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REPAIRING EMBEDDINGS OF 3-CELLS WITH MONOTONE MAPS OF E^3 (1)

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Abstract. If S_1 is a 2-sphere topologically embedded in Euclidean 3-space E^3 and S_2 is the unit sphere about the origin, then there may not be a homeomorphism of E^3 onto itself carrying S_1 onto S_2. We show here how to construct a map f of E^3 onto itself such that $f|S_1$ is a homeomorphism of S_1 onto S_2, $f(E^3-S_1)=E^3-S_2$ and $f^{-1}(x)$ is a compact continuum for each point x in E^3. Similar theorems are obtained for 3-cells and disks topologically embedded in E^3.

1. **Introduction.** In this paper we show that, for any 2-sphere S wildly embedded in Euclidean 3-space E^3, there is a monotone upper semicontinuous decomposition G of E^3 whose nondegenerate elements miss S such that E^3/G is E^3 and S is taken to a tame 2-sphere in E^3/G. If X is a wildly embedded set in a 3-manifold M^3, we will say that the embedding of X can be repaired (see [1]) if there exists a monotone upper semicontinuous decomposition G of M^3 such that each nondegenerate element of G is disjoint from X, $M^3/G=M^3$, and the image of X under the natural projection of M^3 onto M^3/G is tamely embedded in M^3/G. The main theorem of this paper, Theorem 1, says that any 3-cell in E^3 can be repaired. It follows as a corollary of this theorem and a theorem of Hosay [11] and Lininger [14] that any wild embedding of a 2-sphere can be repaired. Another corollary using recent results of Daverman and Eaton [8] is that any 2-cell in E^3 and many arcs in E^3 can be repaired. In §3, we construct a decomposition of the complement of a 3-cell in S^3. It is a kind of triangulation respecting wild embeddings which is difficult to state as a theorem. Therefore, we have been content just with giving a loose description of the decomposition and then proceeding with the construction.

The notation and terminology is largely standard. A *cube-with-handles* is a space homeomorphic to a regular neighborhood in the 3-sphere S^3 of a finite 1-complex and a *cube-with-holes* is a space homeomorphic to the closure of the complement of a cube-with-handles in S^3. The distance between two points x and y in any

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metric space under consideration will be denoted by $\rho(x, y)$ and $N(A, r)$ will denote the set of all points x such that $\rho(x, A) < r$. If σ is a simplex in a space X with triangulation T, we will use St (σ) to denote the point set interior in X of the star of σ in the triangulation T. The j-skeleton of a triangulation T_i will be denoted by T_i^j. A *Sierpinski curve* is the space obtained from a 2-sphere S by deleting the interiors of a null sequence of mutually disjoint disks in S whose union is dense in S. If X is a Sierpinski curve in S obtained by removing the interiors of the disks $\{D_i\}$, then the *accessible part* of X is the set $\bigcup$ Bd D_i and the *inaccessible* part of X, here denoted by Inacc (X), is the set of all points of X which do not lie in the accessible part of X. We have frequently abbreviated piecewise linear to pwl.

2. **Some preliminary lemmas.** Lemma 3 below is needed in the construction in §3. Lemma 1 can be proved as in Theorem 4.1 of [3].

LEMMA 1. *Let D be a disk, X a Sierpinski curve lying in a 2-sphere S, $D \cap X = (\mathrm{Bd}\ D) \cap X = A$, A an arc lying in the inaccessible part of X. Then there is a null sequence of mutually disjoint disks $E_1, E_2, E_3, \ldots$ on $D - A$ such that $D \cap S \subset A \cup (\bigcup E_i)$ such that, for any $\varepsilon > 0$ and any point $p \in A$, there is a neighborhood N of p in D so that only disks E_i of diameter $< \varepsilon$ intersect N.*

LEMMA 2. *Let $\varepsilon > 0$. Let C be a wild cell in E^3, X a tame Sierpinski curve in* Bd C, *and S a 2-sphere. Suppose that $G\colon S \times [0, 1] \to E^3$ is a homeomorphism which is locally piecewise linear* mod $S \times 0$, *$G(S \times (0, 1])$ lies in the unbounded complementary domain of $G(S \times 0)$, $G(S \times 0) \cap \mathrm{Bd}\ C = X$, and $G(S \times 0)$ is tame. Let T_1 and T_2 be triangulations of S such that T_2 refines T_1 and $G(T_2^1 \times 0)$ lies in the inaccessible part of X. Then, for some integer ξ, there is a homeomorphism H from $S \times [0, 1/2^\xi]$ into E^3, which is locally piecewise linear* mod $S \times 0$, *such that*

(1) *$\rho(H(x, t), G(x, t)) < \varepsilon$ for all $x \in S$ and all $t \in [0, 1/2^\xi]$,*

(2) *for all $v \in T_2^0$, $H(v \times (0, 1/2^\xi]) \cap C = \varnothing$,*

(3) *for all $\sigma \in T_2^1$ and $n = 0, 1, 2, 3, \ldots$, $H(\sigma \times 1/2^{\xi+n}) \cap C = \varnothing$,*

(4) *for all $x \in S$, $H(x, 0) = G(x, 0)$,*

(5) *if G has properties* (2) *and* (3) *with respect to T_1, then $H(x, t) = G(x, t)$ for all $(x, t) \in T_1^0 \times (0, 1/2^\xi]$ and $H(T_1^1 \times (0, 1/2^\xi]) = G(T_1^1 \times (0, 1/2^\xi])$.*

Proof. First we obtain condition (2). Let $v_1, v_2, v_3, \ldots, v_k, v_{k+1}, \ldots, v_l$ be the vertices of T_2 with $v_1, v_2, \ldots, v_k$ being those vertices for which $G(v_i \times (0, 1]) \cap C = \varnothing$. We will show how to adjust G so that $G(v_{k+1} \times (0, 1/2^\eta]) \cap C = \varnothing$ for some nonnegative integer η, so that $G(v_i \times (0, 1]) \cap C = \varnothing$, $i = 1, \ldots, k$, and so that, if $T_1^0 \subset \{v_1, v_2, \ldots, v_k\}$, then $G | T_1^0 \times [0, 1]$ is left unaltered.

To do this, let $v = v_{k+1}$ and suppose that σ and τ are 1-simplexes in T_2^1 such that $\sigma \cap \tau = v$. Let $A = \sigma \cup \tau$ and D be the disk $G(A \times [0, 1])$. By Lemma 1, there is a null sequence $E_1, E_2, E_3, \ldots$ of mutually disjoint disks in D such that $D \cap C \subset G(A \times 0) \cup (\bigcup E_i)$, and, for each $i = 1, 2, 3, \ldots$, $G(A \times 0) \cap E_i = \varnothing$. Let α be a polygonal arc in $D-\bigcup E_i$ joining the endpoints of $G(A\times 0)$ such that

$$\alpha \cap G(v\times[0,1])$$

is a single point p. Let β be the subarc of $G(v\times[0,1])$ joining $G(v\times 0)$ and p. If D' is the disk in D bounded by $\alpha\cup G(A\times 0)$, then there is a homeomorphism f of D' onto itself, which is locally piecewise linear mod α, fixed on Bd D', so that $f(\beta)\cap(\bigcup E_i)=\varnothing$. By extending f piecewise linearly in a sufficiently close neighborhood N of Int D' so that f is fixed on Bd N and then extending this map to all of E^3 by the identity, we obtain a map f such that $f\circ G$ satisfies requirement (2) for some integer ξ and the vertex v_{k+1}. Similarly, we alter G near each of the vertices $v_{k+2}, v_{k+3},\dots, v_l$ to obtain a homeomorphism G_1 of $S\times[0,1]$, locally piecewise linear mod $S\times 0$, such that $G_1|S\times 0=G|S\times 0$ and, for all $v\in T_2^0$ and some sufficiently large integer η, $G_1(v\times(0,1/2^\eta])\cap C=\varnothing$.

Adjusting G_1 to obtain a homeomorphism G_2 satisfying conditions (2) and (3) is similar. Let $\sigma\in T_2^1$ with Int $\sigma\cap T_1^1=\varnothing$ if G_1 satisfies (2) and (3) (replacing T_2 with T_1 and H with G_1). Note that we may suppose that G_1 satisfies (2) and (3) if G does. Let $\{v,v'\}=$ Bd σ. Lemma 1 can be used to obtain a sequence of "horizontal" arcs in $G_1(\sigma\times[0,1/2^\eta])$ spanning from $G_1(v\times[0,1/2^\eta])$ to $G_1(v'\times[0,1/2^\eta])$ and converging to $G_1(\sigma\times 0)$ and "vertical" arcs from $G_1(\sigma\times 0)$ to the interiors of the horizontal spanning arcs. By a suitable choice of these arcs, it is possible to define G_2 on $\sigma\times 1/2^{\nu+n}$ for some $\nu\geqq\eta$ and all $n=0,1,2,\dots$ in such a way that it extends $G_2|S\times 0=G_1|S\times 0$. The "vertical" arcs are used to make the "horizontal" arcs converge on $G_2(S\times 0)=G_1(S\times 0)$ homeomorphically and together they decompose $G_1(\sigma\times[0,1/2^\eta])$ into disks so that G_2 can then be extended to take all of $\sigma\times[0,1/2^\eta]$ onto $G_1(\sigma\times[0,1/2^\eta])$. Doing this for each $\sigma\in T_2^1$, we then have

$$G_2\colon T_2^1\times[0,1/2^\eta]\to G_1(T_2^1\times[0,1/2^\eta])$$

such that $G_2|T_2^1\times 0=G|T_2^1\times 0$ and, for some $\nu\geqq\eta$, $G_2(T_2^1\times 1/2^{\nu+n})\cap C=\varnothing$ for each $n=0,1,2,\dots$. Furthermore, G_2 can be taken to be locally piecewise linear mod $T_1^2\times 0$. Let $G_2|S\times 0=G_1|S\times 0$ and $G_2|S\times 1/2^\eta=G_1|S\times 1/2^\eta$. Then G_2 is defined on the boundary of each cell $\tau\times[0,1/2^\eta]$, $\tau\in T_2^2$, and can be extended to take this cell into $G_1(\tau\times[0,1/2^\eta])$ so that G_2 satisfies all the conditions of the lemma except possibly (1). Condition (1) is met by using the fact that $G_2(x,0)=G(x,0)$ for all $x\in S$ and choosing $\xi\geqq\nu\geqq\eta$. For this choice of ξ, we set $H=G_2|S\times[0,1/2^\xi]$. This completes the proof of Lemma 2.

