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*-TAMING SETS FOR CRUMPLED CUBES. II: HORIZONTAL SECTIONS IN CLOSED SETS(1)

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Abstract. We prove that a closed subset X of E^3 is a *-taming set if no horizontal section of X has a degenerate component. This implies, for example, that a 2-sphere S in E^3 is tame if no horizontal section of S has a degenerate component. It also implies (less obviously) that a 2-sphere S in E^3 is tame if it can be touched at each point from each side of S by a pencil.

We assume familiarity with the definitions and results from the first paper [5] of this series in which the basic properties of *-taming sets are developed. We recall for the reader's convenience, however, our definitions of crumpled cubes, 3-cells, taming sets, and *-taming sets. A crumpled cube is the union of a 2-sphere and one of its complementary domains in E^3. If C is a crumpled cube in E^3, then the closure $\mathrm{cl}\,(E^3-C)$ of E^3-C is also a crumpled cube, which we denote by C^*. We write $\mathrm{Bd}\, C$ for the 2-sphere $C \cap C^*$ and $\mathrm{Int}\, C$ for the set E^3-C^*. A 3-cell is a crumpled cube homeomorphic with either the standard unit ball B in E^3 or the crumpled cube B^*. Suppose that X is a compact subset of some 2-sphere in E^3. Then one may consider in particular those 2-spheres S in E^3 which contain X and are locally tame at each point of $S-X$. If each such 2-sphere is tame, then X is said to be a taming set. According to [4], a compact set X in E^3 is a taming set if and only if it lies on some tame 2-sphere in E^3 and has no degenerate components. In short, the notion of taming set allows one to answer the natural question, "Is a 2-sphere in E^3 tame if it is tame modulo a tame subset X?" The notion of *-taming set results when one tries to free X from the requirement that it lie in a surface in E^3; i.e., one has the more general question, "Is a 2-sphere S in E^3 tame if it is tame modulo some given set X which is not necessarily a subset of S?" The most interesting theory results when one requires that X not intersect both complementary domains of S in E^3. A *-taming set X in E^3 is a closed (not necessarily compact) subset of E^3 having the following property: if C is a crumpled cube in E^3, $X \subset C$, and $\mathrm{Bd}\, C$ is locally tame at each point of $\mathrm{Bd}\, C-(X \cap \mathrm{Bd}\, C)$, then C^* is a 3-cell.

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If r and s are real numbers with $r<s$, then we set $P(r)=\{(x, y, z) \in E^3 : z=r\}$ and $P[r, s]=\{(x, y, z) \in E^3 : r \leqq z \leqq s\}$. If X is a closed subset of E^3 and t a positive number, then we set $X(r)=X \cap P(r)$, $X[r, s]=X \cap P[r, s]$, and $X(r)^t=\bigcup\{K : K$ a component of $X(r)$, Diam $K \geqq t\}$. Note that $X(r)$, $X[r, s]$, and $X(r)^t$ are closed sets. We let $p(r)\colon E^3 \to P(r)$ denote the projection map defined by $p(r)(x, y, z)=(x, y, r)$.

A cubical neighborhood of a point (x, y, z) in E^3 is a set of the form $I_1 \times I_2 \times I_3$, where I_1, I_2, and I_3 are finite open subintervals of the real line R containing x, y, and z respectively.

THEOREM 1. *Suppose that X is a closed subset of E^3 and that, for some $t>0$, $X=\bigcup\{X(r)^t : r \in R\}$. Then X is a *-taming set.*

Proof. Let C be a crumpled cube in E^3 with $X \subset C$. Let p be an arbitrarily chosen point of Bd C. By [5, Lemma 2.1, Theorems 3.1 and 3.4], in order to show that X is a *-taming set it suffices to show that Int C^* is 1-LC in E^3-X at p. To this end, let N denote an open set containing p. We may assume, making N smaller if necessary, that N is a cubical neighborhood of p and that Diam $N<t$. Using [5, Proposition 3.2], we choose a cubical neighborhood N' of p with $N' \subset N$ such that if J is any loop in $N' \cap$ Int C^* and Y is any *-taming set in C, then J bounds a singular disk in $N-Y$. We shall complete the proof that Int C^* is 1-LC in E^3-X at p by showing that if J is any loop in $N' \cap$ Int C^*, then J bounds a singular disk in $N-X$. This will complete the proof of Theorem 1.

Let Δ denote a triangular disk and $J\colon \text{Bd}\,\Delta \to N' \cap \text{Int}\,C^*$ a loop. Our plan to shrink J to a point in $N-X$ proceeds roughly as follows. We first carefully choose certain planes $P(r_1), \ldots, P(r_n)$ $(r_1<\cdots<r_n)$; we then shrink J to a point in $N-\bigcup_{i=1}^n X(r_i)$, say by a map $D\colon \Delta \to N-\bigcup_{i=1}^n X(r_i)$; we put D in general position with respect to $\bigcup_{i=1}^n P(r_i)$; and we use portions of $J(\text{Bd}\,\Delta) \cup (D(\Delta) \cap \bigcup_{i=1}^n P(r_i))$, appropriately adjusted, as a guideline for shrinking J to a point in $N-X$. The procedure is a refinement of the procedures followed in [10] and [11].

