THE BIFURCATION OF SOLUTIONS IN BANACH SPACES

BY

WILLIAM S. HALL

Abstract. Let $L: D \subset X \to D \subset X^*$ be a densely defined linear map of a reflexive Banach space X to its conjugate X^*. Define M and M^* to be the respective null spaces of L and its formal adjoint L^*. Let $f: X \to X^*$ be continuous. Under certain conditions on L^* and f there exist weak solutions to $Lu = f(u)$ provided for each $w \in X$, $v(w) \in M$ can be found such that $f(v(w) + w)$ annihilates M^*. Neither M and M^* nor their annihilators need be the ranges of continuous linear projections. The results have applications to periodic solutions of partial differential equations.

1. **Introduction.** Sufficient conditions are given for the existence of solutions in a reflexive Banach space X for the equation

$$(1.1) \qquad\qquad Lu = f(u)$$

where f is a (nonlinear) continuous map of X to its conjugate X^*. We assume the linear operator L has a nontrivial manifold of solutions. Hence one condition is that a bifurcation equation can be solved. Results of these kinds are not new; what is of interest here is that none of the frequently assumed hypotheses of complementing subspaces, direct sum decompositions, and continuous projections are made. Rather, we show that if the solution to the bifurcation equation satisfies a continuity condition, then the quotient space and the natural map provide the necessary tools for solving (1.1).

While thus broadening the class of problems that can be considered via bifurcation theory, only weak solutions are obtained. Hence our work neither replaces nor entirely includes the researches of others in this area. Of these, we mention the work of Hale, Bancroft, and Sweet [1] for general theory, the paper of Mawhin [2] for an application to ordinary differential equations, and the efforts of Cesari [3], Hale [4], Vejvoda [5], Rabinowitz [6], and the author [7], [8] for applications to periodic solutions of partial differential equations.

The paper is divided into four sections. First, the notation is introduced and a linear equation is solved. The nonlinear equation is then considered, and the next section is devoted to remarks on the previous. An example from partial differential equations concludes the paper.

2. **A related linear problem.** Let $(D, \|\cdot\|)$ be a normed linear space and X its isometric completion. $(X^*, \|\cdot\|^*)$ will be its conjugate, and for $u \in X$, $v \in X^*$, put (u, v) to be the value of v at u. X is assumed reflexive, and we suppose $D \subset X^*$.

$L: D \to D$ is a given linear operator which we assume has a formal adjoint L^* defined by $(\phi, L\psi) = (\psi, L^*\phi)$ for all ϕ, ψ in D. Note that L is regarded as having range in X^*, and because X is reflexive, the range of L^* is also in X^*.

When $\psi \in D$ satisfies $L\psi = 0$, then $0 = (\phi, L\psi) = (\psi, L^*\phi)$ for all $\phi \in D$. This motivates the definition

$$M = \{u \in X; (u, L^*\phi) = 0, \forall \phi \in D\}$$

as the generalized null space of L. Similarly, the generalized null space of L^* is the set

$$M^* = \{u \in X; (u, L\phi) = 0, \forall \phi \in D\}.$$

$°M$ and $°M^*$ are their respective annihilators in X^*, while X/M and X/M^* are the corresponding quotient spaces. $\Pi: X \to X/M$ will denote the natural map $\Pi u = u + M$.

Note that the conjugate of X/M is $°M$ and that the adjoint $\Pi^*: °M \to X^*$ is the inclusion. Hence $(u+M, \phi) = (\Pi u, \phi) = (u, \Pi^*\phi) = (u, \phi)$ whenever $\phi \in °M$. Also, because X is reflexive, the dual of $°M$ is congruent to X/M, and the range of L^*,

$$E^* = \{\psi \in D; \psi = L^*\phi, \phi \in D\},$$

is dense in $°M$. If not, and if $v \neq 0$ is in $°M \backslash \operatorname{cl} E^*$ then by the Hahn-Banach Theorem and reflexivity, there exists $u+M$ such that $(u+M, v) = \operatorname{dist}(v, \operatorname{cl} E^*) > 0$, $\|u+M\| = 1$, and u vanishes on $\operatorname{cl} E^*$. Hence $(u+M, L^*\phi) = (u, L^*\phi) = 0$ for all $\phi \in D$. Thus $u \in M$ and so $\|u+M\| = 0 \neq 1$.