The following lemma is a modification of Lemma 2 of a paper by D. R. McMillan, Jr. [15].

LEMMA 3. *Let C be a 3-cell and $h\colon C\to E^3$ a homeomorphism. There is a monotone decreasing sequence $\{\zeta_n\}$, $0<\zeta_n\leqq 1/n$, and for each n, a pwl homeomorphism*

$$H_n\colon \text{Bd } C\times[-\zeta_n,\zeta_n]\to E^3$$

with the following properties:

(i) $\rho(h(x), H_n(x, t)) < 1/n$, *for all* $x \in \operatorname{Bd} C$ *and* $t \in [-\zeta_n, \zeta_n]$,

(ii) $H_n(\operatorname{Bd} C, -\zeta_n) \subset \operatorname{Int} h(C)$,

(iii) $h(C) \cap H_n(\operatorname{Bd} C, \zeta_n)$ *is covered by the interiors of a finite disjoint collection of* 2-*cells in* $H_n(\operatorname{Bd} C, \zeta_n)$ *each of diameter less than* $1/n$,

(iv) *for all n, there exists a finite disjoint collection of topological* 3-*cells* $C_1^n, C_2^n, \ldots, C_k^n$ *in* $h(C)$ *such that* C_i^n *has diameter less than* $1/n$ *and meets* $h(\operatorname{Bd} C)$ *precisely in a* 2-*cell such that* $h(\operatorname{Bd} C) - H_n(\operatorname{Bd} C \times [-\zeta_n, \zeta_n])$ *is covered by the interiors of these* 2-*cells and such that*

$$\operatorname{Bd} C_i^n - \operatorname{Int}(C_i^n \cap h(\operatorname{Bd} C)) \subset H_n(\operatorname{Bd} C \times [-\zeta_n, \zeta_n]).$$

Furthermore, there is a sequence of triangulations $T_1, T_2, \ldots,$ *of* Bd C *such that* mesh $h(T_n) < 1/n$, $h(T_n^1)$ *is a tame finite graph and* T_{n+1} *refines* T_n; *there is a sequence of homeomorphisms* G_n: $\operatorname{Bd} C \times [-\zeta_n, \zeta_n] \to E^3$ *which are locally piecewise linear* mod $\operatorname{Bd} C \times 0$ *satisfying the following properties*:

(1) $\rho(h(x), G_n(x, t)) < 1/n$, *for all* $x \in \operatorname{Bd} C$, $t \in [-\zeta_n, \zeta_n]$,

(2) $G_n | \operatorname{Bd} C \times \zeta_n = H_n | \operatorname{Bd} C \times \zeta_n$,

(3) $G_n(\operatorname{Bd} C \times [-\zeta_n, \zeta_n]) = H_n(\operatorname{Bd} C \times [-\zeta_n, \zeta_n])$,

(4) $G_n(x, 0) = h(x)$, *for any* $x \in T_n^1$,

(5) $G_n(v \times (0, \zeta_n]) \cap h(C) = \varnothing$, *for all* $v \in T_n^0$,

(6) $G_n(T_n^1 \times \zeta_i) \cap h(C) = \varnothing$, *for all* $i \geqq n$,

(7) *for each n and each* n', $n' = 1, 2, \ldots, n-1$,

$$G_n(T_{n'}^1 \times [0, \zeta_n]) = G_{n'}(T_{n'}^1 \times [0, \zeta_n]),$$

(8) *for each n and each* n', $n = 1, 2, \ldots, n-1$, *each component of*

$$G_n(\operatorname{Bd} C \times [-\zeta_n, \zeta_n]) \cap G_{n'}(T_{n'}^1 \times (\zeta_n, \zeta_{n'}])$$

is a closed set missing

$$G_{n'}(T_{n'}^1 \times \zeta_{n'}) \cup G_{n'}(T_{n'}^1 \times \zeta_n) \cup G_{n'}(T_{n'}^0 \times (\zeta_n, \zeta_{n'}]).$$

Proof. *Step* 1. *Construction of* G_1, H_1, T_1. Let $\varepsilon = 1$ and let δ be a positive number such that for any homeomorphism g: Bd $C \to E^3$ differing from $h|\operatorname{Bd} C$ by less than δ and for any compact set Y in $g(\operatorname{Bd} C)$ whose components have diameter less than δ, then there is a finite collection of ε-disks in $g(\operatorname{Bd} C)$ such that Y lies in the union of the interiors of these disks. Let T_1^1 be a triangulation of Bd C such that $h(T_1)$ has mesh less than ε and $h(T_1^1)$ is tame [2]. Let X_1 be a tame Sierpinski curve in $h(\operatorname{Bd} C)$ such that $h(T_1^1) \subset \operatorname{Inacc}(X_1)$ and the diameter of each component of $h(\operatorname{Bd} C) - X_1$ is less than δ [6, Theorem 9.1]. Let g_1: Bd $C \to E^3$ be a homeomorphism obtained by pushing $h(\operatorname{Bd} C) - X_1$ slightly into $h(C)$ so that g_1 is locally pwl mod $(h^{-1}(X_1))$, differs from h by less than δ, $g_1|h^{-1}(X_1) = h|h^{-1}(X_1)$, and the closures of components of $h(C) - g_1(\operatorname{Bd} C)$ form a null sequence of 3-cells $C_1^1, C_2^1, C_3^1, \ldots$.

Since $g_1(\text{Bd } C)$ is locally tame mod a tame Sierpinski curve, it is tame [6, Theorem 8.2]. It follows from the tameness of $g_1(\text{Bd } C)$ and Theorem 2 of [5] that there is a homeomorphism $G_1: \text{Bd } C \times [-1, 1] \to E^3$ which is locally pwl mod $\text{Bd } C \times 0$ satisfying $G_1(x, 0) = g_1(x)$ for all $x \in \text{Bd } C$, $G_1(\text{Bd } C \times -1) \subset \text{Int } h(C)$, and condition (1). By Lemma 2, we may suppose that G_1 satisfies conditions (5) and (6). Take H_1 to be a sufficiently close pwl approximation to G_1 using Theorem 3 of [5] in order to obtain conditions (2) and (3). There is a k such that $C_1^1, C_2^1, \ldots, C_k^1$ are the only cells of the null sequence not lying in $H_1(\text{Bd } C \times (-1, 1))$ and these cells are the ones of condition (iv). By our choice of δ, H_1 satisfies condition (iii).

Step 2. *Construction of* G_n, H_n, T_n. Choose δ as in Step 1, but with $\varepsilon = 1/n$. Choose a Sierpinski curve X_n by adding on to X_{n-1} in the following way. Let $D_1, D_2, \ldots, D_m$ be those component disks of $h(\text{Bd } C) - \text{Inacc } (X_{n-1})$ such that the diameter $D_i \geqq \delta$ or $\rho(x, G_{n-1}(x, 0)) \geqq \delta$ for some $x \in D_i$. We add these disks back on to X_{n-1} and remove a null sequence of disks from their interiors to obtain X_n such that components of $h(C) - X_n$ have diameter $< \delta$. Let T_n be a triangulation of $\text{Bd } C$ such that $h(T_n^1)$ is a finite graph in the inaccessible part of X_n, T_n refines T_{n-1}, and mesh $h(T_n^1) < 1/n$ [2], [6].

We obtain g_n, as we did g_1, but in a more careful way to get $g_n: \text{Bd } C \to h(C)$ such that $g_n | (\text{Bd } C - h^{-1}(\bigcup \text{Int } D_i)) = G_{n-1} | (\text{Bd } C - h^{-1}(\bigcup \text{Int } D_i)) \times 0$ by pushing the little disks in $\bigcup D_i$ into $\text{Int } h(C)$ but not so far as D_i was pushed by $G_{n-1} | \text{Bd } C \times 0$ nor as far as δ. Thus $\rho(g_n(x), h(x)) < \delta$ and $g_n(h^{-1}(D_i)) \cup G_{n-1}(h^{-1}(D_i) \times 0)$ bounds a little cell C_i' containing $g_n(h^{-1}(D_i))$. Let N be a neighborhood of $h(T_{n-1}^1)$ in E^3 such that $(\text{Cl } N) \cap (\bigcup C_i') = \varnothing$. Let N_1 be a neighborhood of $\bigcup C_i'$ missing $\text{Cl } N$. We take a space homeomorphism f fixed outside N_1 which moves

$$G_{n-1}(\text{Bd } C \times 0)$$

onto $g_n(\text{Bd } C)$ as follows: The C_i''s are tame [6, Theorem 8.2], so fatten the C_i''s in N_1 except at $\text{Bd } D_i$'s to form cells and move $G_{n-1}(h^{-1}(D_i) \times 0)$ onto $g_n(h^{-1}(D_i))$. We do this inside the fattened C_i''s in such a way that f is fixed on $h(\text{Bd } C) - \bigcup \text{Int } (D_i)$ and on $G_{n-1}((\text{Bd } C - h^{-1}(\bigcup \text{Int } D_i)) \times 0)$, and so that $f \circ G_{n-1}(x, 0) = g_n(x)$ for all $x \in \text{Bd } C$. Extend f to a homeomorphism of E^3 onto itself which is fixed outside of the fattened C_i''s.

We obtain G_n from $f \circ G_{n-1}$. Choose a power t_n of $\frac{1}{2}$, $0 < t_n \leqq \zeta_{n-1}$, so small that $G_{n-1}(T_{n-1}^1 \times [-t_n, t_n]) \subset N$ and, for all $x \in \text{Bd } C$, $\rho(h(x), G_{n-1}(x, t)) < 1/n$. In order to get property (8), we also choose t_n so small that

$$f \circ G_{n-1}(h^{-1}(\bigcup D_i) \times [-t_n, t_n]) \cap G_{n'}(T_{n'}^1 \times \zeta_{n'}/2^j) = \varnothing,$$

for $n' = 1, 2, \ldots, n-1$ and $j = 0, 1, 2, \ldots$, and so small that

$$f \circ G_{n-1}(h^{-1}(\bigcup D_i) \times [-t_n, t_n]) \cap G_{n'}(v \times (0, \zeta_{n'}]) = \varnothing,$$

for $n' = 1, 2, \ldots, n-1$ and $v \in T_{n'}^0$. These last two conditions can be met, because $h(C)$ misses $G_{n'}(T_{n'}^1 \times \zeta_{n'}/2^j)$ and $G_{n'}(v \times (0, \zeta_{n'}])$, while $f \circ G_{n-1}(h^{-1}(\bigcup D_i) \times 0)$

$=g_n(h^{-1}(\bigcup D_i))$ lies in $h(C)$. Set $G_n' = f \circ G_{n-1} | \mathrm{Bd}\, C \times [-t_n, t_n]$ and make it locally pwl mod Bd $C \times 0$ without changing its values on

$$(\mathrm{Bd}\, C \times 0) \cup (T_{n-1}^1 \times [-t_n, t_n])$$

by using Theorem 3 of [5] in such a manner as preserve property (8).

