TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 161, November 1971

*-TAMING SETS FOR CRUMPLED CUBES. I: BASIC PROPERTIES([1])

BY
JAMES W. CANNON

Abstract. Is a surface in a 3-manifold tame if it is tame modulo a tame set? This question was answered by the author through the introduction and characterization of taming sets. The purpose of this paper is to introduce and establish the basic properties of the more general and more flexible, but closely related, *-taming set.

1. **Introduction.** A crumpled cube is the union of a 2-sphere and one of its complementary domains in E^3. If C is a crumpled cube in E^3, then the closure cl (E^3-C) of E^3-C is also a crumpled cube which is said to be complementary to C and is denoted by C^* (with $C=(C^*)^*$). We write Bd C (=Bd C^*) for the 2-sphere $C \cap C^*$ and Int C for the set E^3-C^*. A crumpled cube in E^3 is said to be a 3-cell if it is homeomorphic either with the solid round ball B of radius 1 centered at the origin in E^3 or with the complementary crumpled cube B^* of B. (This nonstandard terminology allows us the descriptive advantages of having well-defined interiors and exteriors for all 2-spheres and of having horizontal and other geometrical-linear objects, as in E^3, while having a complementary crumpled cube for each crumpled cube, as in S^3.)

A *-taming set X in E^3 is a closed subset of E^3 having the following property: if C is a crumpled cube in E^3, $X \subset C$, and Bd C is locally tame at each point of Bd $C-(X \cap \text{Bd } C)$, then C^* is a 3-cell. Speaking roughly, a *-taming set X in C "tames" C^*.

The concept of *-taming set arises naturally as one seeks the underlying principle which unifies theorems of the following sort:

Example 1. A 2-sphere S in E^3 which contains a tame arc A and is locally tame modulo A is tame [14]. (A set which shares this "taming" property with tame arcs has been called a taming set [9]. We shall show that the notion of *-taming set is a true generalization of the notion of taming set (Corollary 3.8).)

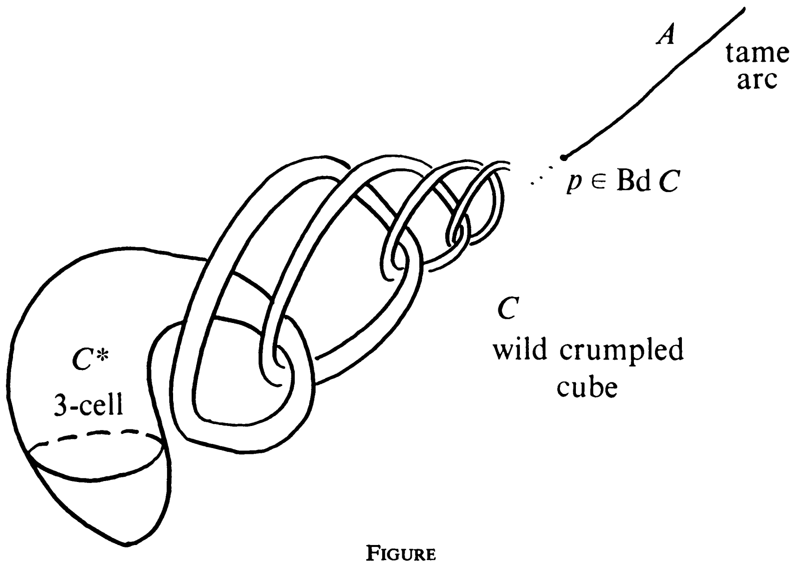
Example 2. If a crumpled cube C has boundary which is tame modulo a point p and if p is accessible with a tame arc A from Int C, then C^* is a 3-cell. (This is an easy consequence of [22, Theorems 3 and 1] and [2].) For the person intent on

Received by the editors October 14, 1970.

AMS 1970 *subject classifications*. Primary 57A10.

Key words and phrases. Tame modulo tame sets, crumpled cubes, surfaces in 3-manifolds, taming sets, *-taming sets, 2-spheres.

([1]) This research was supported by NSF graduate and postdoctoral fellowships.



FIGURE

understanding and remembering the content of the *-taming set definition, this is a particularly good example. (See the Figure.)

The tame arc A in both Example 1 and Example 2 plays the "taming" role of a *-taming set. The unification of such results as Examples 1 and 2 provides a tool which is surprising in its flexibility. As examples of this, we mention two new theorems which are both corollaries to the same *-taming set theorem [11, Theorem 2].

EXAMPLE 3 [11, COROLLARY 3]. If S is a 2-sphere in E^3 and no horizontal section of S has a degenerate component, then S is tame.

EXAMPLE 4 (COROLLARY 4.6 AND [11, COROLLARY 5]). If S is a 2-sphere in E^3, U and V are the components of E^3-S, and S can be touched from V at each point of S with a pencil, then S is tame from U. (That is, cl U is a 3-cell if, in the terminology of analysis, cl U satisfies an exterior cone condition.)

In this first paper of a series (see also [11] and [12]) we set up the basic machinery for studying *-taming sets. Our main theorems are Theorems 3.1, 3.4, 3.7, and 3.10. We give a number of examples of *-taming sets in §4. In later papers we extend the list of *-taming sets and find what information one can obtain about the tameness of a 2-sphere from the structure of its horizontal sections.

Loveland has announced applications of *-taming set theory [20], [21]. Burgess [4], [6] authored two early papers which contain what can easily be recognized as *-taming set theorems. One of Burgess' lemmas (our Lemma 3.3), which we obtained independently before we realized the connection of our work with that of Burgess, will play a key role in Theorem 3.4. We are deeply indebted to Burgess for his instruction and friendship.