If $u \in D$ satisfies $Lu = g$, $g \in D$, then g annihilates $M^* \cap D$ since for such ϕ, $(\phi, g) = (u, L^*\phi) = 0$. Conversely, we have

THEOREM 1. *Suppose for all $\phi \in D$, $\|\phi + M^*\| \leq k\|L^*\phi\|^*$. Then for each g in $°M^*$ there is a unique coset $u+M = Kg$ such that any representative is a weak solution of $Lu = g$. The operator $K: °M^* \to X/M$ is linear and bounded with $\|K\| \leq k$.*

Proof. Let $\mathscr{L}^*: D/M^* \to X^*$ be the induced operator defined by $\mathscr{L}^*(\phi + M^*) = L^*\phi$, $\phi \in D$. By hypothesis,

$$\|\phi + M^*\| \leq k\|L^*\phi\|^* = k\|\mathscr{L}^*(\phi + M^*)\|$$

and hence $(\mathscr{L}^*)^{-1}$ is a continuous bijection of E^* onto D/M^* with norm not exceeding k.

Define the functional l on E^* by $l(\psi) = ((\mathscr{L}^*)^{-1}\psi, g)$. Then l is linear, and from the above

$$|l(\psi)| \leq \|(\mathscr{L}^*)^{-1}\psi\| \|g\|^* \leq k\|\psi\|^* \|g\|^*.$$

Let l also denote the unique extension to $^\circ M$, the closure of E^*. By reflexivity, there is a unique $u + M$ such that $l(\psi) = (u + M, \psi)$ for all $\psi \in E^*$, and $\|u + M\| = \|l\| \le k \|g\|^*$.

Let $K: {}^\circ M \to X/M$ be given by $u + M = Kg$. Then K is linear and $\|K\| \le k$.

Now $\psi = \mathscr{L}^*(\phi + M^*) = L^*\phi$, for some $\phi \in D$. Thus, $l(\psi) = (u + M, \psi) = (u + M, L^*\phi)$, and since $L^*\phi \in {}^\circ M$, $(u + M, L^*\phi) = (u, L^*\phi)$ for any representative $u \in u + M$. But, $l(\psi) = ((\mathscr{L}^*)^{-1}\psi, g) = (\phi + M^*, g) = (\phi, g)$ since g annihilates M^*. Therefore, $(u, L^*\phi) = (Kg, L^*\phi) = (\phi, g)$. This last relation holds for all $\phi \in D$ since $(\mathscr{L}^*)^{-1}$ is bijective. Hence any $u \in u + M$ is a weak solution of $Lu = g$. This proves the theorem.

3. **The nonlinear equation.** We now consider the problem $Lu = f(u)$. The following properties for f are assumed:

(i) f is a continuous, bounded map of X to X^* such that

$$(3.1) \qquad \|f(u)\|^* \le c_1[\|u\|]$$

where c_1 is increasing in $\|u\|$.

(ii) For each $u \in X$, there exists $v(u) \in M$ such that $f(v(u) + u)$ is in $^\circ M^*$. The element $v: X \to M$ is continuous and

$$(3.2) \qquad \|v(u)\| \le c_2[\|u\|]$$

where c_2 is increasing in $\|u\|$. This is the assumption that the bifurcation equation can be solved, and that the solution $v(u)$ is a bounded continuous function of u.

THEOREM 2. *Let f be as above and suppose $\|\phi + M^*\| \le k\|L^*\phi\|^*$ for all $\phi \in D$. Assume the operator K of Theorem 1 is completely continuous. If c_1 is "sufficiently small", then $Lu = f(u)$ has a weak solution.*

The statement, "if c_1 is sufficiently small" will be clarified in the proof. We shall need the following result:

THEOREM (MICHAEL [9, p. 6]). *Let Z and W be Banach spaces and $T: Z \to W$ a continuous epimorphism. For each $\lambda > 1$, there exists a continuous $h: W \to Z$ such that $(T \circ h)(w) = w$ for all $w \in W$. In addition,*

$$\|h(w)\| \le \lambda \inf \{\|z\|; Tz = w\}.$$

We shall use Michael's result with $Z = X$, $W = X/M$, and $T = \Pi$. In this case

$$(3.3) \qquad \|h(u + M)\| \le \lambda \|u + M\|.$$

Proof of Theorem 2. Let λ and h be as above. Let $B(a) = \{u \in X; \|u\| \le a\}$ and consider the map

$$Tu = (h \circ K)(f(v(u) + u))$$

where K is as in Theorem 1, and where $v \in M$ is chosen as in (ii) above. Then T is continuous, and by (3.1) to (3.3),

$$\|Tu\| \;\leqq\; \lambda \|Kf(v(u)+u)\| \;\leqq\; \lambda \|f(v(u)+u)\|^*$$
$$\leqq\; \lambda k c_1 [\|v(u)+u\|] \;\leqq\; \lambda k c_1 [c_2[a]+a].$$

Hence, if $\lambda > 1$ and $a > 0$ can be chosen so that

$$(3.4) \qquad\qquad\qquad \lambda k c_1 [c_2[a]+a] \;\leqq\; a$$

then T maps $B(a)$ to itself. Since K is completely continuous, Schauder's Theorem gives a fixed point w. Thus, $w = (h \circ K)f(v(w)+w)$ and $\Pi w = Kf(v(w)+w)$. Let $u = v(w)+w$. Then

$$(u, L^*\phi) = (\Pi u, L^*\phi) = (\Pi w, L^*\phi)$$
$$= (Kf(v(w)+w), L^*\phi) = (\phi, f(v(w)+w)) = (\phi, f(u)).$$

Hence u is the desired solution and the theorem is proved.