We use Lemma 2 to get G_n: Bd $C \times [-\zeta_n, \zeta_n] \to E^3$, for some power ζ_n of $\frac{1}{2}$, from G_n'. In applying Lemma 2 we choose ε sufficiently small to preserve properties (1) and (8). Lemma 2 gives us properties (5) and (6) without destroying (4) or (7).

We obtain H_n from G_n as we obtained H_1 from G_1 using Theorem 3 of [6]. This also gives properties (2) and (3).

3. **A decomposition.** *Cube-with-holes decompositions.* For convenience, we make the following definition: A *cube-with-holes decomposition* of a space X is a "triangulation" of X with cubes-with-holes replacing 3-simplexes and disks-with-handles replacing their 2-faces. In a cube-with-holes decomposition we will allow each cube-with-holes to have any finite number of faces, not just four as in a simplicial 3-complex. In this section we construct a cube-with-holes decomposition of the complement in S^3 of a wild 3-cell. In this case all but one cube-with-holes has five faces; the one has many more faces. Some of the disks-with-handles have four 1-faces, each a 1-simplex, whereas, others have three 1-faces.

Let C be a 3-cell and let $h: C \to S^3$ be a topological embedding of C. We construct a sequence of triangulations $T_1, T_2, \ldots$ of Bd C with mesh $h(T_i) \to 0$ as $i \to \infty$ and, with T_{i+1} refining T_i. Our decomposition of $S^3 - h(C)$ is into small cubes-with-holes $\Gamma_{\sigma,m}$ with σ being a 2-simplex of T_{m-1}. For a fixed m, $m \geqq 2$, the $\Gamma_{\sigma,m}$ may be thought of as lying in a shell, S_m, about $h(C)$ and this shell, which is a 3-manifold with two boundary components (actually a cell-with-handles with a cell-with-handles containing $h(C)$ removed from its interior), consists of

$$\bigcup \{\Gamma_{\sigma,m} : \sigma \in T_{m-1}^2\}.$$

The shell S_{m+1} formed by $\bigcup \{\Gamma_{\sigma,m+1} : \sigma \in T_m^2\}$ is the next shell in toward $h(C)$ from the one formed by $\bigcup \{\Gamma_{\sigma,m} : \sigma \in T_{m-1}^2\}$, and $S_{m+1} \cap S_m$ is the outer boundary of S_{m+1} and the inner boundary of S_m. $\Gamma_{\sigma,m}$, for $m=1$, is a single cube-with-holes Γ_1, with Γ_1 being the closure of the complementary domain of $S_2 = \bigcup \Gamma_{\sigma,2}$ not containing $h(C)$. Schematically the situation is shown in Figure 1.

Each shell S_m is a thickened sphere or hollow ball with holes and handles (see Figure 2). The shell's two surfaces are divided into disks-with-handles in the same pattern into which $h(T_{m-1})$ divides Bd $h(C)$; the shell itself is like a Cartesian product of a sphere and an interval with handles added to the "outer" boundary and removed from the "inner" boundary so that the shell lies in $S^3 - h(C)$. Figure 3 shows what a cross-section of such a shell might look like. It is a union of cubes-with-holes, each having five faces, with each face being a disk-with-handles. Any two of the cubes-with-holes intersect along a disk-with-handles face or a

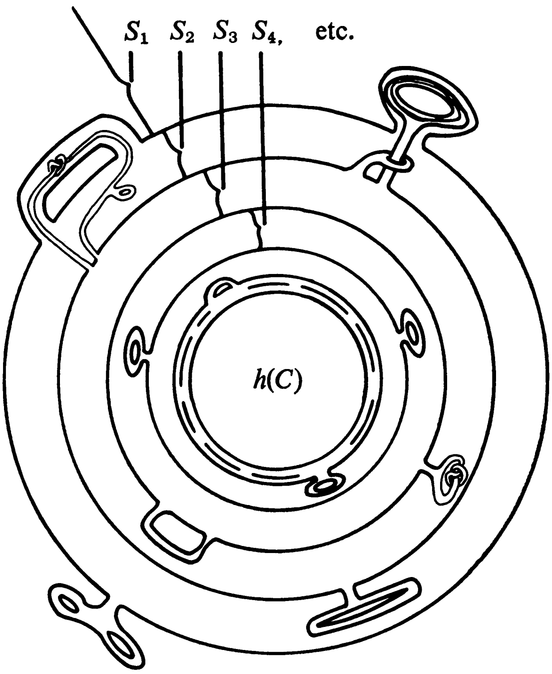

FIGURE 1

1-simplex in the boundary of such a face or not at all. If σ is a 2-simplex of T_{m-1}, then $\Gamma_{\sigma,m}$ is the cube-with-holes "above" $h(\sigma)$ in the mth shell S_m.

First we construct a sequence of triangulations $T_1, T_2, \ldots$ of Bd C and a sequence of cubes-with-handles $M_1, M_2, \ldots$ converging to $h(C)$ such that $M_m = L_m \cup (\bigcup_{i=1}^{k_m} H_i^m)$, L_m is a tame 3-cell and each H_i^m, $i=1, 2, \ldots, k_m$, is a small cube-with-handles such that $L_m \cap H_i^m = F_i^m$ is a disk in Bd $L_m \cap$ Bd H_i^m. On each L_m we will

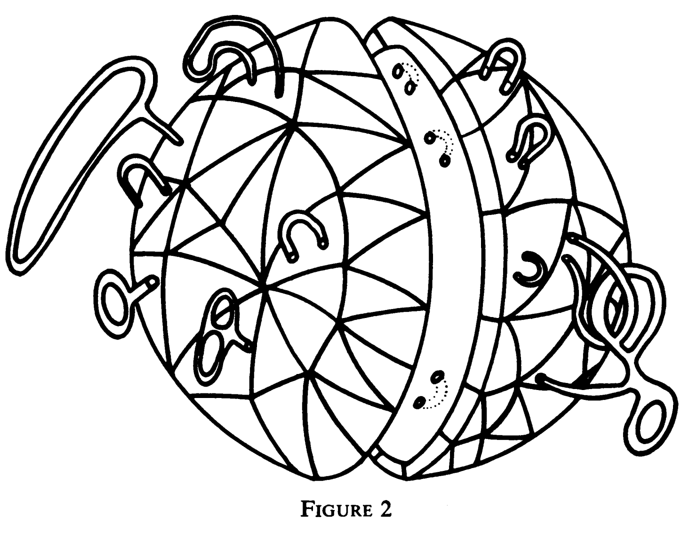

FIGURE 2

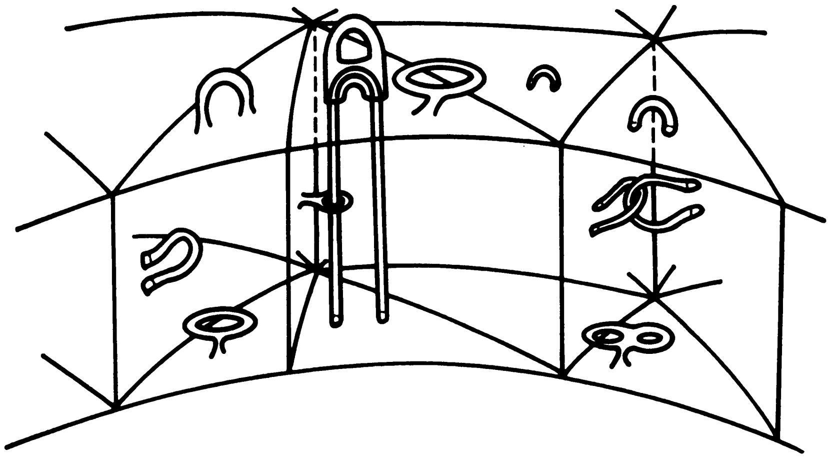

FIGURE 3

put a copy of T^1_{m-1} which will have a collar running down to a copy of T^1_{m-1} in T^1_m on L_{m+1} dividing up the space between Bd M_m and Bd L_{m+1}. By adjusting this collar it will miss Bd M_{m+1} except in the copy of T^1_{m-1} on Bd L_{m+1} and divide up the space between Bd M_m and Bd M_{m+1} into cubes-with-holes in such a fashion that any two will intersect along a common disk-with-handles in their boundaries or along an arc in the boundary of such a disk-with-handles, or not at all. In this fashion we get a cube-with-holes decomposition of $S^3-h(C)$. Each cube-with-holes, $\Gamma_{\sigma,m}$, will be named by the triangulation T_{m-1} and the 2-simplex $\sigma\in T_{m-1}$ associated with it by a map G_{n_m}, given by Lemma 3. The size of $\Gamma_{\sigma,m}\to 0$ as $m\to\infty$ and if $\sigma_1, \sigma_2, \ldots$ is a sequence of 2-simplexes with $\sigma_i\in T_i^2$ and σ_{i+1} lying in σ_i, then $\Gamma_{\sigma_m,m}\to\bigcap \sigma_m$ as $m\to\infty$.

Construction of M_1. Consider G_1: Bd $C\times[-\zeta_1,\zeta_1]\to E^3$ and T_1 from Lemma 3. Choose ε_1 as follows:

(a) $\varepsilon_1<\eta_1$, where η_1 is less than $\frac{1}{3}$ the distance from $G_1(v\times[0,\zeta_1])$ to

$$G_1(\{T_1^1-\mathrm{St}\,(v)\}\times[0,\zeta_1])$$

for each $v\in T_1^0$.

(b) ε_1 less than $\frac{1}{3}$ the distance from $G_1(\sigma\times[0,\zeta_1])-N(G_1(\mathrm{Bd}\,\sigma\times[0,\zeta_1]),\eta_1)$ to $G_1(\{T_1^1-\mathrm{Int}\,\sigma\}\times[0,\zeta_1])$ for every $\sigma\in T_1^1$.