Choosing the planes $P(r_1), \ldots, P(r_n)$. Let M denote a compact connected neighborhood of J in $N'-X$. Let I denote the closed interval

$$\{z : \text{there exist } x, y \text{ in } R \text{ such that } (x, y, z) \in M\}.$$

Fix $r \in I$. There are only finitely many components of $P(r) \cap (N-X)$ which contain points of M since M is compact. It is therefore an easy matter to construct a compact set $Q(r)$ in $P(r) \cap (N-X)$ which contains $P(r) \cap M$, which has precisely one component in each of the components of $P(r) \cap (N-X)$ which contains a point of M, and which has only arcwise connected components. Given $d>0$, examine the set $Q(r, d)=Q(r) \cup (M \cap P[r-d, r+d])$. For some $d(r)>0$, $p(r)[Q(r, d(r))] \cap p(r)[X[r-d(r), r+d(r)]]=\varnothing$. For otherwise we find that $X \cap Q(r) \neq \varnothing$, a contradiction. It is therefore possible to find real numbers $r_1, \ldots, r_n$ in I, $r_1<\cdots<r_n$, such that $I \subset \bigcup_{i=1}^n (r_i-d(r_i), r_i+d(r_i))$. We may assume

that there are real numbers $s_1, \ldots, s_{n-1}$, $r_1 < s_1 < r_2 < \cdots < r_{n-1} < s_{n-1} < r_n$, such that $r_{i+1} - d(r_{i+1}) < s_i < r_i + d(r_i)$ $(i = 1, \ldots, n-1)$. (We may have to delete certain unnecessary r_i's to obtain the situation we have just hypothesized.) Then the planes $P(r_1), \ldots, P(r_n)$ are the desired planes.

Shrinking J to a point in $N - \bigcup_{i=1}^n X(r_i)$. The set $X(r_i)$ is a *-taming set for each i by [5, Theorem 3.7]. Hence $Y = \bigcup_{i=1}^n X(r_i)$ is a *-taming set by the same theorem. Since Y is a *-taming set in C and J is a loop in $N' \cap \operatorname{Int} C^*$, J bounds a singular disk in $N - Y$ by our choice of N'. Let $D\colon \Delta \to N - Y$ denote a singular disk bounded by the loop $J\colon \operatorname{Bd} \Delta \to N' \cap \operatorname{Int} C^*$. After a slight homotopy in $N - X$, we may assume that D is piecewise-linear and in general position with respect to $P(r_1), \ldots, P(r_n)$. The adjustment may be realized in such a manner that the adjusted J is a loop in M since M is a neighborhood of the original J. It suffices to show that the adjusted J can be shrunk to a point in $N - X$.

Finding guidelines near $J(\operatorname{Bd} \Delta) \cup (D(\Delta) \cap \bigcup_{i=1}^n P(r_i))$ for shrinking J to a point in $N - X$. Examine the set $D^{-1}[D(\Delta) \cap \bigcup_{i=1}^n P(r_i)]$, which is a subset of Δ. Since D and $\bigcup_{i=1}^n P(r_i)$ are in general position, the set $D^{-1}[D(\Delta) \cap \bigcup_{i=1}^n P(r_i)]$ is a union of finitely many disjoint simple closed curves in $\operatorname{Int} \Delta$, which curves we ignore, and of finitely many disjoint spanning arcs $A_1, \ldots, A_m$ of Δ. For each j, $D(A_j)$ is a path in $P(r_i) \cap (N - X)$ for some i, hence a subset of some single component of $P(r_i) \cap (N - X)$. The endpoints of A_j are in $\operatorname{Bd} \Delta$, hence are mapped by D into M. We conclude that the endpoints of A_j are mapped by D into a single component of $Q(r_i)$. Since the components of $Q(r_i)$ are arcwise connected, we may define a map $D'\colon (\operatorname{Bd} \Delta) \cup (\bigcup_{j=1}^m A_j) \to N - X$ by setting $D' | \operatorname{Bd} \Delta = D | \operatorname{Bd} \Delta = J$ and by sending each A_j into the appropriate $Q(r_i)$. The map D' will serve as a guideline for shrinking J in $N - X$.

Shrinking J to a point in $N - X$. Notice that the m spanning arcs $A_1, \ldots, A_m$ divide Δ into $m+1$ disks $\Delta_1, \ldots, \Delta_{m+1}$ with disjoint interiors. For each j, the map D' is defined on $\operatorname{Bd} \Delta_j$ and takes $\operatorname{Bd} \Delta_j$ into $N - X$. In order to shrink J to a point in $N - X$, it suffices to find for each j a map $D_j'\colon \Delta_j \to N - X$ which extends the map $D' | \operatorname{Bd} \Delta_j\colon \operatorname{Bd} \Delta_j \to N - X$.