The hypothesis that K be a compact mapping can be removed if both f and v are Lipschitz continuous in u. Even so, the function $Tu = (h \circ K)(f(v(u)+u))$ is not necessarily Lipschitzian since h may not possess this property. Hence a method other than the application of a nonexpansive mapping theorem is required. This is provided by the following result.

From the interior mapping principle, if $T: Z \to W$ is an epimorphism, then there is a $t > 0$ such that for each $w \in W$, $\exists z \in Z$ with $w = Tz$ and $\|z\| \leqq t\|w\|$.

THEOREM (GRAVES [10, p. 111]). *Let $G: Z \to W$ be continuous for $\|z\| < \gamma$ with $G(0) = 0$. Let T be as above and suppose*

$$\|G(z) - G(z') - T(z-z')\| \;\leqq\; \delta \|z-z'\|$$

whenever $\|z\| < \gamma$, $\|z'\| < \gamma$. If $t\delta < 1$ and $\|w\| < \gamma(1-t\delta)/t$, then there exists $z \in Z$ such that $G(z) = w$.

We shall apply this theorem with $Z = X$, $W = X/M$, and $T = \Pi$. In this case any $t > 1$ will suffice.

Assume f satisfies the conditions (i)', (ii)' below:

(i)' f is a continuous, bounded map of X to X^*, and there are constants d_1 and d_2 such that

$$(3.5) \qquad\qquad\qquad \|f(0)\|^* \;\leqq\; d_1,$$

$$(3.6) \qquad\qquad\qquad \|f(u)-f(u')\|^* \;\leqq\; d_2\|u-u'\|,$$

whenever $\|u\| \leqq a$ and $\|u'\| \leqq a$.

(ii)' For each $u \in X$, there exists $v \in M$ such that $f(v(u)+u) \in {}^\circ M^*$. The element $v: X \to M$ satisfies

$$(3.7) \qquad\qquad\qquad \|v(0)\| \;\leqq\; d_3,$$

$$(3.8) \qquad\qquad\qquad \|v(u)-v(u')\| \;\leqq\; d_4\|u-u'\|,$$

whenever $\|u\| \leq a$ and $\|u'\| \leq a$. Here, $v(0)$ is an element in M such that $f(v(0)) \in {}^\circ M^*$. ($v(0)$ is the bifurcation point for $Lu = f(u)$.)

THEOREM 3. *Suppose for all* $\phi \in D$, $\|L^*\phi\|^* \geq \|\phi + M^*\|$. *If the constants* d_i, *$i = 1, 2, 3, 4$, are "sufficiently small" then* $Lu = f(u)$ *has a weak solution.*

Proof. Define for $u \in X$, $\|u\| \leq a$, the function $G: X \to X/M$ by

$$G(u) = \Pi u - Kf(v(u) + u) + Kf(v(0))$$

where K is as in Theorem 1. $G(0) = 0$, and if $\|u\| \leq a$, $\|u'\| \leq a$, then by (3.5) to (3.8),

$$
\begin{aligned}
\|G(u) - G(u') - \Pi(u - u')\| &\leq k\|f(v(u) + u) - f(v(u') + u')\|^* \\
&\leq kd_2\|v(u) + u - v(u') - u'\| \\
&\leq kd_2\{\|u - u'\| + \|v(u) - v(u')\|\} \\
&\leq kd_2\{1 + d_4\}\|u - u'\|.
\end{aligned}
$$

Next, $\|v(0)\| \leq d_3$. Hence

$$
\begin{aligned}
\|Kf(v(0))\| &\leq k\|f(v(0))\|^* \\
&\leq k\{\|f(v(0)) - f(0)\|^* + \|f(0)\|^*\} \leq k\{d_2 d_3 + d_1\}.
\end{aligned}
$$

Hence if there exists $a > 0$ and $t > 1$ such that

$$(3.9) \qquad\qquad kd_2(1 + d_4) = t\delta < 1,$$

$$(3.10) \qquad\qquad k\{d_2 d_3 + d_1\} < a(1 - t\delta)/t.$$

Then by Grave's theorem, there is an element $w \in X$ with $\|w\| \leq a$ such that $G(w) = Kf(v(0))$. Hence $\Pi w = Kf(v(w) + w)$, and as before, if $u = v(w) + w$, $(u, L^*\phi) = (\phi, f(u))$ for all $\phi \in D$. Thus, the theorem is proved.