We remark that by standard procedures (see [8, §11]) one can generalize all of our results in more or less obvious ways to cover embeddings of arbitrary 2-manifolds in 3-manifolds.

2. **Local homotopy properties near surfaces.** We use Δ to denote a standard 2-simplex. A loop (in E^3) is a map f: Bd $\Delta \to E^3$; a simple closed curve is a homeomorphism J: Bd $\Delta \to E^3$. A singular disk is a map $g: \Delta \to E^3$; a disk is a homeomorphism $D: \Delta \to E^3$. We shall at times purposely ignore the distinction between a loop (or singular disk) and its image in E^3. A singular disk g is bounded by $g|$Bd Δ. A simple closed curve is unknotted if it bounds a disk (in E^3).

We use ρ for the Euclidean distance function and Diam for diameter. We use $N(X, \varepsilon)$ to denote the ε-neighborhood of X ($\varepsilon > 0$). If Diam $X < \varepsilon$, then X is called an ε-set. Let $X \subset E^3$ and let $\varepsilon: X \to [0, \infty)$ be a map. If $g: X \to E^3$ is a map such that $x = g(x)$ if $\varepsilon(x) = 0$ and $\rho(x, g(x)) < \varepsilon(x)$ if $\varepsilon(x) > 0$, then g is called an $\varepsilon(x)$-map or ε-map.

Let A and B denote sets in E^3 and x a point in E^3. Then A is 1-LC in B at x if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each loop in $N(x, \delta) \cap A$ bounds a singular disk in $N(x, \varepsilon) \cap B$. If A is 1-LC in B at each point x of cl A and if for each $\varepsilon > 0$ the corresponding δ can be chosen to be independent of x, then A is said to be 1-ULC in B. If A is 1-ULC in itself, then we say simply that A is 1-ULC. We define A to be weakly 1-ULC in B at x if small unknotted curves in A near x bound small singular disks in B. Weakly 1-ULC is defined in a natural manner.

The following easy and, for the most part, well-known lemmas are very useful.

LEMMA 2.1. *If C is a crumpled cube in E^3, X is a closed subset of C, and* Int C^* *is 1-LC in $E^3 - X$ (alternatively: weakly 1-LC in $E^3 - X$; 1-LC in $C^* - X$) at each point of $X \cap C^*$, then* Int C^* *is 1-ULC in $E^3 - X$ (respectively: weakly 1-ULC in $E^3 - X$; 1-ULC in $C^* - X$).*

Proof. One uses the Lebesgue number of an open covering of $X \cap C^*$.

LEMMA 2.2. *If C is a crumpled cube in E^3, X is a subset of C (not necessarily closed), and* Int C^* *is 1-ULC in $C^* - X$, then $C^* - X$ is 1-ULC.*

Proof. Since Int C^* is also 0-ULC [24, p. 66], Lemma 2.2 is a consequence of [15, Theorem 4]. The reader is advised however to devise his own proof of this simple fact.

LEMMA 2.3. *Let S be a 2-sphere in E^3, and let $\{X_i\}_{i=1}^{\infty}$ be a family of closed subsets of S such that, for each i, ($S \cup$* Int *$S) - X_i$ is 1-ULC. Then ($S \cup$* Int *$S) - \bigcup_{i=1}^{\infty} X_i$ is 1-ULC.*

Proof. This is Lemma 2.3 of [10].

LEMMA 2.4. *If C is a crumpled cube in E^3 and X is a closed subset of C, then there are an open set U in $E^3 - X$ which contains $C^* - X$ and a retraction*

$$r: U \cup X \to C^* \cup X$$

such that $r(U) = C^* - X$.

Proof. This is an immediate consequence of the fact that Bd C is an absolute neighborhood retract.

LEMMA 2.5. *Let C be a crumpled cube in E^3, U an open subset of E^3, and $\varepsilon(x)$: Bd $C \to [0, \infty)$ a continuous function with $\varepsilon(x) \leqq \eta \cdot \rho(x, E^3 - U)$ for some constant η $(0 \leqq \eta < 1)$. If h: Bd $C \to E^3$ is an $\varepsilon(x)$-homeomorphism, then h(Bd C) separates two points of $E^3 - U$ if and only if* Bd C *does. In particular, if $p \in$ (Int C) $- U$ and V is the component of $E^3 - h$(Bd C) which contains p, then $V \subset U \cup$ Int C.*

Proof. Note that Bd C and h(Bd C) are homotopic under a homotopy whose image lies in (Bd C) $\cup$ U. The result thus follows from [18, p. 97].

3. **Basic properties of *-taming sets.** Our first theorem is a preliminary characterization of *-taming sets and corresponds to Loveland's preliminary characterization of taming sets [19, Theorem 16].

THEOREM 3.1. *A closed subset X of E^3 is a *-taming set if and only if, for each crumpled cube C in E^3 which contains X, $C^* - X$ is* 1-*ULC.*

Proof. Suppose that C is a crumpled cube in E^3 which contains a given closed set X and suppose that Bd C is tame modulo X. If $C^* - X$ is 1-ULC, we claim that Int C^* is also 1-ULC, hence that C^* is a 3-cell [2]. Indeed, (Bd C)$-X$ may be expressed as a union $\bigcup_{i=1}^{\infty} F_i$, where each F_i is closed in Bd C. Since Bd C is tame modulo X, $C^* - F_i$ is clearly 1-ULC for each i. Thus $C^* - (X \cup \bigcup_{i=1}^{\infty} F_i) =$ Int C^* is 1-ULC by Lemma 2.3. It follows that if $C^* - X$ is 1-ULC for each such crumpled cube C, then X is a *-taming set by definition. Note that in this half of the theorem we do not need the full strength of the hypothesis.