Condition (a) says any ε_1-set intersecting $N(G_1(v\times[0,\zeta_1]),\eta_1)$ cannot intersect $G_1(T_1^1\times[0,\zeta_1])$ outside $G_1(\mathrm{St}\,(v)\times[0,\zeta_1])$. Condition (b) says any ε_1-set intersecting $G_1(\sigma\times[0,\zeta_1])$ but not $N(G_1(\mathrm{Bd}\,(\sigma)\times[0,\zeta_1]),\eta_1)$ cannot intersect $G_1(T_1^1\times[0,\zeta_1])$ except in $G_1(\mathrm{Int}\,(\sigma)\times[0,\zeta_1])$. Together, these conditions imply that any ε_1-set intersecting $G_1(T_1^1\times[0,\zeta_1])$ lies in $G_1((\mathrm{St}\,(\sigma^0)\cap T_1^1)\times[0,\zeta_1])$ for some $\sigma^0\in T_1^0$—that is, any ε_1-set intersecting $G_1(T_1^1\times[0,\zeta_1])$ intersects it only in fins radiating from one post $G_1(\sigma^0\times[0,\zeta_1])$.

Following McMillan's Theorem 1 of [15] and using Lemma 3, we find an integer n_1 such that $1/n_1 < \delta_1/3$, where δ_1 is a positive number chosen as McMillan does δ in his Theorem 1 for $\varepsilon = \varepsilon_1$. We use H_{n_1}, as given by Lemma 3 above, for his H in his Theorem 1. This gives a cube-with-handles $M_1 = L_1 \cup (\bigcup_{i=1}^{k_1} H_i^1)$, where L_1 is a cube with Bd L_1 ε_1-homeomorphic to $h(\text{Bd } C)$ and each H_i^1 is an ε_1-cube-with-handles for each $i = 1, 2, \ldots, k_1$; $h(C)$ lies in Int M_1. By Lemma 3, condition (7),

$$G_{n_1}(T_1^1 \times [0, \zeta_{n_1}]) = G_1(T_1^1 \times [0, \zeta_{n_1}]) \subset G_1(T_1^1 \times [0, \zeta_1])$$

so that the ε_1-set intersection properties prescribed by conditions (a) and (b) above for G_1 and T_1 hold also for G_{n_1} and T_1.

Construction of M_m. We assume M_{m-1} is already constructed. We construct M_m just as M_1 but with additional restrictions on the closeness of M_m to $h(C)$. Choose ε_m as follows:

(a) $\varepsilon_m < \eta_m$, where η_m is less than $\frac{1}{3}$ the distance from $G_m(v \times [0, \zeta_m])$ to

$$G_m(\{T_m^1 - \text{St}(v)\} \times [0, \zeta_m])$$

for each $v \in T_m^0$.

(b) ε_m less than $\frac{1}{3}$ the distance from $G_m(\sigma \times [0, \zeta_m]) - N(G_m(\text{Bd } \sigma \times [0, \zeta_m], \eta_m)$ to $G_m(\{T_m^1 - \text{Int } \sigma\} \times [0, \zeta_m])$ for every $\sigma \in T_m^1$.

Conditions (a) and (b) insure that any ε_m-set intersecting $G_m(T_m^1 \times [0, \zeta_m])$ lies in $G_m(\text{St}(\sigma^0) \times [0, \zeta_m])$ for some $\sigma^0 \in T_m^0$.

(c) $\varepsilon_m < \rho(h(C), \text{Bd } M_{m-1})$.

(d) $\varepsilon_m < 1/m$.

(e) $\varepsilon_m < \varepsilon_{m-1}$.

We choose an integer $n_m > n_{m-1} > \cdots > n_1$ such that $1/n_m < \delta_m/3$, where δ_m is chosen as δ was in McMillan's Theorem 1 for $\varepsilon = \varepsilon_m$. We use H_{n_m} from Lemma 3 for his H in his Theorem 1. With this H his Theorem 1 gives $M_m = L_m \cup (\bigcup_{i=1}^{k_m} H_i^m)$ with Bd L_m ε_m-homeomorphic to $h(\text{Bd } C)$, L_m a cell in an ε_m-neighborhood of $h(C)$, and each H_i^m, $i = 1, 2, \ldots, k_m$, an ε_m-cube-with-handles. We also have

$$h(C) \subset \text{Int } M_m \subset M_m \subset \text{Int } M_{m-1}.$$

By Lemma 3, $G_{n_m}(T_m^1 \times [0, \zeta_{n_m}]) = G_m(T_m^1 \times [0, \zeta_{n_m}])$ so that conditions (a) and (b) tell us that any ε_m-set intersecting $G_{n_m}(T_m^1 \times [0, \zeta_{n_m}])$ lies in $G_{n_m}(\text{St}(\sigma^0) \times [0, \zeta_{n_m}])$ for some $\sigma^0 \in T_m^0$. According to McMillan's theorem, $H_i^m \cap L_m = F_i^m$ is a disk in Bd H_i^m and in Bd L_m. The rest of the proof will consist mostly of simplifying intersections between the M_m and the "collars" $G_{n_{m-1}}(T_{m-1}^1 \times [0, \zeta_{n_{m-1}}])$ of $h(C)$ by altering the "collars".

Before going on let us make the following simplification in notation. Rename G_{n_m} and ζ_{n_m}. We will use G_m instead of G_{n_m} and ζ_m instead of ζ_{n_m}.

Simplifying intersections with F_i^m. We would like to say that $G_m(T_m^1 \times \zeta_m)$ lies in $L_m - \bigcup F_i^m$. To achieve this we must look at how McMillan arrives at the F_i^m.

Each H_i^m comes from a W_i^m, a polyhedral cube-with-handles such that each component of $W_i^m \cap L_m$ is a 2-cell in the common boundary of W_i^m and of L_m. He finds an $\varepsilon_m/2$-cell, F_i^m, in Bd L_m such that $W_i^m \cap L_m \subset F_i^m$. (No two of these F_i^m intersect.) Then H_i^m is obtained by adding to W_i^m a cell obtained by thickening F_i^m (in L_m). This pushes Bd L_m into L_m slightly so that $H_i^m \cap L_m = F_i^m$ (or rather the pushed-in F_i^m) and $H_i^m \cap L_m$ is a single 2-cell F_i^m.

With an ε_m-homeomorphism of S^3, we can adjust $G_m(T_m^1 \times [0, \zeta_m])$ [before the assumption just prior to this section this set would have been written

$$G_{n_m}(T_m^1 \times [0, \zeta_{n_m}])]$$

in a sufficiently small neighborhood of F_i^m so that $G_m(T_m^1 \times \zeta_m)$ lies in $L_m - \bigcup_{i=1}^{k_m} F_i^m$. This homeomorphism also adjusts $G_{m-1}(T_{m-1}^1 \times [0, \zeta_{m-1}])$ so that $G_{m-1}(T_{m-1}^1 \times \zeta_m) = G_m(T_{m-1}^1 \times \zeta_m)$ lies in $L_m - \bigcup_i F_i^m$. In constructing this space homeomorphism we just take a homeomorphism of Bd L_m onto itself which is fixed outside a small neighborhood of F_i^m and slips $G_m(T_m^1 \times \zeta_m)$ off F_i^m and extend to a homeomorphism of S^3 onto itself that is also fixed outside a small neighborhood of F_i^m. These neighborhoods are to be so small that nothing is moved near any other F_i^m and so small that nothing is moved near any other Bd M_m. The foregoing shows that we may assume that $G_m(T_m^1 \times \zeta_m)$ lies in $L_m - \bigcup F_i^m$ and that $G_m(T_m^1 \times \zeta_{m+1})$ lies in $L_{m+1} - \bigcup F_i^{m+1}$ and $F_i^m \cap G_m(T_m^1 \times [\zeta_{m+1}, \zeta_m]) = \varnothing$.

If σ^1 is a 1-simplex of T_{m-1}, let us use the notation $\sigma^1(m-1)$ to denote the disk

$$G_{m-1}(\sigma^1 \times [\zeta_m, \zeta_{m-1}])$$

and $T(m-1)$ to denote $G_{m-1}(T_{m-1}^1 \times [\zeta_m, \zeta_{m-1}])$. G_m (as adjusted) imposes a triangulation Q_m on Bd L_m such that $Q_m = \{G_m(\sigma \times \zeta_m) : \sigma \in T_{m-1}\}$. Each simplex of Q_m is $(\varepsilon_m + 1/n_m)$-homeomorphic to its corresponding simplex of T_{m-1}. The 1-skeleton of Q_m^1 is a sub-polyhedron of Bd L_m. By construction, if $\sigma^2 \in Q_m^2$, then $\sigma^2 \cap G_m(T_{m-1}^1 \times [0, \zeta_m])$ lies in $G_m(T_{m-1}^1 \times \zeta_m)$ and in Bd σ^2. If $\sigma^1 \in T_{m-1}^1$, then, assuming general position, a component of

$$\sigma^2 \cap \sigma^1(m-1) = \sigma^2 \cap G_{m-1}(\sigma^1 \times [\zeta_m, \zeta_{m-1}])$$

is a simple closed curve in Int $\sigma^2 \cap$ Int $\sigma^1(m-1)$ or is possibly an arc lying in the boundary of each of the 2-cells or a point in the boundary of each. This is a consequence of condition (8) of Lemma 3.

By trading disks we can change $\sigma^1(m-1)$ so that $\sigma^1(m-1) \cap \sigma^2$ contains no simple closed curves. Then $\sigma^1(m-1) \cap \sigma^2$ is a common arc of boundary or a common point in the boundary of each. We do *not* adjust σ^2 for fear of uncovering $h(C)$. Suppose $\sigma_1, \sigma_2, \ldots, \sigma_l$ are the 2-simplexes of Q_m. First we adjust $T(m-1)$ so that it misses Int σ_1 as follows: Let J be any simple closed curve in Int $\sigma_1 \cap T(m-1)$ that bounds a disk D in Int σ_1 such that Int D does not intersect $T(m-1)$. Replace the disk J bounds in $T(m-1)$ by D and push off Int σ_1. Proceeding in this manner $T(m-1)$ may be adjusted so that it misses Int σ_1. Note that the adjusted $T(m-1)$ (also denoted by $T(m-1)$) is homeomorphic to the $T(m-1)$ we started with. We did not introduce any new self intersections. Now we adjust $T(m-1)$ to miss Int σ_2, then to miss Int σ_3, and so on. Thus $T(m-1)$ may be adjusted to miss $L_m - Q_m^1$. Thus, we have that $T(m-1) \cap L_m = Q_m^1$.