4. Remarks. Special situations, such as when f contains a parameter, M splits X, X is uniformly convex, etc., deserve comment.

REMARK 1. If $f(u) = g(u, \varepsilon)$ where $\|g(u, \varepsilon)\|^* \leq \eta(\varepsilon, \|u\|)$, and if $\eta(\varepsilon, a) \to 0$ uniformly in a with ε, then inequality (3.4) holds for small ε. Similarly, suppose whenever $\|u\| \leq a$, $\|u'\| \leq a$,

$$\|g(0, \varepsilon)\|^* \leq \rho_1(\varepsilon), \qquad \|g(u, \varepsilon) - g(u', \varepsilon)\|^* \leq \rho_2(\varepsilon)\|u - u'\|,$$

where ρ_1 and ρ_2 are $O(\varepsilon)$ as $\varepsilon \to 0$. Then f will satisfy (3.9) and (3.10). However, see the example, where such a restriction is not required.

REMARK 2. Suppose M splits X so that $X = M \oplus N$ with N closed in X. Assume f and v satisfy the Lipschitz conditions (i)' and (ii)' of §3. Then the contraction principle can be applied to the mapping $Tu = (h \circ K)(f(v(u) + u))$ where h is the linear isometry between N and X/M. By (3.8) v is uniquely determined by w. Put $u = v(w) + w$ to obtain the locally unique weak solution to $Lu = f(u)$.

In this case w is the limit of a sequence of Picard iterates

$$w_n = (h \circ K)(f(v(w_{n-1}) + w_{n-1}))$$

where w_0 is any element in N with $\|w_0\| \leqq a$. Thus, to compute w_n,

    (i) find $v_n = v(w_{n-1}) \in M$ such that $f(v_n + w_{n-1})$ annihilates M^*,

    (ii) put $w_n = (h \circ K)(f(v_n + w_{n-1}))$.

In the special case where $X = X^*$ is a Hilbert space, $M = M^*$, and $w_0 = 0$, we obtain the sequence used by Rabinowitz in [6, p. 159].

REMARK 3. If X is uniformly convex, then corresponding to each $u + M$ in X/M, there is a unique $w \in X$ such that $\|u + M\| = \|w\|$. w is orthogonal to M in the sense that $\|w + v\| \geqq \|w\|$ for all $v \in M$ (Nirenberg [11, p. 30]). In addition, w depends continuously on the coset $u + M$. To see this, let $u_n + M \to u + M$ with $\|u + M\| = 1$. Let $\delta \in (0, 1)$ and choose n so large that

$$\|u_n - u + M\| < \delta \quad \text{and} \quad \big| \|u_n + u + M\| - 2\|u + M\| \big| < 2\delta.$$

If w_n is the element in X with $\|w_n\| = \|u_n + M\|$ then $\|w_n\| < 1 + \delta$, and from the above, $\|u_n + u + M\| > 2(1 - \delta)$.

Put $x_n = w_n/(1 + \delta)$ and $x = w/(1 + \delta)$ where $\|w\| = \|u + M\|$. Then $\|x_n\| \leqq 1$, $\|x\| \leqq 1$, and

$$\begin{aligned}
\|(x_n + x)/2\| &= \|w_n + w\|/2(1 + \delta) \\
&\geqq \|u_n + u + M\|/2(1 + \delta) \geqq (1 - \delta)/(1 + \delta).
\end{aligned}$$

By uniform convexity, $\|x_n - x\| \leqq \eta\{(1 - \delta)/(1 + \delta)\}$ where $\eta(\theta) \to 0$ as $\theta \to 1$. Hence $w_n \to w$.

Thus, when X is uniformly convex we can define $h: X/M \to X$ by $h(u + M) = w$, where $\|w\| = \|u + M\|$, and search for a fixed point to $Tu = (h \circ K)(f(v(u) + u))$. If, for example, (3.4) holds (with $\lambda = 1$), then one exists, and the solution to $Lu = f(u)$ is of the form $u = v(w) + w$ where $v(w) \in M$ and w is orthogonal to M.

In general, h is not linear, but if X is a Hilbert space then h is just the linear isometry relating X/M and $M^\perp$, the orthogonal complement of M.

REMARK 4. Using some theory from monotone maps, it is possible to give conditions insuring that when $M = M^*$, the bifurcation equation can be solved.

Let u be fixed in X with $\|u\| \leqq a$. Consider the functional $l_v(\phi) = (\phi, f(v + u))$ where ϕ and v are in M. By (3.1),

$$|l_v(\phi)| \leqq \|f(v + u)\|^* \|\phi\| \leqq \{c_1[\|v\| + a]\}\|\phi\|$$

and so l_v is continuous on M for each such v. Hence a mapping $v \to Bv$ to the dual of M is defined by $(\phi, Bv) = (\phi, f(v + u))$. Suppose f is such that

    (i) B is monotone, $(\phi - \psi, B\phi - B\psi) \geqq 0$ for all $\phi, \psi \in M$,

    (ii) there exists a closed ball $C \subset M$ with $0 \in C$ such that $(\phi, B\phi) \geqq 0$ on ∂C.