Suppose conversely that X is a *-taming set and that C is a crumpled cube in E^3 which contains X. Let x be arbitrarily chosen from $X \cap C^*$. Since $X \cap C^*$ is compact, in order to show that $C^* - X$ is 1-ULC it suffices to show that Int C^* is 1-LC in $C^* - X$ at x (Lemmas 2.1 and 2.2). The strategy in capsule form is to approximate C by a crumpled cube whose boundary is a 2-sphere that is tame modulo X, apply the definition of *-taming set, and then use a retraction (Lemmas 2.4 and 2.5) to obtain information about C.

Step 1. Suppose $\varepsilon > 0$ given. We seek a $\delta > 0$ such that any loop in (Int C^*) $\cap$ $N(x, \delta)$ bounds a singular disk in $(C^* - X) \cap N(x, \varepsilon)$. By Lemma 2.4, there is a retraction r: $X \cup U \to X \cup C^*$, where U is an open subset of $E^3 - X$ which contains $C^* - X$, such that $r(U) = C^* - X$. We choose a $\delta_0 > 0$ such that r: $U \cap N(x, \delta_0) \subset (C^* - X) \cap N(x, \varepsilon)$ and a $\delta > 0$ such that

(1) $0 < \delta < \delta_0/3$ and

(2) any δ-subset of a δ-approximation S to Bd C lies in a $\delta_0/3$-disk on S.

We show in two further steps that a loop J: Bd $\Delta \to$ (Int C^*) $\cap$ $N(x, \delta)$ bounds a singular disk in $(C^* - X) \cap N(x, \varepsilon)$.

Step 2. We now carefully choose a crumpled cube K in E^3 which contains X and whose boundary is tame modulo X. Let

$$f(y) = \min\{\delta, (1/2)\cdot\rho[y, (E^3-U)\cup J(\mathrm{Bd}\,\Delta)]\} \qquad (y \in \mathrm{Bd}\, C).$$

Let h: Bd $C \to E^3$ be an $f(y)$-homeomorphism of Bd C such that h(Bd C) is tame modulo X [1, Theorem 7]. Let K be the crumpled cube in E^3 bounded by h(Bd C) and not containing J(Bd Δ). Then $X \subset K$ and $J(\mathrm{Bd}\,\Delta) \subset \mathrm{Int}\, K^* \subset [U - J(\mathrm{Bd}\,\Delta)] \cup \mathrm{Int}\, C^* = U$ by Lemma 2.5. The crumpled cube K^* is therefore a 3-cell because X is a *-taming set in K.

Step 3. We use the crumpled cube K^* and the retraction r to contract J to a point. Since J is a δ-loop in Int K^*, since δ-sets in Bd K^* lie in $\delta_0/3$-disks in Bd K^* by (2), and since K^* is a 3-cell, it is easy to see that J bounds a singular $2\delta_0/3$-disk in Int K^*. One simply cuts off a singular disk bounded by J near Bd K^*. If $D_0\colon \Delta \to E^3$ is such a singular disk, then

$$D_0(\Delta) \subset (\mathrm{Int}\, K^*) \cap N(x, \delta_0) \subset U \cap N(x, \delta_0)$$

and

$$D = r \circ D_0\colon \Delta \to r[U \cap N(x, \delta_0)] \subset (C^* - X) \cap N(x, \varepsilon).$$

Since $J = D_0|\mathrm{Bd}\,\Delta = r \circ D_0|\mathrm{Bd}\,\Delta = D|\mathrm{Bd}\,\Delta$, we conclude that J bounds a singular disk in $(C^* - X) \cap N(x, \varepsilon)$. This completes the proof that $C^* - X$ is 1-ULC.

Proposition 3.2_1. *Let C be a crumpled cube in E^3 and ε a positive number. Then there is a $\delta > 0$ such that if X is any *-taming set in C and J is any δ-loop in $C^* - X$, then J bounds a singular ε-disk in $C^* - X$.*

(*Note.* If $X = \varnothing$, then Proposition 3.2_1 reduces to the well-known result that C^* is 1-ULC.)

Proof. Choose $\delta > 0$ such that δ-subsets of Bd C lie in $\varepsilon/2$-disks in Bd C. Let X be a *-taming set in C and J: Bd $\Delta \to C^* - X$ a δ-loop. Then J bounds a singular δ-disk $D_0\colon \Delta \to E^3$. The set $D_0(\Delta) \cap \mathrm{Bd}\, C$ lies in an $\varepsilon/2$-disk E in Bd C. By the Tietze extension theorem, D_0 can be cut off on E so as to form a singular disk $D_1\colon \Delta \to (D_0(\Delta) \cup E) \cap C^*$. Then D_1 is clearly a singular ε-disk. Since $C^* - X$ is 1-ULC by Theorem 3.1, we may apply [10, Lemma 2.2] to conclude that D_1 may be adjusted slightly, with $J = \mathrm{Bd}\, D_1$ fixed, so as to form a singular ε-disk D bounded by J in $C^* - X$.

Essentially the same proof can also be used to prove the following proposition.

