zero-regular.*

BIBLIOGRAPHY

1. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965. MR **31** #4927.

2. L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N. J., 1966.

3. H. Kneser, *Zur Theorie der gebrochenen Funktionen mehrer Verandlicher*, Jber. Deutsch. Math.-Verein. **48** (1938), 1–28.

4. E. Lindelöf, *Sur les fonctions entières d'ordre entier*, Ann. Sci. École Norm. Sup. (3) **22** (1905), 369–396.

5. L. A. Rubel and B. A. Taylor, *A fourier series method for meromorphic and entire functions*, Bull. Soc. Math. France **96** (1968), 53–96.

6. W. Stoll, *About entire and meromorphic functions of exponential type*, Proc. Sympos. Pure Math., vol. 11, Amer. Math. Soc., Providence, R. I., 1968, pp. 392–430. MR **38** #4706.

7. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Univ. Press, London, 1939.

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