By a theorem of Strauss [12, p. 118], B sends an element $v(u)$ of M to zero. But the dual of M is congruent to $X^*/^\circ M$. Hence $f(v(u) + u)$ annihilates $M^* = M$.

REMARK 5. When X has a Schauder basis of eigenvectors of L^*, it may be possible to find a simple criterion insuring that $\|\phi + M^*\| \leqq k\|L^*\phi\|^*$ for all $\phi \in D$.

Thus, suppose $u \in X$ has the unique expansion $u = \sum u(s)e(s)$ where $e(s) \in D$ for each s in a (countable) index set Γ. Assume $(e(s), e(t)) = \delta_{st}$, so that the Fourier coefficients of u are given by $u(s) = (u, e(s))$. Let $L^*e(s) = v(s)e(s)$, and put $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \{s \in \Gamma; v(s) = 0\}$ and Γ_2 is its complement.

Without loss of generality we can suppose that the members of D are finite linear combinations of the basis elements $\{e(s)\}$. Then $\phi = v + w$ with

$$(4.1) \qquad v = {\sum}' \phi(s)e(s) \quad \text{and} \quad w = {\sum}'' \phi(s)e(s)$$

where $\sum'$ is the sum over Γ_1 and $\sum''$ is over Γ_2. Because $v \in M^*$, $\|\phi + M^*\| \leq \|\phi - v\| = \|w\|$.

Now suppose $(\mathfrak{X}, \|\| \cdot \|\|)$ is a sequence space forming a Hausdorff-Young pair with X; that is, there exists a linear map B to u in X from its sequence of Fourier coefficients $\mathfrak{u} = \{u(s)\}$ in $\mathfrak{X}$ such that $\|u\| \leq \|B\mathfrak{u}\| \leq \|\|\mathfrak{u}\|\|$. If $B^*: X^* \to \mathfrak{X}^*$ is the induced map then for $v \in X^*$ and corresponding $\mathfrak{v} = B^*v$ in $\mathfrak{X}^*$, $\|\|\mathfrak{v}\|\|^* \leq \|v\|^*$.

For $\phi \in D$ define a map $\mathfrak{L}^*$ on $\mathfrak{X}$ by $\{\phi(s); s \in \Gamma\} \to \{v(s)\phi(s); s \in \Gamma_2\}$. The image under $\mathfrak{L}^*$ is precisely the sequence of Fourier coefficients for $L^*\phi$ since

$$L^*\phi = L^*(v + w) = L^*w = {\sum}'' v(s)\phi(s)e(s).$$

Let w be as in (4.1). Then $\mathfrak{w} = \{\phi(s); s \in \Gamma_2\}$. Hence if $\|\|\mathfrak{w}\|\| \leq \|\|\mathfrak{L}^*\mathfrak{w}\|\|^*$, then by the properties of B and its adjoint,

$$\|\phi + M^*\| \leq \|w\| \leq \|\|\mathfrak{w}\|\| \leq k\|\|\mathfrak{L}^*\mathfrak{w}\|\|^* \leq k\|L^*\phi\|^*.$$

For example, let L be a partial differential operator with constant coefficients. Let $X = L_r(T)$ where $r \in (2, \infty)$ and T is the period cube of side 2π in R_n centered at 0. A suitable choice for D is the set of trigonometric polynomials

$$(4.2) \qquad \phi(x) = \sum \phi(s)e^{is \cdot x}$$

where $s = (s_1, s_2, \ldots, s_n)$ is an n-tuple of integers, $x = (x_1, x_2, \ldots, x_n)$ is a point in R_n, $s \cdot x = s_1 x_1 + \cdots + s_n x_n$, and $\phi(-s) = \overline{\phi(s)}$.

If $L = \sum_{|p| \leq m} a_p D^p$, then the formal adjoint L^* is defined via integration by parts,

$$\int_T \phi L \psi = \int_T \psi L^* \phi,$$

and so $L^* = \sum_{|p| \leq m} (-1)^p a_p D^p$ with

$$L^*e^{is \cdot x} = v(s)e^{is \cdot x} = \Big\{ \sum_{|p| \leq m} (-1)^p (is)^p a_p \Big\} e^{is \cdot x}.$$