Proposition 3.2_2. *Let C and ε be as in the statement of Proposition* 3.2_1. *Then there is a $\delta > 0$ having the following properties*:

(1) *If X is any *-taming set in C, possibly with $X = \varnothing$, and if $\alpha > 0$, then any unknotted δ-loop in* Int C^* *bounds an ε-disk in* $(E^3 - X) \cap N(C^*, \alpha)$.

(2) *If C^* is a* 3-*cell, then δ-loops in* Int C^* *bound singular ε-disks in* Int C^* *and unknotted δ-loops in* Int C^* *bound ε-disks in* Int C^*.

The following technical lemma is an easy adaptation of [4, Theorem 3] and is used in the proof of Theorem 3.4. We omit the proof.

LEMMA 3.3. *Suppose C is a crumpled cube in E^3, X is a closed subset of C which has no component of diameter less than $1/i>0$, R is a disk in* Bd C, *and D is a disk such that*

(1) *R is locally polyhedral at each point of $R-X$,*

(2) $D\subset E^3-X$,

(3) Bd $D\subset \operatorname{Int} C^*$,

(4) $D \cap \operatorname{Bd} C\subset \operatorname{Int} R$, *and*

(5) $3\cdot \operatorname{Diam}(D\cup R)<1/i$.

Then for each positive number ε there is a disk D' such that

(6) Bd $D'=$ Bd D,

(7) $D'\subset \operatorname{Int} C^*$, *and*

(8) *D' is in an ε-neighborhood of $D\cup R$.*

THEOREM 3.4. *Suppose that C is a crumpled cube in E^3, that X is a closed subset of C which has no degenerate components, and that* Int C^* *is weakly* 1-*ULC in E^3-X. Then C^*-X is* 1-*ULC.*

Proof. Let X_i $(i=1, 2, \ldots)$ denote the closed subset of X whose components are the components of X of diameter equal to or greater than $1/i$. Note that Int C^* is weakly 1-ULC in E^3-X_i for each i. If C^*-X_i were 1-ULC for each i, then C^*-X would be 1-ULC by Lemma 2.3 since $X=\bigcup_{i=1}^{\infty} X_i$. Hence it suffices to prove our theorem under the special assumption that $X=X_i$, where i is some positive integer which we assume fixed for the remainder of this proof.

By [10, Theorem 2.6], it suffices to prove that Int C^* is weakly 1-ULC in C^*-X. To this end, assume $\varepsilon>0$ given. We suppose that $\varepsilon<1/i$. Choose δ, $0<\delta<\varepsilon/9$, such that each δ-subset of Bd C lies in an $\varepsilon/9$-disk in Bd C. We claim that each unknotted δ-simple closed curve in Int C^* bounds a singular ε-disk in C^*-X. A proof of this assertion will complete the proof of this theorem.

Let J be an unknotted δ-simple closed curve in Int C^*. Then J bounds a δ-disk D in E^3 such that D is locally polyhedral at each point of $D-J$ [1, Theorem 7]. If $D\cap \operatorname{Bd} C=\varnothing$, we are done. Otherwise, there is an $\varepsilon/9$-disk R in Bd C such that $D\cap\operatorname{Bd} C\subset\operatorname{Int} R$ and $\operatorname{Diam}(R\cup D)\leqq \operatorname{Diam} R+\operatorname{Diam} D<\varepsilon/9+\delta<\varepsilon/3$. It follows from [10, Lemma 2.5] that there is a map $h\colon D\to \operatorname{Int} C^*\cup \operatorname{Int} R$ such that $h(D)$ is a singular $\varepsilon/3$-disk, h is the identity in a neighborhood of J, $h(D)\cap\operatorname{Bd} C$ is 0-dimensional, and $h|D-h^{-1}(\operatorname{Bd} C)$ is a homeomorphism.

There is an α, $0<\alpha<\delta$, such that $N(h(D),\alpha)\cap \operatorname{Bd} C\subset \operatorname{Int} R$, $\rho(R,J)>\alpha$, and $\operatorname{Diam} N(h(D)\cup R,\alpha)<\varepsilon/3$. Using the hypothesis that Int C^* is weakly 1-ULC in E^3-X, we find a β, $0<\beta<\alpha/2$, such that each unknotted β-simple closed curve in Int C^* bounds a singular $\alpha/2$-disk in E^3-X.

Because $h(D)\cap\operatorname{Bd} C$ is 0-dimensional, there is a finite collection $J_1,\ldots,J_k$ of disjoint simple closed curves in D such that $\bigcup_{j=1}^{k} J_j$ separates J from $h^{-1}(\operatorname{Bd} C)$ in D and such that each $h(J_j)$ is a β-loop in a β-neighborhood of the set $h(D)\cap \operatorname{Bd} C$. We assume that no proper subcollection of $J_1,\ldots,J_k$ separates J from $h^{-1}(\operatorname{Bd} C)$

in D. Because $h|D-h^{-1}(\text{Bd } C)$ is a homeomorphism and $J_j \subset D-h^{-1}(\text{Bd } C)$, it follows from Dehn's Lemma [23] that each $h(J_j)$ is an unknotted β-simple closed curve in Int C^*. Hence by the way in which β was chosen, there is a map $h_1: D \to E^3-X$ such that h_1 agrees with h on that component of $D-\bigcup_{j=1}^{k} J_j$ which contains J and such that h_1 takes the disk in D bounded by J_j into an $\alpha/2$-subset of E^3-X. We note that $h_1(D) \cap \text{Bd } C \subset \text{Int } R$, that $h_1(D)$ has no singularities near J, and that Diam $(R \cup h_1(D)) < \varepsilon/3$. Hence by Dehn's Lemma, we may replace $h_1(D)$ by a nonsingular disk E bounded by J such that $E \cap \text{Bd } C \subset \text{Int } R$, Diam $(R \cup E) < \varepsilon/3$, and $E \subset E^3-X$. We may further require that E be locally polyhedral at each point of $E-J$.