Before we calculate how much $T(m-1)$ is moved by the process, let us point out one precaution we wish to make in the "pushing off" part of this disk trading procedure. Each H_i^m intersects L_m in a disk F_i^m. From the point of view of the H_i^m, the disk trading occurs only near the F_i^m. By pushing F_i^m off itself in H_i^m we get a new disk disjoint from F_i^m having its boundary in Bd $H_i^m - F_i^m$. This new disk together with the F_i^m and an annulus on Bd H_i^m bounds a cell K_i^m in H_i^m. K_i^m may be thought of as a cylinder $F_i^m \times [0, 1]$. In pushing off during the above disk trading procedure, we wish not to push anything into $H_i^m -$ Int K_i^m. When part of the disk to be pushed off lies in F_i^m and is to be pushed to the H_i^m side of L_m, we wish to push along the lines perpendicular to F_i^m in the representation of K_i^m as $F_i^m \times [0, 1]$. Thus, any new disks resulting from such disk trading will intersect H_i^m as shown in Figure 4. This will be convenient later.

Now to show that the disk trading does not enlarge $G_m(T_m^1 \times [\zeta_{m+1}, \zeta_m]) = T(m)$ too much. Each $T(m)$ has already had an ε_m-adjustment to move it off the F_i^m. Recall that $\rho(h(x), G_m(x, t)) < 1/n_m$ for all $x \in C$ and $t \in [-\zeta_m, \zeta_m]$ and that $H_m = G_m$ on Bd $C \times \{-\zeta_m, \zeta_m\}$. This latter condition says $G_m(\text{Bd } C \times \zeta_m) = L_m$. All this was

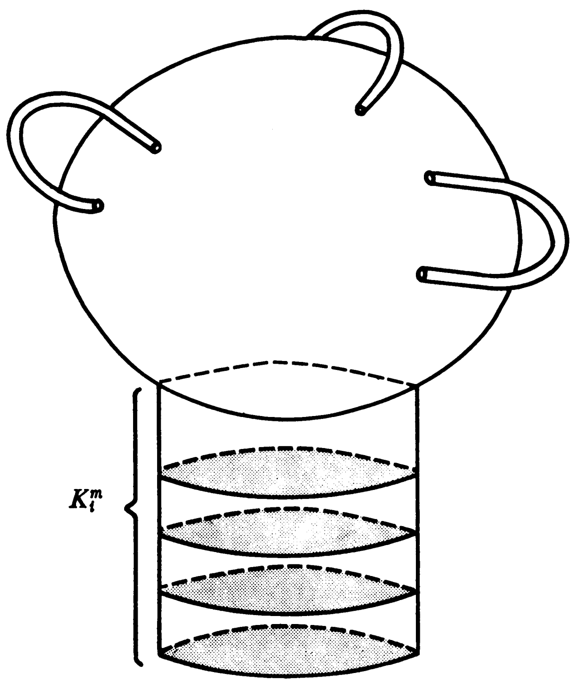

FIGURE 4

true *before* the alteration of G_m to push $G_m(T_m^1 \times \zeta_m)$ off F_i^m. Now G_m is an $(\varepsilon_m + 1/n_m)$-homeomorphism of $\{\bigcup T_i^1 \times [0, \zeta_m]\} \cup \{\bigcup_{i=m}^{\infty} \text{Bd } C \times \zeta_i\}$ instead of a $1/n_m$-homeomorphism. Since $\sigma^2 \in Q_m^2$ is $(\varepsilon_m + 1/n_m)$-homeomorphic to some $\sigma \in T_{m-1}^2$, then mesh Q_m is less than

$$1/(m-1) + 2\varepsilon_m + 2/n_m.$$

Since $1/n_m < \delta_m/3$ and $\delta_m < \varepsilon_m/2$ (see McMillan's Theorem 1 and the beginning of this construction) and $\varepsilon_m < 1/m < 1/(m-1)$, then mesh $Q_m < 4/(m-1)$. Thus no point of $T(m-1)$ gets moved by more than $4/(m-1)$ in the disk trading procedure.

Naming the cubes-with-holes. Now let us do some naming. $T(m-1)$ separates the set $M_{m-1} - \text{Int } L_m$ into little 3-manifolds with connected boundary. See Figure 5. We want to name these manifolds and their sides. Each 3-manifold with boundary is a cube-with-handles with a "top" (which is a disk-with-handles), 3 "sides" (disks) and a "bottom" (disk).

$\alpha_{\sigma,m} \subset \text{Bd } M_{m-1}$. Let $\sigma \in T_{m-1}^2$. σ corresponds to some set $\sigma_{m-1} \subset L_{m-1}$ under G_{m-1}, namely $G_{m-1}(\sigma \times \zeta_{m-1})$. It does not correspond to an element of Q_{m-1}, for each such element is the image of elements of T_{m-2}^2 under G_{m-1}. However, each element of Q_{m-1} is a union of such σ_{m-1}'s. Let $\alpha_{\sigma,m}$ be the disk-with-handles obtained by replacing those F_i^{m-1} in σ_{m-1} with $\text{Bd } H_i^{m-1} - \text{Int } F_i^{m-1}$.

$\beta_{\sigma,m} \subset \text{Bd } L_m$. Let $\sigma \in T_{m-1}^2$. Under G_m there corresponds some $\sigma_m \in Q_m^2$, namely $G_m(\sigma \times \zeta_m)$. Let $\beta_{\sigma,m}$ be this σ_m.

$\gamma_{\sigma_1,m}; \gamma_{\sigma_2,m}; \gamma_{\sigma_3,m}$. If $\sigma \in T_{m-1}^2$, let $\text{Bd } \sigma$ be $\sigma_1 \cup \sigma_2 \cup \sigma_3$, where $\sigma_i \in T_{m-1}^1$, $i = 1, 2, 3$. Define $\gamma_{\sigma_i,m} = \sigma_i(m-1)$.

Each of the $\beta_{\sigma,m}$ and $\gamma_{\sigma_i,m}$ ($i = 1, 2, 3$) is a disk. If any two among $\alpha_{\sigma,m}$, $\beta_{\sigma,m}$ and $\gamma_{\sigma_i,m}$ intersect, it is along an arc of boundary. Recall that

$$T(m-1) = \bigcup \{\gamma_{\sigma^1} : \sigma^1 \in T_{m-1}^1\} = G_{m-1}(T_{m-1}^1 \times [\zeta_m, \zeta_{m-1}]).$$

Then $\alpha_{\sigma,m} \cup \beta_{1,m} \cup \gamma_{\sigma_1,m} \cup \gamma_{\sigma_2,m} \cup \gamma_{\sigma_3,m}$ is a 2-manifold separating S^3. Denote

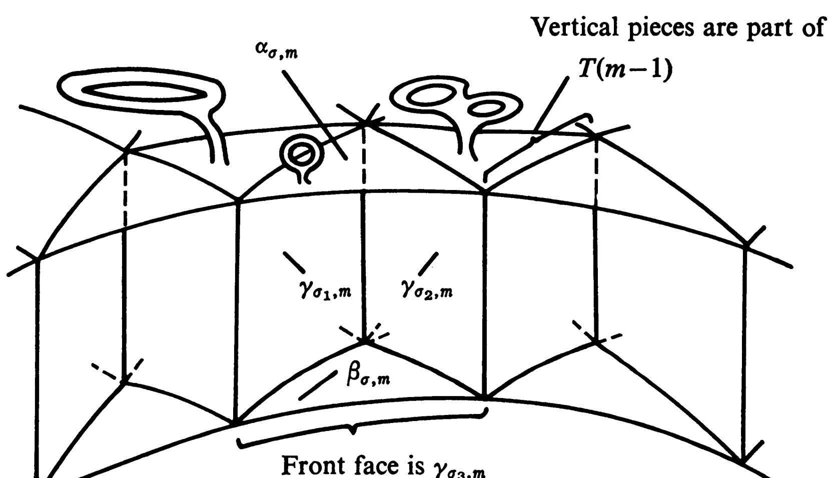

FIGURE 5

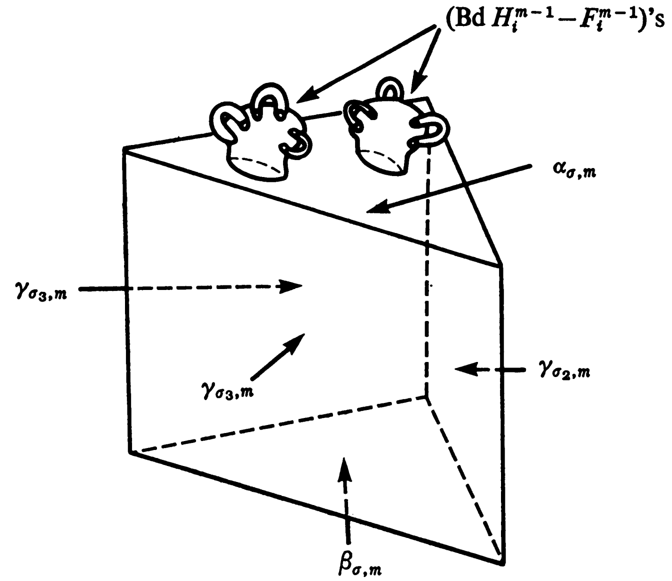

FIGURE 6

by $\Gamma_{\sigma,m}$ the closure of that component of S^3 minus this 2-manifold in Int M_{m-1} (see Figure 6). Then $M_{m-1} - \text{Int } L_m = \bigcup \{\Gamma_{\sigma,m} : \sigma \in T^2_{m-1}\}$. See Figure 5. For if $p \in \text{Int}(M_{m-1} - \text{Int } L_m)$ take an arc pq from p to Bd $M_{m-1} \cup$ Bd L_m such that $\text{Int}(pq) \subset \text{Int}(M_{m-1} - \text{Int } L_m)$ and $pq \cap T(m-1) = \varnothing$. Then $q \in \alpha_{\sigma,m}$ or $\beta_{\sigma,m}$ for some $\sigma \in T^2_{m-1}$. Since $pq - q$ misses Bd $\Gamma_{\sigma,m}$, and points near q on the other side of Bd $\Gamma_{\sigma,m}$ can be joined by an arc to $S^3 - M_{m-1}$ missing Bd $\Gamma_{\sigma,m}$, then $p \in \Gamma_{\sigma,m}$. Hence, $M_{m-1} - \text{Int } L_m \subset \bigcup \{\Gamma_{\sigma,m} : \sigma \in T^2_{m-1}\}$. The other inclusion is obvious.