By the Hausdorff-Young Theorem [13, p. 247], $\mathfrak{X} = l_{r'}$, $1/r + 1/r' = 1$. Hence if w is as in (4.1), the Hölder inequality gives

$$\|\|\mathfrak{w}\|\| = \Big\{ {\sum}'' |\phi(s)|^{r'} \Big\}^{1/r'} = \Big\{ {\sum}'' |v(s)\phi(s)/v(s)|^{r'} \Big\}^{1/r'}$$
$$\leq k \Big\{ {\sum}'' |v(s)\phi(s)|^r \Big\}^{1/r} = k\|\|\mathfrak{L}^*\mathfrak{w}\|\|^*$$

where $k = \{\sum'' 1/|v(s)|^q\}^{1/q}$, $q = p/(p-2)$. Hence if $\{1/v(s); s \in \Gamma_2\}$ is in l_q, $q = p/(p-2)$, then $\|\phi + M^*\| \leq k\|L^*\phi\|^*$ for all $\phi \in D$.

REMARK 6. Let $\{g_n\}$ be a norm 1 sequence converging weakly in $^\circ M^*$ to g. We want to find conditions insuring that $Kg_n \to Kg$ strongly in X/M and hence that K is compact.

By the Hahn-Banach Theorem there is an element $x_n \in {}^\circ M^*$ with $\|x_n\|^* = 1$ such that $\|K(g_n - g)\| = (K(g_n - g), x_n)$. Since the range of L^* is dense in $^\circ M^*$ there exists $\psi_n = L^*\phi_n$ such that $\|x_n - \psi_n\|^* < 1/n$. Let $k = \|K\|$, and recall from Theorem 1 that $(Kg, L^*\phi) = (\phi, g)$ whenever $\phi \in D$ and $g \in {}^\circ M^*$. Hence

$$\begin{aligned}
\|K(g_n - g)\| &= (K(g_n - g), x_n) \\
&= (K(g_n - g), x_n - \psi_n) + (K(g_n - g), L^*\phi_n) \\
&< 2k/n + (\phi_n, g_n - g).
\end{aligned}$$

Hence the convergence of Kg_n to Kg depends on showing that $(\phi_n, g_n - g) \to 0$.

Let X, $\mathfrak{X}$, etc. be as in Remark 5. Assume that for $\mathfrak{u} = \{u(s)\}$ in $\mathfrak{X}$ and $\mathfrak{v} = \{v(s)\}$ in $\mathfrak{X}^*$, $(\mathfrak{u}, \mathfrak{v}) = \sum u(s)v(s)$. Suppose $\{e(s)\}$ is also a basis for X^* so that if $v \in X^*$ then $v = \sum v(s)e(s)$ where $\mathfrak{v} = \{v(s)\}$ is its sequence of Fourier coefficients. It is easy to check that when $u = \sum u(s)e(s)$ is in X, Parseval's Theorem holds, i.e. $(u, v) = \sum u(s)v(s) = (\mathfrak{u}, \mathfrak{v})$.

Suppose $v(s)$ is as previously defined and for g_n and $g \in {}^\circ M^*$, let $\mathfrak{g}_n$ and $\mathfrak{g}$ be their corresponding sequences in $\mathfrak{X}^*$. Define a map $\mathfrak{K}$ by

$$\mathfrak{g} \to \{g(s)/v(s); s \in \Gamma_2\} = \mathfrak{K}\mathfrak{g}.$$

As in (4.1), write $\phi_n = v_n + w_n$, and observe that since g annihilates M^*, $(\phi_n, g_n - g) = (w_n, g_n - g)$. Thus

$$\begin{aligned}
(\phi_n, g_n - g) &= (w_n, g_n - g) = \sum'' \phi_n(s)\{g_n(s) - g(s)\} \\
&= \sum'' \phi_n(s)v(s)\{g_n(s) - g(s)\}/v(s) \\
&\leq \|\|\mathfrak{L}^*\mathfrak{w}_n\|\|^* \|\|\mathfrak{K}(\mathfrak{g}_n - \mathfrak{g})\|\|.
\end{aligned}$$

Now $\|\|\mathfrak{L}^*\mathfrak{w}_n\|\|^* \leq \|L^*w_n\|^* = \|L^*\phi_n\|^* = \|\psi_n\|^*$ is bounded. Hence K is compact if $g_n \to g$ weakly implies $\mathfrak{K}\mathfrak{g}_n \to \mathfrak{K}\mathfrak{g}$.

For instance, suppose X, D, L, etc. are as in the example of Remark 5. If $g_n \to g$ weakly in X^*, then

$$g_n(s) = \int_T g_n e^{-is \cdot x} \to \int_T g e^{-is \cdot x} = g(s)$$

for each s. Now,

$$\|\|\mathfrak{K}(\mathfrak{g}_n - \mathfrak{g})\|\| = \left\{ \left(\sum_I'' + \sum_{II}''\right) |(g_n(s) - g(s))/v(s)|^{r'} \right\}^{1/r'}$$

where $\sum_I''$ is the sum $\sum''$ over $\{s; |s| \leq r\}$ and the second is over its complement. Again suppose $\{1/v(s); s \in \Gamma_2\}$ is in l_q, $q = r/(r-2)$. If r is sufficiently large, the

Hölder inequality shows that $\sum_{II}''$ can be made arbitrarily small. This fixes r, and since $g_n(s) \to g(s)$ uniformly in s for $|s| \leq r$, $\sum_{I}''$ also goes to zero as $n \to \infty$. Hence in this case, the assumption $v^{-1}(s) \in l_q$ insures both the continuity and complete continuity of K.