By Lemma 2.4, there are an open set U in E^3-X containing C^*-X and an $\varepsilon/3$-retraction $r: U \to C^*-X$. Define $f: \text{Bd } C \to [0, \varepsilon/3]$ by the formula $f(x) = \min \{\varepsilon/3, (1/2)\cdot\rho(x, (E^3-U) \cup J)\}$. It follows from [1, Theorem 7] that there is an $f(x)$-homeomorphism $g: \text{Bd } C \to E^3$ such that $g(\text{Bd } C)$ is locally polyhedral at each point of $g(\text{Bd } C-X)$. We may clearly require that g be so near the identity that $E \cap g(\text{Bd } C) \subset g(\text{Int } R)$ and that Diam $(E \cup g(R)) < \varepsilon/3$.

We now apply Lemma 3.3 in order to conclude that J bounds an $\varepsilon/3$-disk F in the complement of $g(\text{Bd } C)$. Here we use our disk E in place of the disk D of Lemma 3.3, $g(R)$ in place of R, and $g(\text{Bd } C)$ in place of Bd C. By Lemma 2.5 and our choice of g, the disk F lies in U.

Finally rF is a singular disk in C^*-X bounded by J since $r|J=\text{identity}$ and $F \subset U = r^{-1}(C^*-X)$. The singular disk rF has diameter less than Diam $F+2\varepsilon/3 < \varepsilon$ since F is an $\varepsilon/3$-disk and r is an $\varepsilon/3$-retraction. This completes the proof that J bounds a singular ε-disk in C^*-X and thereby completes the proof of Theorem 3.4.

COROLLARY 3.5. *Suppose that C is a crumpled cube in E^3, that X is a closed subset of C which has no degenerate components, and that* Int C^* *is weakly* 1*-ULC in E^3-X. Then if* Bd C *is tame modulo $X \cap$* Bd C, *C^* is a* 3*-cell.*

The following technical lemma is an easy adaptation of [9, Lemma 3.3] and is used in the proof of Theorem 3.7, condition (2). Since our proof of Theorem 3.7, condition (2) will refer to the proof of [9, Theorem 1.2], we omit the definitions which are found in [9] and are necessary for a complete understanding of Lemma 3.6.

LEMMA 3.6. *Let M denote a subcontinuum of a circular disk D in $E^2=E^2\times\{0\} \subset E^3$ such that M contains* Bd D *and $M-$*Bd $D \subset C$, *where C is a crumpled cube in E^3. Let H denote a polyhedral handlebody (cube-with-handles) formed by thickening a disk-with-holes D_0 in E^2 which has the following properties*:

(1) $M \subset \text{Int } D_0$.

(2) *No two components of E^2-D_0 lie in the same component of E^2-M.*

Then given $\varepsilon>0$, there is a set generator G for $\pi_1(H)$ such that $G-N(\text{Bd } D, \varepsilon)$ is a subset of Int C.

THEOREM 3.7. *A closed subset X of E^3 is a *-taming set if either of the following two conditions is satisfied*:

(1) *X is a countable union of *-taming sets.*

(2) *X has no degenerate components and lies on a tame 2-sphere in E^3 (i.e., X is a taming set* [9]).

Proof. Suppose that $X=\bigcup_{i=1}^{\infty} X_i$, where each X_i is a *-taming set. Let C be any crumpled cube in E^3 which contains X. By Theorem 3.1, C^*-X_i is 1-ULC for each i. Hence by Lemma 2.3, $C^*-X=C^*-\bigcup_{i=1}^{\infty} X_i$ is 1-ULC. We conclude from Theorem 3.1 that X is a *-taming set. This proves that condition (1) is sufficient.

Suppose now that X has no degenerate components and lies on a tame 2-sphere in E^3. The fact that X is a *-taming set if it is a countable union of *-taming sets allows us to assume first that X is not a 2-sphere, hence without loss of generality that it lies in the plane $E^2=E^2\times\{0\}\subset E^3$, and second that X has no component of diameter less than $1/i$ for some positive integer i (since $X=\bigcup_{i=1}^{\infty} X_i$, where X_i is the union of the components of X having diameter at least $1/i$). That is, the general case of condition (2) will follow from the special case $X=X_i\subset E^2=E^2\times\{0\}\subset E^3$ by condition (1).

Let C be a crumpled cube in E^3 which contains X and let $x\in C^*\cap X$. By Theorems 3.1 and 3.4 and Lemma 2.1, in order to show that X is a *-taming set it suffices to show that Int C^* is weakly 1-LC in E^3-X at x. That this actually is the case follows from Proposition $3.2_2(1)$ (applied to the empty *-taming set), Lemma 3.6, and [9, Lemma 3.2] as the corresponding fact in the proof of [9, Theorem 1.2] followed from [9, Lemmas 2.4, 3.3, and 3.2]. This completes the proof.

COROLLARY 3.8. *If X is a compact subset of a 2-sphere S in E^3, then X is a *-taming set if and only if X is tame and has no degenerate components (i.e., if and only if X is a taming set* [9]).