We must now alter the γ so that we can replace $\beta_{\sigma,m}$ with the union of the appropriate $\alpha_{\sigma',m+1}$'s—that is, replace disks on $\beta_{\sigma,m}$ with Bd $H_i^m - \text{Int } F_i^m$'s in the manner we did to make the $\alpha_{\sigma,m}$. To do this we adjust the γ to miss the H_i^m. We cannot do this, however, while the γ remain disks, so we add handles to the γ.

Simplifying intersections with H_i^m. Before we can adjust $T(m-1)$ so that no H_i^m can intersect it, we must be sure that no handle of H_i^m loops a "fence post" $G_{m-1}(v \times [\zeta_m, \zeta_{m-1}])$, where v is a vertex of T_{m-1}. In Figure 7, H_i^m is shown as a torus growing out of Bd L_m. Bd L_m is shown jutting up through two "walls" in $T(m-1)$. The walls are shown as they were adjusted to remove $T(m-1)$ from Bd L_m.

Let N_i^m be a regular neighborhood in $\text{Cl}(S^3 - L_m)$ of H_i^m. We want the N_i^m to be mutually disjoint, each N_i^m to be an ε_m-set, and $E_i^m = N_i^m \cap L_m$ to be a disk. We want each E_i^m, and hence each N_i^m, to miss $G_m(T^1_m \times [0, \zeta_m])$. This can be done, because the F_i^m have this property. In particular, we want each E_i^m to miss the 1-skeleton of Q_m. We also want $N_i^m \subset \text{Int } M_{m-1}$.

We want to look at each H_i^m as a fattened up wedge of simple closed curves $J_r^{i,m}$, $r = 1, 2, \ldots, R_{i,m}$, with $J_r^{i,m}$ in general position with respect to $\bigcup \gamma_{\sigma,m}$, with the wedge point $x_{i,m}$ in Int F_i^m, and with $J_r^{i,m} - x_{i,m} \subset \text{Int } H_i^m$. Furthermore, we

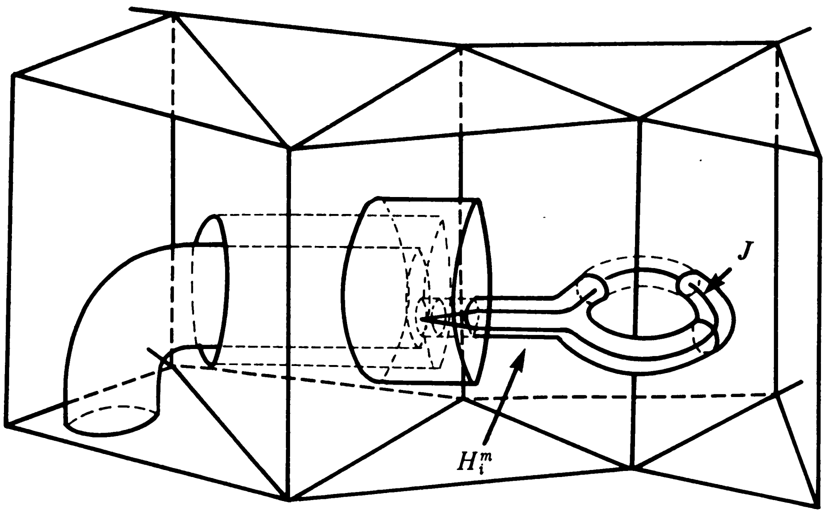

FIGURE 7

want to choose the $J_r^{i,m}$ so that they intersect the disks $\gamma \cap K_i^m$ exactly twice (see Figure 4). Define a pseudo-isotopy f_i^m: $N_i^m \times I \to N_i^m$ such that

(1) $f_i^m(x, 0) = x$, for all $x \in N_i^m$,

(2) $f_i^m(x, t) = x$, for all $x \in J_r^{i,m}$, $r = 1, 2, \ldots, R_{i,m}$, and for all $x \in \mathrm{Bd}\, N_i^m$,

(3) $f_i^m|(N_i^m - (H_i^m - F_i^m)) \times t$ is a homeomorphism for all $t \in [0, 1]$,

(4) $f_i^m(H_i^m, 1) = \bigcup \{J_r^{i,m} : r = 1, 2, \ldots, R_{i,m}\} \cup F_i^m$,

(5) $f_i^m(N_i^m, t) = N_i^m$ for all t in $[0, 1]$.

The plan is to adjust the $\gamma_{\sigma,m}$'s to miss $\bigcup_r J_r^{i,m}$ and use f_i^m to push them off all of H_i^m. By choosing a stage t of f_i^m close enough to the end stage that $f_i^m(H_i^m, t)$ lies so close to $(\bigcup_r J_r^{i,m}) \cup F_i^m$ that it misses all the $\gamma_{\sigma,m}$'s, too, we can use

$$(f_i^m \mid N_i^m \times t)^{-1}$$

to push all the $\gamma_{\sigma,m}$'s off H_i^m.

First, we make a few definitions. Consider each $J_r^{i,m}$ and each $\gamma_{\sigma,m}$ as being oriented. Let J be any $J_r^{i,m}$ and γ any $\gamma_{\sigma,m}$. Let $p(J)$ be the number of positive crossings of J through γ and let $n(J)$ be the number of negative crossings of J through γ. Define the piercing number of J with respect to γ to be

$$p \# J = p(J) - n(J)$$

and the intersection number of J with respect to γ to be

$$I(J) = p(J) + n(J).$$

We will refer to an arc of boundary of a $\gamma_{\sigma,m}$ spanning from $\mathrm{Bd}\, L_{m-1}$ to $\mathrm{Bd}\, L_m$ as a post. We will refer to $\gamma_{\sigma,m}$ as a fin from each of its posts. We will assume that each $J_r^{i,m}$ is in general position with respect to $T(m-1)$ so that $J \cap T(m-1)$ is finite, misses all the posts of $T(m-1)$, and crosses at each point of intersection.

Let us look back to $T(m-1)$, which is the union of the $\gamma_{\sigma,m}$'s. We made two adjustments to $T(m-1)$ to get the $\gamma_{\sigma,m}$'s. Before the adjustments, each H_i^m was an ε_m-set so that, by our choice of ε_m at the very beginning of this proof, $H_i^m \cap T(m-1)$ lay in the union of the fins radiating from some post P. Thus $p \# J_r^{i,m}=0$ with respect to all γ'' not radiating from this post P, because $I(J_r^{i,m})=0$ with respect to such γ''. The first adjustment (moving $G_{m-1}(T_{m-1}^1 \times [0, \zeta_{m-1}])$ off the F_i^m and F_i^{m-1}) does not change this. The next adjustment, the disk trading, did. It caused new γ'' to hit H_i^m in K_i^m. But $p \# J$ with respect to γ'' is the linking number of J and Bd γ''. Since Bd γ'' and J were not moved in the disk trading procedure, $p \# J$ with respect to any γ'' which is not a fin of P remained 0. Hence neither adjustment made $p \# J$ with respect to those γ which are not fins from P nonzero.

If P is a post in $T(m-1)$, γ and γ' are fins from P, and $(H_i^m - K_i^m) \cap T(m-1)$ lies in the union of the fins from P, then, for any $J_r^{i,m}$ in H_i^m, $p \# J_r^{i,m}=0$ with respect to γ implies $p \# J_r^{i,m}=0$ with respect to γ'. For consider the set

$$X = \bigcup \{\Gamma_{\sigma,m} : P \subset \Gamma_{\sigma,m}\} \cup K_i^m.$$

Then $\gamma \cup \gamma'$ separates X, K_i^m lies in one component, and $J_r^{i,m} \subset X$. Since $J_r^{i,m}$ is a simple closed curve, it crosses $\gamma \cup \gamma'$ as many times in one direction as another. Since $p \# J_r^{i,m}=0$ with respect to γ, it crosses γ as many times in one direction as another. Thus it must cross γ' as many times in one direction as another. Hence $p \# J_r^{i,m}=0$ with respect to γ'. This shows, in fact, that any simple closed curve in

$$\bigcup \{\Gamma_{\sigma,m} : P \subset \Gamma_{\sigma,m}\}$$

links Bd γ iff it links Bd γ'.

We want $p \# J=0$ with respect to all γ making up $T(m-1)$. To accomplish this we must adjust $T(m-1)$. For each post P we choose a fin γ such that $P \subset \text{Bd}\,\gamma$, and for each J such that $p \# J \neq 0$ with respect to γ and $J \subset H_i^m$ such that $(H_i^m - K_i^m) \cap T(m-1)$ lies in those fins radiating from P, we will make $I(J)=0$ with respect to γ by an adjustment of $T(m-1)$. The manner in which we do this says that $p \# J=0$ with respect to all γ radiating from P and hence for all γ making up $T(m-1)$.

We take a collection of disjoint polygonal arcs from P minus its endpoints to points of $J \cap \gamma$ such that each arc misses J' for all $J' \neq J$, each arc intersects J at only one point, and each arc lies, except for its endpoint on P, in Int γ.

Choose a disjoint collection of neighborhoods $N_1, N_2, \ldots, N_k$ of the arcs joining P to $J \cap \gamma$. We choose the N_i so that none contains but one such arc, none contains a point of J' for any $J' \neq J$, none gets outside of Int $(M_{m-1} - L_m)$, none intersects any post other than P, none intersects any $\gamma_{\sigma,m}$ not radiating from P, none gets outside Int $(\bigcup \{\Gamma_{\sigma,m} : P \subset \Gamma_{\sigma,m}\})$ and none gets outside the ε_m-neighborhood of the arc it contains. With a homeomorphism of S^3, fixed outside $\bigcup_{i=1}^k N_i$, and taking $\gamma \cup \gamma'$ onto $\gamma \cup \gamma'$ (here γ' is any other fin from P), we move P so that all the arcs lie in γ'. We also want the new γ' to contain all the old γ'. This homeomorphism adjusts $T(m-1)$ so that $I(J)=0$ with respect to γ.

We do the above process to all the J of the type under consideration intersecting that particular γ so that no such J has nonzero piercing number with respect to any fin from P. We do this for every post P. After doing this, no J will loop any post P. In other words, $p \# J=0$ with respect to any γ making up $T(m-1)$ and for all $J_r^{i,m}$ for all H_i^m.