5. An example.   Consider the equation

$$(5.1) \qquad c^2 w_{tt}(t, y) + \Delta^{2p} w(t, y) = b|w(t, y)|^{r-1} \operatorname{sgn} w(t, y) + g(t, y)$$

where $y = (y_1, \ldots, y_n)$, Δ is the n-dimensional Laplacean, r is a real number in $(2, \infty)$, p is a positive integer, b and c are constants, sgn is the signum function, and g is $2\pi/\omega_0$ periodic in t and $2\pi/\omega_i$ periodic in y_i.

The change of variable $x_0 = \omega_0 t$, $x_i = \omega_i y_i$ reduces (5.1) to the form $Lu = f(u)$ with $x = (x_0, x_1, \ldots, x_n)$, $u(x) = w(t, y)$, $f(u) = b|u|^{r-1} \operatorname{sgn} u + h$, where $h(x) = g(t, y)$ is 2π-periodic in each variable x_i, $i = 0, 1, \ldots, n$. L is the operator

$$L = c^2 \omega_0^2 \, \partial^2/\partial x_0^2 + \{\omega_1^2 \, \partial^2/\partial x_1^2 + \cdots + \omega_n^2 \, \partial^2/\partial x_n^2\}^{2p}.$$

We put $X = L_r(T)$, as defined in Remark 5. Then f is a bounded, continuous map of X to its conjugate provided $h \in L_{r'}(T)$, $r' = r/(r-1)$.

For this choice of X, Remark 6 also applies. Hence we must check that $1/v(s)$, where

$$v(s) = -c^2 \omega_0^2 s_0^2 + \{\omega_1^2 s_1^2 + \cdots + \omega_n^2 s_n^2\}^{2p}$$

is an element of l_q, $q = r/(r-2)$.

Write

$$v(s) = -c^2 \omega_0^2 \{s_0^2 - (c_1^2 s_1^2 + \cdots + c_n^2 s_n^2)^{2p}\}$$

where $c_i = \omega_i/(c\omega_0)^{1/2p}$. Assume each c_i is of the form $(k_i/m_i)^{1/2}$ where k_i and m_i are positive integers. Let m be the least common multiple of the m_i. Then

$$v(s) = -(c^2 \omega_0^2/m^{2p})(m^{2p} s_0^2 - \rho^{2p}(s))$$

where $\rho(s) = d_1 s_1^2 + \cdots + d_n s_n^2$ and the d_i are positive integers. Clearly, $v(s) = 0$ iff $\mu(s) = m^{2p} s_0^2 - \rho^{2p}(s) = 0$.

If $\rho(s) \neq 0$,

$$\mu^{-1}(s) = 2^{-1} \rho^{-p}(s)\{(m^p s_0 - \rho^p(s))^{-1} - (m^p s_0 + \rho^p(s))^{-1}\}.$$

Hence, if $\alpha = r/(r-2)$, then

$$|\mu^{-\alpha}(s)| \leq |\rho(s)|^{-p\alpha}\{|m^p s_0 - \rho^p(s)|^{-\alpha} + |m^p s_0 + \rho^p(s)|^{-\alpha}\}$$

where we have used the fact that $|A + B|^p \leq 2^p\{|A|^p + |B|^p\}$. Thus

$$\sum_{\mu(s) \neq 0} |\mu(s)|^{-\alpha} = \left\{ \sum_{\mu(s) \neq 0; \rho(s) = 0} + \sum_{\mu(s) \neq 0; \rho(s) \neq 0} \right\} |\mu(s)|^{-\alpha}$$

$$\leq \operatorname{const} \left\{ \sum_{t=1}^{\infty} 1/t^{2\alpha} + \left[\sum_{s_1, \ldots, s_n; \rho(s) \neq 0} |\rho(s)|^{-p\alpha} \right] \sum_{t=1}^{\infty} 1/t^{\alpha} \right\}.$$

The sum over $\rho(s) \neq 0$ converges if $p\alpha = pr/(r-2) > n/2$. If $n \leq 2p$, any $r \in (2, \infty)$ will do, while if $n > 2p$, then r must be strictly less than $2n/(n-2p)$. With these restrictions, $\nu(s)$ satisfies the desired property.