Proof. If X is tame and has no degenerate components, then X is a *-taming set by Theorem 3.7. Conversely, if X is a *-taming set and S' is any 2-sphere which contains X and is locally tame modulo X, then, by the definition of *-taming set, $S'\cup$ Int S' is a 3-cell (since $X\subset S'\cup$ Ext S' and S' is tame modulo X). Similarly, $S'\cup$ Ext S' is a 3-cell. Thus S' is tame, and X is a taming set by definition. Since taming sets are tame and have no degenerate components [9], our proof is complete.

THEOREM 3.9. *If C is a crumpled cube in E^3, X is a closed subset of C, C^*-X is 1-ULC, and $\varepsilon>0$, then there is an embedding $h\colon C\to E^3$ such that h moves no point as far as ε, $h|X=$ identity, and $h(C)^*$ is a 3-cell.*

Proof. The theorem follows by a standard argument from the following lemma (see [13, Lemma and the remarks following the proof of that lemma]).

LEMMA. *Given $\varepsilon>0$, there exist an ε-homeomorphism h' from C into E^3, fixed on a neighborhood of X in C, and a polyhedral 2-sphere S', homeomorphically within ε of* Bd $C=S$, *such that $h'(C)\cap S'=\varnothing$.*

This lemma contains two obvious changes from [13, Lemma]. The first is the inclusion of X in the statement; the requirement that h' be fixed on a neighborhood of X in C ensures that $h'(C)^*-X$ will be 1-ULC, hence that the lemma can be applied iteratively. The second change is that of requiring only that $h'(C) \cap S' = \varnothing$ rather than $h'(C) \subset \operatorname{Int} S'$; this simply reflects the fact that, in our case, C may be a noncompact crumpled cube. The proof of [13, Lemma], with minor changes in the first part of the proof only, suffices to establish our lemma as well. We refer the reader to the proof of [13, Lemma] and simply indicate the changes that must be made in that proof [13, pp. 52–53]:

Since C^*-X is 1-ULC, property $(*, X, \operatorname{Int} C^*)$ is satisfied (see [10, Theorem 2.4]). All this means is that in Daverman's proof [13] the disks $D_1, D_2, \ldots, D_m$ in S may be chosen to lie in $S-X$. Choose a connected neighborhood N of X in C such that $\operatorname{cl} N \subset C - \bigcup_{i=1}^m D_i$. Alter the homeomorphism g on the disks $D_1, D_2, \ldots, D_m$ by (1) using the Tietze's extension theorem (see [5, Lemma 1]) to map $g(D_i)$ into $[g(D_i) \cap C^*] \cup \operatorname{Int} D_i$ and (2) using Dehn's Lemma [23] to change the singular disk thus obtained into a new polyhedral disk $g(D_i)$ very near the old $[g(D_i) \cap C^*] \cup \operatorname{Int} D_i$ in $E^3 - \operatorname{cl} N$. Use the new g rather than the old to complete Daverman's proof. The small 3-cell B_i of that proof, which is formed by thickening, in our case, the new $g(D_i)$, can therefore be chosen very near the old $[g(D_i) \cap C^*] \cup \operatorname{Int} D_i$ in $E^3 - \operatorname{cl} N$. If we use the terminology $\operatorname{Int} g(S)$ for the component of $E^3-g(S)$ which contains N, then Daverman's homeomorphism h' will move C into $\operatorname{Int} g(S)$ and fix N.

THEOREM 3.10. *If C and K are homeomorphic crumpled cubes in E^3, X is a *-taming set contained in C, and $h: C \to K$ a homeomorphism, then the following are equivalent*:

(1) *$K^*-h(X)$ is 1-ULC.*

(2) *X and $h(X)$ are equivalently embedded in E^3.*

(3) *$h(X)$ is a *-taming set.*

Proof. (1) implies (2). Note that C^*-X is 1-ULC by Theorem 3.1. Hence by Theorem 3.8, there are embeddings $f: C \to E^3$ and $g: K \to E^3$ such that $f(C)^*$ and $g(K)^*$ are homeomorphic 3-cells, $f|X = \text{identity}$, and $g|h(X) = \text{identity}$. Since $f(C)^*$ and $g(K)^*$ are homeomorphic 3-cells and $ghf^{-1}: f(C) \to g(K)$ is a homeomorphism, ghf^{-1} can be extended to a homeomorphism $H: E^3 \to E^3$. But H takes X to $h(X)$, and we conclude that X and $h(X)$ are equivalently embedded in E^3.

(2) implies (3). Clear.

(3) implies (1). Theorem 3.1.

4. **Miscellaneous examples of *-taming sets.** The following theorem is [6, Theorem 2].

THEOREM 4.1. *A crumpled cube X in E^3 is a *-taming set if X^* is a 3-cell.*

Proof. Let C denote a crumpled cube in E^3 which contains X. If X^* is a 3-cell, then $\operatorname{Int} X^* = E^3 - X$ is 1-ULC. Hence $\operatorname{Int} C^*$ is clearly 1-ULC in $E^3 - X$. By Theorems 3.1 and 3.4, X is a *-taming set.

An arc A in E^3 is said to be locally peripherally unknotted (l.p.u.) if for each $p \in A$ and each $\varepsilon > 0$ there is a 2-sphere S of diameter less than ε in E^3 such that $p \in \operatorname{Int} S$ and $S \cap A$ consists of one point if p is an endpoint of A and two points otherwise.

THEOREM 4.2. *A locally peripherally unknotted arc is a *-taming set.*

Proof. Let A denote an l.p.u. arc in E^3. Let C be a crumpled cube in E^3 which contains A and whose boundary is tame modulo A. As noted in the first paragraph of the proof of Theorem 3.1, in order to show that A is a *-taming set it suffices to show that for this special kind of crumpled cube C that $C^* - A$ is 1-ULC. By Lemma 2.1 and Theorem 3.4, it suffices in turn to show that $\operatorname{Int} C^*$ is 1-LC in $E^3 - A$ at each point x in $A \cap C^*$.