We are now in a position to alter the γ so that they miss the $J_r^{i,m}$. We choose such a J and show how it is done. Let x_0 be the point at which J is attached to L_m. Suppose for the time being that $J \subset H_i^m$ and K_i^m misses all the walls γ. Proceed along J to the first point of intersection with a γ, say γ_0. Now proceed to the first point q_0 at which J pierces this γ_0 in the opposite direction and back up to the last point p_0 at which J pierced γ_0 in the original direction. We would like for the arc p_0q_0 to be disjoint from all the γ's except for its endpoints p_0 and q_0. If not we connect q_0 to p_0 by an arc A_0 lying in γ_0 and push the resultant simple closed curve J_0 off γ_0. $J_0 \cap \gamma_0 = \varnothing$ so that J_0 does not link Bd γ_0 and hence does not link Bd γ for any γ radiating from P. Hence there are points p_1, q_1 of p_0q_0 lying in some fin γ_1 such that $p_1q_1 \cap \gamma_1 = \{p_1, q_1\}$. Continuing in this way we can find two points p_n, q_n such that p_nq_n is a subarc of J, p_n and q_n lie on some wall γ_n, J pierces this wall γ_n in different directions at p_n and q_n, and Int (p_nq_n) misses all the γ.

Take a small regular neighborhood R of p_nq_n in the $\Gamma_{\sigma,m}$ containing p_nq_n, remove it from that $\Gamma_{\sigma,m}$ and attach it to the $\Gamma_{\sigma,m}$ which J_n leaves at p_n and enters at q_n. Replace the two disks of $R \cap \gamma_n$ with Bd R minus the interiors of those disks to get a new γ_n (the new γ_n now has an oriented handle). The number of points at which J hits the $\bigcup \gamma$ is now reduced by two.

We must, of course, take R sufficiently close to p_nq_n so it misses all of $J - p_nq_n$ as well as all the other $J_r^{i,m}$ and lies inside the H_i^m containing J. Note that the size of a γ after a finite number of changes of this sort is not increased as much as $2\varepsilon_m$.

Now consider the case in which $J \subset H_i^m$ and K_i^m intersects some wall γ. Then $H_i^m \cap \gamma = K_i^m \cap \gamma$ is a collection of mutually disjoint disks in K_i^m, each of which J intersects once in each direction. Since the component disks are linearly ordered from X_0, they may now be treated in a manner similar to the above.

Repeating this process a finite number of times, adjusts the walls γ so that *no* $J_r^{i,m}$ hits any of them. Thus a wedge of simple closed curves $J_r^{i,m}$, $r=1, 2, \ldots, R_{i,m}$, lies in the interior of the $\Gamma_{\sigma,m}$ containing $x_{i,m}$ (except that $x_{i,m}$ lies on Bd $\Gamma_{\sigma,m}$). By choosing a number $\xi_{i,m} > 0$ close enough to 1 we have that

$$f_i^m(H_i^m, \xi_{i,m}) \subset \Gamma_{\sigma,m}$$

and lies so close to $\bigcup J_r^{i,m}$ that it misses all the walls $\gamma_{\sigma j,m}$ of $\Gamma_{\sigma,m}$. Then

$$(f_i^m \mid N_i^m \times \xi_{i,m})^{-1}$$

pushes all the walls off of H_i^m, is fixed outside N_i^m, and moves no point more than diameter $N_i^m < \varepsilon_m$. Piecing together all these $(f_i^m \mid N_i^m \times \xi_{i,m})^{-1}$'s we move all the walls γ off all the H_i^m by a space homeomorphism fixed outside $\bigcup_{i=1}^{k_m} N_i^m$.

Naming new $\Gamma_{\sigma,m}$. At present, a $\Gamma_{\sigma,m}$ is a cube-with-holes with $\mathrm{Bd}\,\Gamma_{\sigma,m}=\alpha_{\sigma,m} \cup \beta_{\sigma,m} \cup \gamma_{\sigma_1,m} \cup \gamma_{\sigma_2,m} \cup \gamma_{\sigma_3,m}$, where the $\gamma_{\sigma_i,m}$ are disks-with-handles as obtained in the previous section. The $\beta_{\sigma,m}$ is a disk lying in $\mathrm{Bd}\,L_m$. We want to change $\beta_{\sigma,m}$ to a disk-with-handles by replacing each F_i^m lying in $\beta_{\sigma,m}$ with $\mathrm{Bd}\,H_i^m - \mathrm{Int}\,F_i^m$. Note that the new $\beta_{\sigma,m}$ lies in $\Gamma_{\sigma,m}$. We remove from $\Gamma_{\sigma,m}$ the interiors of the H_i^m lying in $\Gamma_{\sigma,m}$ to get a new $\Gamma_{\sigma,m}$ for which $\mathrm{Bd}\,\Gamma_{\sigma,m}=\alpha_{\sigma,m} \cup \beta_{\sigma,m} \cup \gamma_{\sigma_1,m} \cup \gamma_{\sigma_2,m} \cup \gamma_{\sigma_3,m}$ where the $\beta_{\sigma,m}$ is the new $\beta_{\sigma,m}$. Now we have that $\bigcup_{\sigma \in T^2_{m-1}} \Gamma_{\sigma,m} = M_{m-1} - \mathrm{Int}\,M_m$ rather than $M_{m-1} - \mathrm{Int}\,L_m$ as before. Thus $\bigcup \{\Gamma_{\sigma,m} : \sigma \in T^2_{m-1}, m=2, 3, \ldots\} = M_1 - h(C)$. Define Γ_1 to be the closure of $S^3 - M_1$. Then

$$\Gamma_1 \cup \left(\bigcup \Gamma_{\sigma,m}\right) = S^3 - h(C).$$

We wish now to calculate diameter $\Gamma_{\sigma,m}$ for $m>1$. Clearly, diameter $\Gamma_{\sigma,m}$ equals diameter $\mathrm{Bd}\,\Gamma_{\sigma,m}$. Recall

$$\mathrm{Bd}\,\Gamma_{\sigma,m} = \alpha_{\sigma,m} \cup \beta_{\sigma,m} \cup \gamma_{\sigma_1,m} \cup \gamma_{\sigma_2,m} \cup \gamma_{\sigma_3,m},$$

where $\sigma \in T^2_{m-1}$ and σ_1, σ_2, σ_3 are the 1-faces of σ.

(i) *diameter* $\alpha_{\sigma,m} < 4/(m-1)$.

$$\begin{aligned} \text{diameter } \alpha_{\sigma,m} &< \text{mesh } T_{m-1} + \text{expansion due to } G_m + \text{diameter } H_i^m \\ &\qquad + \text{adjustment to move } Q_{m-1} \text{ off the } F_i^{m-1} \\ &< 1/(m-1) + 2/n_{m-1} + 2\varepsilon_m + 2\varepsilon_{m-1} < 6/(m-1). \end{aligned}$$

We call the reader's attention to the notation change from G_{n_m} to G_m and ζ_{n_m} to ζ_m following the construction of M_m.

(ii) *diameter* $\beta_{\sigma,m} < 4/(m-1)$. We have here

$$\text{diameter } \beta_{\sigma,m} < \text{mesh } Q_m + 2\varepsilon_m < \text{mesh } Q_m + 2/(m-1) < 6/(m-1).$$

The mesh Q_m is calculated prior to the first naming of the $\Gamma_{\sigma,m}$.

(iii) *diameter* $\gamma_{\sigma_i,m} < 16/(m-1)$. We have diameter $\sigma_i < 1/(m-1)$, which is the mesh T_{m-1}; each point of the original $\gamma_{\sigma_i,m}$ is within $1/n_{m-1}$ of a point of σ_i. We moved $\gamma_{\sigma_i,m}$ by ε_m and ε_{m-1}, respectively, to get it off of F_i^m and F_i^{m-1}, by $4/(m-1)$ in the disk trading and by ε_m in pushing them off of the H_i^m. Thus

$$\begin{aligned} \text{diameter } \gamma_{\sigma_i,m} &< 1/(m-1) + 2/n_{m-1} + 2\varepsilon_m + 2\varepsilon_{m-1} + 8/(m-1) + 2\varepsilon_m \\ &< 1/(m-1) + 1/(m-1) + 2/(m-1) + 2/(m-1) + 8/(m-1) + 2/(m-1) \\ &< 16/(m-1). \end{aligned}$$

Putting all these parts together, we have that

$$\text{diameter } \mathrm{Bd}\,\Gamma_{\sigma,m} < 6/(m-1) + 6/(m-1) + 3(16/(m-1)) = 60/(m-1).$$

We see then that diameter $\Gamma_{\sigma,m} < 60/(m-1)$ and, thus, diameter $\Gamma_{\sigma,m} \to 0$ as $m \to \infty$.

4. **Repairing embeddings.** The following lemma will be useful in proving Theorem 1:

LEMMA 4. *Let N be a connected closed 2-manifold and let $K_1, K_2, \ldots, K_n$ be a finite collection of disjoint connected 1-complexes in N. Suppose there is a map h taking N onto a 2-sphere S whose nondegenerate point inverses are the K_i. Then there is an extension f of h taking $N \times [0, 1]$ onto $S \times [0, 1]$ such that*

(a) $f(x, 0) = h(x)$ *for all* $x \in N$,

(b) $f^{-1}(S \times t) = N \times t$,

(c) $f|N \times 1$ *has just one nondegenerate point inverse, K, and K is a connected 1-complex,*

(d) *each nondegenerate point inverse of f is a connected 1-complex,*

(e) *the image of the nondegenerate point inverses under f is n arcs, disjoint except a common endpoint $f(K)$.*

Proof. Since each K_i goes to a point in S under h, the K_i do not separate N. Thus there is a collection of disjoint polygonal arcs $A_1, A_2, \ldots, A_{n-1}$ in N such that A_i joins K_1 to K_{i+1} and Int A_i lies in $N - \bigcup_{j=1}^{n} K_j$. Set $K = (\bigcup K_i) \cup (\bigcup A_j)$. Then $h(K)$ is a wedge of $n-1$ arcs in S emanating from $h(K_1)$. Define a pseudo-isotopy $H\colon S \times [0, 1] \to S$ that shrinks $h(K)$ to a point in S. Then

$$f\colon N \times [0, 1] \to S \times [0, 1]$$

defined by

$$f(x, t) = H(h(x), t) \times t$$

is the required mapping.

The following result is our main theorem. It says that the embedding of a 3-cell in S^3 can be repaired.