Next, we must show that the bifurcation equation can be solved. In this case, because L is symmetric, this means finding $v(u)$ in M such that $f(v(u)+u)$ annihilates M. We also require v to be bounded and continuous in u. Use will be made of the inequality

$$(5.2) \quad (x-y)(|x+u|^{r-1} \operatorname{sgn} (x+u) - |y+u|^{r-1} \operatorname{sgn} (y+u)) \geq |x-y|^r/2^{r-1}$$

a proof of which can be found in [7].

Assume $v(u)$ exists. Then

$$(v(u), b|v(u)+u|^{r-1} \operatorname{sgn} (v(u)+u)+h) = 0$$

and thus from (5.2),

$$(5.3) \qquad\qquad \|v(u)\|^{r-1} \leq 2^{r-1}(|b| \|u\|^{r-1} + \|h\|^*).$$

Next, if both $v(u)$ and $v(w)$ are solutions to the bifurcation equation then by (5.2) and the mean value theorem,

$$\|v(u)-v(w)\|^{r-1} \leq (r-1)\|(v(w)+w+\theta(u-w))^{r-2}(u-w)\|^*$$

where $0 < \theta < 1$. Hence by Hölder's inequality, $\|v(u)-v(w)\|^{r-1} \leq \operatorname{const} \|u-w\|$ where the constant depends on the norms of $v(w)$, u, and w. Hence v, if it exists, is a bounded, Hölder continuous function of u.

For the existence we apply Remark 4. By (5.2)

$$(\phi, B\phi) = (\phi, f(\phi+u)) = (\phi, b|\phi+u|^{r-1} \operatorname{sgn} (\phi+u)+h)$$
$$\geq (\|\phi\|^{r-1}/2^{r-1} - |b| \|u\|^{r-1} - \|h\|^*)\|\phi\|.$$

Hence if C is the set of $v \in M$ satisfying inequality (5.3) then $(\phi, B\phi) \geq 0$ on ∂C. Clearly B is monotone. Thus v exists and satisfies the requirements for Theorem 2.

To conclude, it must be shown that inequality (3.4) can be obtained. Remark 3 is applicable so λ may be chosen equal to 1.

Let

$$c_1[t] = |b|t^{r-1}+d, \qquad c_2[t] = 2(c_1[t])^{1/(r-1)}$$

where $d = \|h\|^*$. Clearly, (3.1) is satisfied by this choice of c_1, while (3.2) holds by virtue of (5.3). We must show, therefore, that $kc_1[c_2[a]+a] \leq a$. As we shall see, this can be done by restricting d, the amplitude of the external excitation.

Now,

$$kc_1[c_2[a]+a] = k\{|b|(c_2[a]+a)^{r-1}+d\}$$
$$\leq k\{2^{r-1}|b|(c_2^{r-1}[a]+a^{r-1})+d\} = ma^{r-1}+nd$$

where $m=2^{r-1}|b|k(1+2^{r-1}|b|)$ and $n=(1+4^{r-1}|b|)k$. So if we can find positive a_0 such that $w(a_0)=ma_0^{r-1}+nd-a_0<0$ then (2.6) will be met.

In fact, choose $a_0=(1/m(r-1))^{1/(r-2)}$. Because $r>2$, $w(a)$ has a minimum here and $w(a_0)=nd-a_0(r-2)/(r-1)$. With a thus fixed, restrict d to be less than $a_0(r-2)/n(r-1)$. Then $w(a_0)<0$, and we have satisfied all of the conditions for (5.1) to have a periodic solution.

We can now state

THEOREM 4. *The partial differential equation*

$$c^2w_{tt}+\Delta^{2p}w = b|w|^{r-1}\,\mathrm{sgn}\,w+g(t, y_1, \ldots, y_n)$$

where p is a positive integer and g is $2\pi/\omega_0$ periodic in t and $2\pi/\omega_i$ periodic in y_i, $i=1, 2, \ldots, n$, has a weak, rth power integrable solution with the same period as g providing

    (i) *for each i, $w_i^2/(cw_0)^{1/p}$ is rational,*

    (ii) *$r \in (2, j)$ where $j= +\infty$ if $n\leq 2p$ and $j=2n/(n-2p)$ if $n>2p$,*

    (iii) *either b, or $\int_0^{2\pi/\omega_0} \int_0^{2\pi/\omega_1} \cdots \int_0^{2\pi/\omega_n} |g(t, y_1, \ldots, y_n)|^{r/(r-1)}\, dt\, dy_1 \cdots dy_n$ is sufficiently small.*

Furthermore, the solution is of the form $v+w$ where v is in the generalized null space of the operator $c^2\, \partial^2/\partial t^2 + \Delta^{2p}$, and w is orthogonal to v.

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UNIVERSITY OF PITTSBURGH, PITTSBURGH, PENNSYLVANIA 15213