We assume therefore that a point $x \in A \cap C^*$ and a number $\varepsilon > 0$ are given. We assume $x \in \operatorname{Int} A$ and leave the other (slightly easier) case where $x \in \operatorname{Bd} A$ to the reader. Let J be a simple closed curve in C which contains A. Let B be an arc in $N(x, \varepsilon)$ such that $x \in \operatorname{Int} B \subset B \subset \operatorname{Int} A$. Let δ_0, $0 < \delta_0 < \varepsilon$, be such that $N(x, \delta_0) \cap J \subset \operatorname{Int} B$. Let D be a disk such that $x \in \operatorname{Int} D \subset D \subset \operatorname{Bd} C \cap N(x, \delta_0)$, and let δ, $0 < \delta < \delta_0$, be such that $N(x, \delta) \cap \operatorname{Bd} C \subset \operatorname{Int} D$. We show in the next paragraph that a loop $f\colon \operatorname{Bd} \Delta \to \operatorname{Int} C^* \cap N(x, \delta)$ bounds a singular disk in $N(x, \delta) - A$. This will complete the proof.

There is an extension of f which takes Δ into $[\operatorname{Int} C^* \cup D] \cap N(x, \delta_0)$. We call the extension f also. Since $\operatorname{Bd} C$ is tame modulo A and $N(x, \delta_0) \cap J \subset \operatorname{Int} B$, we may require that $f(\Delta) \cap \operatorname{Bd} C \subset \operatorname{Int} B$. By the proof to [16, Theorem 1] (since A is an l.p.u. arc), there is a 2-sphere S in $N(x, \varepsilon)$ with $(S \cup \operatorname{Int} S) \cap J$ being an arc B' such that B' contains B, $\operatorname{Int} B'$ lies in $\operatorname{Int} S$, and S is polyhedral modulo $\operatorname{Bd} B'$. Note that since $f(\Delta) \cap J = f(\Delta) \cap \operatorname{Int} B = f(\Delta) \cap \operatorname{Bd} C \subset \operatorname{Int} S$ and $f(\operatorname{Bd} \Delta) \subset \operatorname{Ext} S$, we may assume f to be adjusted so as to be in general position with respect to S. If J' is a component of $f^{-1}(f(\Delta) \cap S)$, then J' is a simple closed curve and $f(J')$ is a loop in $(S - \operatorname{Bd} B) \cap \operatorname{Int} C^*$. Since $J \subset C$ and $f(J') \subset C^*$, J and $f(J')$ cannot link [8, Theorem 4.7.1]. Thus $f(J')$ must bound a singular disk in $S - A$. Let $J_1, J_2, \ldots, J_m$ be the components of $f^{-1}(f(\Delta) \cap S)$ which are also boundary components of that component K of $\Delta - f^{-1}(f(\Delta) \cap S)$ which contains $\operatorname{Bd} \Delta$. Let Δ_i be the subdisk of Δ bounded by J_i. Define $g\colon \Delta \to f(K) \cup (S - \operatorname{Bd} B') \subset N(x, \varepsilon) - A$ by defining $g|K = f|K$, and by letting $g|\Delta_i$ be any extension of $f|J_i$ which takes Δ_i into $S - \operatorname{Bd} B'$. This completes the proof.

COROLLARY 4.3 (SEE [17, THEOREM VI]). *An arc A is tame if it lies on a 2-sphere in E^3 and is l.p.u.*

Proof. This is a consequence of Theorem 4.2 and Corollary 3.8.

THEOREM 4.4. *A compact set X in E^3 which has no degenerate components is a *-taming set if there exists a sequence $\{M_i\}$ of 3-manifolds with boundary such that*

(1) *K is uniformly described by $\{M_i\}$,*

(2) *each component of each M_i is a polyhedral cube, and*

(3) *$\{M_i\}$ is sequentially 1-ULC.*

Proof. This can be proved by the same methods used to prove Theorem 1 of [7]. (See [7] also for definitions.)

THEOREM 4.5. *A closed set X in E^3 is a *-taming set if it is a union of convex 3-cells.*

Proof. Let C be a crumpled cube in E^3 which contains X. Let $\{d_k\}_{k=1}^{\infty}$ be a dense set of directions in E^3. For each triple (i, j, k) of positive integers, let $A(i, j, k)$ denote the family of cones in X with vertex on Bd X, vertex angle $1/i$, height i/j, and direction d_k (from the base of the cone to the vertex). If A is such a cone, then we let A' denote the subinterval of the axis of A which lies in A, has length $1/(2j)$, and has the vertex of A as one endpoint. We let $B(i, j, k)=\bigcup\{A' \mid A \in A(i, j, k)\}$. We observe that $B(i, j, k)$ is a compact subset of C which has no component of diameter less than $1/(2j)$.