THEOREM 1. *Let C be a (wild) 3-cell in S^3 and let $h\colon C \to S^3$ be an embedding of C such that $h(C)$ is tame. Then h can be extended to a monotone map f of S^3 onto itself such that $f(S^3 - C) = S^3 - f(C)$. Furthermore, each nondegenerate point inverse can be taken to be a finite 1-complex.*

Proof. We may suppose that $h(C)$ is the round unit ball in S^3. Now consider a sequence of triangulations T_i of Bd C as given in §3. Then $h(T_i)$ is a sequence of triangulations of $h(\text{Bd}\, C)$. Let H carry Bd $C \times [0, \frac{1}{2}]$ homeomorphically into $S^3 - \text{Int}\, h(C)$ such that, for all $x \in \text{Bd}\, C$, $H(x, 0) = h(x)$, $H(\text{Bd}\, C \times (0, \frac{1}{2}]) \cap h(C) = \varnothing$, and $H(\text{Bd}\, C \times t)$ is a round sphere concentric with $h(\text{Bd}\, C)$. Let $C_{\sigma,m}$ be the 3-cell $H(\sigma \times [\frac{1}{2}^{m+1}, \frac{1}{2}^{m}])$ for each $\sigma \in T_m^2$, $m \geqq 1$. Let C_1 be the closure of that component of $S^3 - H(\text{Bd}\, C \times \frac{1}{2})$ not containing $h(C)$. We want to map $\Gamma_{\sigma,m}$ (from §3) onto $C_{\sigma,m}$ in such a way as to extend h. First, we define the map on Bd $\Gamma_{\sigma,m}$. Recall from §3 that

$$\text{Bd}\, \Gamma_{\sigma,m} = \alpha_{\sigma,m} \cup \beta_{\sigma,m} \cup \gamma_{\sigma_1,m} \cup \gamma_{\sigma_2,m} \cup \gamma_{\sigma_3,m}.$$

Each of $\alpha_{\sigma,m}$ and $\gamma_{\sigma_i,m}$ is a disk-with-handles, so there is a 1-complex on each missing its boundary such that modding out this 1-complex gives a decomposition space homeomorphic to a disk. Call these 1-complexes $K(\alpha_{\sigma,m})$ and $K(\gamma_{\sigma_i,m})$ ($i=1, 2, 3$), respectively.

Since $\bigcup\{\text{Bd } \alpha_{\sigma,m} : \sigma \in T^2_{m-1}\}$ is a copy of T^1_{m-1}, there is a natural homeomorphism from this set onto $H(T^1_{m-1} \times 1/2^m)$. This homeomorphism can be extended to a monotone mapping of $\bigcup\{\alpha_{\sigma,m} : \sigma \in T^2_{m-1}\}$ onto $H(\text{Bd } C \times 1/2^m)$ collapsing only the $K(\alpha_{\sigma,m})$ to a point. Piecing together these maps we have an extension of h to $C \cup (\bigcup\{\alpha_{\sigma,m} : \sigma \in T^2_{m-1}, m=2, 3, 4, \ldots\})$. This map is clearly continuous at Bd C by construction of the $\alpha_{\sigma,m}$. Now we have h defined on two disjoint arcs of boundary of each $\gamma_{\sigma_i,m}$. Extend to all the $\gamma_{\sigma_i,m}$ in such a manner that only the $K(\gamma_{\sigma_i,m})$ get collapsed to a point and so that $\gamma_{\sigma_i,m}$ gets taken onto $H(\sigma_i \times [1/2^{m+1}, 1/2^m])$. Thus h is extended to $\bigcup\{\text{Bd } \Gamma_{\sigma,m} : \sigma \in T^2_{m-1}, m=2, 3, \ldots\} \cup C$, because $\beta_{\sigma,m}$ is the union of $\alpha_{\sigma',m+1}$'s.

Now we extend the map to collars in each $\Gamma_{\sigma,m}$ of the boundary of $\Gamma_{\sigma,m}$ by using Lemma 4. On the inside of these collars the map collapses a connected 1-complex to a point and there is only one nondegenerate point inverse on the inside of the collar of a $\Gamma_{\sigma,m}$. By Theorem 6.2 of [4], the map can be extended to the rest of $\Gamma_{\sigma,m}$ onto $C_{\sigma,m}$ in such a manner that each point inverse is a connected 1-complex.

The extension to Γ_1 onto C_1 is done in the same manner. The extended map is the required mapping f.

Remark. In Theorem 1, if C is locally tame at each point of an open set U of Bd C, then f can be taken to be a homeomorphism on some neighborhood in S^3 of U. Just push Bd C into S^3-C at all points of U and apply the technique of Theorem 1 to the new 3-cell C' so formed.

A *crumpled cube* C is a space homeomorphic to the union of a 2-sphere and its interior in E^3. If C is a crumpled cube, Int C means the set of all points having a neighborhood homeomorphic to E^3 and Bd C means $C-\text{Int } C$. Thus Bd C is a 2-sphere and Int C is homeomorphic to the interior of Bd C under some embedding in E^3. If C_1 and C_2 are crumpled cubes and h is a homeomorphism of Bd C_1 onto Bd C_2, then $C=C_1 \cup_h C_2$ is the space $C_1 \cup C_2$ with $x \in \text{Bd } C_1$ identified with $h(x) \in \text{Bd } C_2$. C_1 and C_2 are said to be *sewn* along their boundary by h and C is called the *sum* of C_1 and C_2. The following theorem is an immediate corollary to Theorem 2 and a result due to Hosay [11] and to Lininger [14], which says that any crumpled cube may be sewed to a 3-cell in such a way that the sewing gives S^3. (For a relatively easy proof of this theorem, the reader is referred to [7].)

Corollary 1. *If C is a crumpled cube and K is a* 3*-cell, then any homeomorphism of* Bd C *onto* Bd K *can be extended to a monotone mapping f of C onto K such that* $f(\text{Int } C)=\text{Int } K$ *and* $f(\text{Bd } C)=\text{Bd } K$.

Proof. Sew C, K to 3-cells C', K', respectively, to get two copies of S^3. Then use Theorem 1 to get a map of S^3 onto itself taking C' to K' extending the given homeomorphism. The restriction of this map to C is the required extension.

Corollary 2 says any embedding of a 2-sphere in S^3 can be repaired:

COROLLARY 2. *If S_1 and S_2 are 2-spheres in S^3, S_2 tame, and h is a homeomorphism of S_1 onto S_2, then h can be extended to a monotone mapping f of S^3 onto itself such that $f(S^3-S_1)=S^3-S_2$.*

Proof. Consider S^3 as the sewing of two crumpled cubes, C_1 and C_2, along S_1 and also as a sewing of two 3-cells, K_1 and K_2, along S_2, and use Corollary 1.

Professors Daverman and Eaton have pointed out that the following theorem, which says that the embedding of any disk in S^3 can be repaired, is easily proved using a result of theirs and Theorem 1:

COROLLARY 3. *If D_1 and D_2 are disks in S^3, D_2 is tame, and h is a homeomorphism of D_1 onto D_2, then there is an extension of h to a monotone mapping f of S^3 onto itself such that $f(S^3-D_1)=S^3-D_2$.*

Proof. We may suppose that D_2 is the disk $\{(x, y, 0) : x^2+y^2 \leqq 1\}$. Let C be the cell $\{(x, y, z) : x^2+y^2+z^2 \leqq 1\}$. In Theorem 7 of [8], Daverman and Eaton have shown that there is a 3-cell K in S^3 and a monotone mapping g of S^3 onto itself such that $g(K)=D_1$, $g|S^3-K$ is a homeomorphism of S^3-K onto S^3-D_1, and the following diagram commutes for some homeomorphism h_1:

$$\begin{tikzcd} K \arrow[r, "h_1"] \arrow[d, "g|K"'] & C \arrow[d, "g_1"] \\ D_1 \arrow[r, "h"'] & D_2 \end{tikzcd}$$

where $g_1 : C \to D_2$ is given by $g_1(x, y, z)=(x, y, 0)$. By Theorem 1, h_1 can be extended to a monotone mapping h_2 of S^3 onto itself such that $h_2(S^3-K)=S^3-C$. Clearly, g_1 can be extended to a mapping g_2 of S^3 onto itself such that $g_2|S^3-C$ is a homeomorphism of S^3-C onto S^3-D_2. Set $f=g_2 \circ h_2 \circ g^{-1}$. Then f is the required monotone mapping.

In general, it is not known whether an embedding of an arc or a simple closed curve in S^3 can be repaired. However, Theorem 3 of the previously mentioned paper of Daverman and Eaton says that, if C is a 3-cell in S^3, there is a map f of S^3 onto itself such that $f(C)$ is an arc and $f|S^3-C$ is a homeomorphism of S^3-C onto $S^3-f(C)$. Since this can be done for wild 3-cells C in such a manner that $f(C)$ is also wild, then it follows that certain wild arcs in S^3, namely those obtained by squeezing a 3-cell in S^3, can be repaired. However, a converse of this result does not exist so that the following question is still open: Can an embedding of an arc (simple closed curve) in S^3 be repaired?

The above theorems completely repair an embedding. But are questions such as the following also true? If S is a wild sphere in S^3 and U an open subset of S, is there a monotone map $f: S^3 \to S^3$ such that $f|S$ is a homeomorphism, $f(S)$ is made locally tame only at each point of $f(U)$, and $f(S^3-S)=S^3-f(S)$? And if so, does such a map change the wildness of points on S that are well away from U? The proof of Theorem 1 required the extension of the map to Γ_1, the closure of the complement in S^3 of a cube-with-handles. For surfaces in 3-manifolds or for spheres-with-handles in S^3, we do not give the theorem analogous to Theorem 1 because of the difficulty of extending a map of Bd Γ_1 into a 3-manifold other than a cell. See Lambert [13] and Jaco and McMillan [12].

These results enable us to extend monotone upper semicontinuous decompositions of the following variety. Let S be a wild 2-sphere in S^3. Let G_1 be an upper semicontinuous decomposition of S into continua not separating S. By a well known theorem of R. L. Moore [16], S/G_1 is homeomorphic to S. By Corollary 2, there is a monotone decomposition G_2 of S^3 whose nondegenerate elements are disjoint from S, $S^3/G_2=S^3$, and S goes to a tame 2-sphere in S^3. If G is the decomposition whose nondegenerate elements are those of G_1 together with those of G_2, then, by [9, Theorem 8], and the preceding statement, $S^3/G=S^3$ and S goes to a tame 2-sphere in S^3/G. For an example of a decomposition in S^3 that cannot be extended to a decomposition of S^3 giving back S^3, the reader is referred to §8 of [4]. For a theorem analogous to Corollary 2, in the sense that it shows how to unknot simple closed curves in E^3 see Theorem 5 of [10].

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