We claim that $C^*-B(i, j, k)$ is 1-ULC. It suffices to show, by Theorem 3.4, that Int C^* is 1-ULC in $E^3-B(i, j, k)$. Let $\varepsilon>0$ be given and choose $\delta>0$ such that δ-loops in C^* bound singular ε-disks in C^* (Proposition 3.2). Let J: Bd $\Delta \to$ Int C^* be a δ-loop and $D: \Delta \to C^*$ be a singular ε-disk bounded by J. We choose $\delta_1>0$ such that $N(J(\text{Bd }\Delta), \delta_1)\subset \text{Int } C^*$ and such that Diam $N(D(\Delta), \delta_1)<\varepsilon$. Let $T: E^3 \to E^3$ be a translation in the direction d_k which moves points a positive distance less than $\min\{\delta_1, 1/(2j)\}$. We observe that $T(D(\Delta)) \cap B(i, j, k)=\varnothing$, that J is homotopic in $N(J(\text{Bd }\Delta), \delta_1)\subset \text{Int } C^*$ to $T\circ J$ and that $N(J(\text{Bd }\Delta), \delta_1) \cup T\circ D(\Delta)\subset N(D(\Delta), \delta_1)-B(i, j, k)$. We conclude that J is nullhomotopic in an ε-subset of $E^3-B(i, j, k)$, hence that Int C^* is 1-ULC in $E^3-B(i, j, k)$ and finally that $C^*-B(i, j, k)$ is 1-ULC.

We observe that $C^*-X=C^*-\bigcup_{i,j,k} B(i, j, k)$ because X is a union of convex 3-cells. Hence C^*-X is 1-ULC by the previous paragraph and Lemma 2.3. By Theorem 3.1, X is a *-taming set.

Our final corollary generalizes a theorem of Bothe [3]. The history of this corollary is discussed in more detail in [11].

COROLLARY 4.6. *A crumpled cube C in E^3 is a 3-cell if* Bd C *can be touched from C^* at each point of* Bd C *by a pencil, i.e., if for each point p of* Bd C *there is a solid right circular cone with vertex at p which lies in C^*.*

Proof. The hypothesis implies that C^* is a union of convex 3-cells, hence a *-taming set by Theorem 4.5. Thus C is a 3-cell by definition of *-taming set.

REFERENCES

1. R. H. Bing, *Approximating surfaces with polyhedral ones*, Ann. of Math. (2) **65** (1957), 456–483. MR **19**, 300.

2. ———, *A surface is tame if its complement is* 1-*ULC*, Trans. Amer. Math. Soc. **101** (1961), 294–305. MR **24** #A1117.

3. H. G. Bothe, *Differenzierbare Flächen sind zahm*, Math. Nachr. **43** (1970), 161–180.

4. C. E. Burgess, *Properties of certain types of wild surfaces in* E^3, Amer. J. Math. **86** (1964), 325–338. MR **29** #598.

5. ———, *Characterizations of tame surfaces in* E^3, Trans. Amer. Math. Soc. **114** (1965), 80–97. MR **31** #728.

6. ———, *Pairs of* 3-*cells with intersecting boundaries in* E^3, Amer. J. Math. **88** (1966), 309–313. MR **33** #4909.

7. C. E. Burgess and J. W. Cannon, *Tame subsets of spheres in* E^3, Proc. Amer. Math. Soc. **22** (1969), 395–401. MR **39** #3469.

8. ———, *Embeddings of surfaces in* E^3, Rocky Mt. J. Math. **1** (1971), 259–344.

9. J. W. Cannon, *Characterization of taming sets on* 2-*spheres*, Trans. Amer. Math. Soc. **147** (1970), 289–299. MR **41** #2644.

10. ———, *Sets which can be missed by side approximations to* 2-*spheres*, Pacific J. Math. **34** (1970), 321–334.

11. ———, **-taming sets for crumpled cubes.* II: *Horizontal sections in closed sets*, Trans. Amer. Math. Soc. **161** (1971), 441–446.

12. ———, **-taming sets for crumpled cubes.* III: *Horizontal sections in* 2-*spheres*, Trans. Amer. Math. Soc. **161** (1971), 447–456.

13. R. J. Daverman, *A new proof for the Hosay-Lininger theorem about crumpled cubes*, Proc. Amer. Math. Soc. **23** (1969), 52–54. MR **39** #7578.

14. P. H. Doyle and J. G. Hocking, *Some results on tame disks and spheres in* E^3, Proc. Amer. Math. Soc. **11** (1960), 832–836. MR **23** #A4133.

15. S. Eilenberg and R. L. Wilder, *Uniform local connectedness and contractibility*, Amer. J. Math. **64** (1942), 613–622. MR **4**, 87.

16. O. G. Harrold, Jr., *The enclosing of simple arcs and curves by polyhedra*, Duke Math. J. **21** (1954), 615–621. MR **16**, 846.

17. O. G. Harrold, Jr., H. C. Griffith and E. E. Posey, *A characterization of tame curves in three-space*, Trans. Amer. Math. Soc. **79** (1955), 12–34. MR **19**, 972.

18. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Math. Series, vol. 4, Princeton Univ. Press, Princeton, N. J., 1941. MR **3**, 312.

19. L. D. Loveland, *Tame subsets of spheres in* E^3, Pacific J. Math. **19** (1966), 489–517. MR **37** #903.

20. ———, *A* 2-*sphere of vertical order five bounds a* 3-*cell*, Notices Amer. Math. Soc. **17** (1970), 472. Abstract #70T-G54.

21. ———, *The boundary of a linearly connected crumpled cube is tame*, Notices Amer. Math. Soc. **17** (1970), 581. Abstract #70T-G69.

22. D. R. McMillan, Jr., *Some topological properties of piercing points*, Pacific J. Math. **22** (1967), 313–322. MR **35** #7319.

23. C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. (2) **66** (1957), 1–26. MR **19**, 761.

24. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloq. Publ., vol. 32, Amer. Math. Soc., Providence, R. I., 1963. MR **32** #440.

UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706