To this end we assume j fixed. We note that there is an integer i $(1 \leqq i \leqq n)$ such that $D'(\operatorname{Bd} \Delta_j) \subset P[r_i, r_{i+1}]$. (This is not strictly true; we might have one of the exceptional cases where $D'(\operatorname{Bd} \Delta_j) \subset P[r_1 - d(r_1), r_1]$ or $D'(\operatorname{Bd} \Delta_j) \subset P[r_n, r_n + d(r_n)]$, but we leave the simple adjustments needed to handle the exceptional cases to the reader.) The existence of such an i follows from the fact that $\operatorname{Bd} \Delta_j$ lies on the boundary of a single component of $\Delta - D^{-1}(D(\Delta) \cap \bigcup_{i=1}^n P(r_i))$. The loop $D' | \operatorname{Bd} \Delta_j$ is homotopic in a natural way to the loop $p(s_i) \circ D' | \operatorname{Bd} \Delta_j$, where s_i is the real number chosen previously such that $r_i < s_i < r_{i+1}$ and $r_{i+1} - d(r_{i+1}) < s_i < r_i + d(r_i)$. The homotopy we have in mind moves a point x along the vertical interval joining the points $D'(x)$ and $p(s_i) \circ D'(x)$. Our choices of the sets $Q(r_j)$, the numbers $d(r_j)$, and the map D' were dictated precisely by the goal that the track of this homotopy lie in $N - X$. The final image of the loop $D' | \operatorname{Bd} \Delta_j$ lies in $P(s_i) \cap (N - X)$;

but this latter set is simply connected since Diam $N<t$ and $X(s_i)$ has no component smaller in diameter than t. Thus we are enabled to shrink the loop $D'|\text{Bd } \Delta_j$ to a point in $N-X$ and the proof is complete.

THEOREM 2. *Let S be a 2-sphere in E^3, K a compact nowhere dense subset of the real line R, and $S_K=\bigcup\{S(r) : r\in K\}$. Then S_K lies on a tame 2-sphere in E^3.*

Proof. By [7, Theorem 1.1] and a compactness argument, it suffices to show that E^3-S is 1-LC in E^3-S_K at each point $p\in S_K$. The proof of this fact proceeds exactly as the proof of Theorem 1 until the levels s_i are chosen. (The set S_K of Theorem 2 corresponds to the set X of Theorem 1.) In the case of Theorem 2, one must choose the s_i in $R-K$. Then $P(s_i)\cap S_K=\varnothing$ and the remainder of the proof proceeds without change.

THEOREM 3. *A closed subset X of E^3 is a *-taming set if, for each real number r, $X(r)$ has no degenerate component.*

Proof. Note that for each positive integer i, the set

$$X_i=\bigcup\{X(r)^{1/i} : r \text{ a real number}\}$$

is a closed subset of E^3 and satisfies the hypothesis of Theorem 1. Hence X_i is a *-taming set. But $X=\bigcup_{i=1}^{\infty} X_i$. Thus X is a *-taming set since a closed set which is a countable union of *-taming sets is a *-taming set by [5, Theorem 3.7].

COROLLARY 4. *A 2-sphere S in E^3 is tame if no horizontal section of S has a degenerate component.*

Proof. The sphere S is a *-taming set by Theorem 3, but a *-taming set which lies on a 2-sphere is tame (easy exercise or [5, Corollary 3.8]).

Corollary 4 generalizes the principal results of [8], [10], [12], and [11] (with references listed in chronological order, [8] and [10] announcing the same result simultaneously).

We say that a set X is unidirectional if there is a straight line L in E^3 such that X is a union of nondegenerate intervals, each parallel to L.

THEOREM 5. *A closed subset X of E^3 is a *-taming set if it is unidirectional.*

Proof. Let L be a straight line in E^3 such that X is a union of nondegenerate straight line intervals parallel to L. Let P be a plane which contains L. Then if a coordinate system is chosen for E^3 with respect to which P is horizontal, we find that $X=\bigcup_{i=1}^{\infty} X_i$, where $X_i=\bigcup\{X(r)^{1/i} : r \text{ a real number}\}$. As in the proof of Theorem 3 we conclude that X is a *-taming set.

The next corollary generalizes recent results of H. C. Griffith [9], L. D. Loveland [13], and H. G. Bothe [3] and answers questions raised by R. H. Bing [2] and Loveland [13, Questions 1 and 3] in the affirmative. Griffith [9] proved that a 2-sphere S in E^3 is tame if there exists a positive number t such that, for each point p of S, there are round balls of diameter t tangent at p and lying except for p in opposite complementary domains of S. Loveland [13] removed the requirement that the size of the tangent balls be uniformly large and independent of p. Bothe [3] in a manuscript written before Loveland's work, but without Loveland's knowledge, proved a slightly stronger theorem which we shall only paraphrase: A 2-sphere S in E^3 is tame if for each point p of S there is a solid right circular double cone having p as vertex, having extremely large vertex angle, and being separated by S into its two halves.

COROLLARY 6. *A crumpled cube C in E^3 is a 3-cell if it can be touched at each point from C^* by a pencil.*

Proof. One sees easily that C^* is a countable union of closed unidirectional sets, hence a *-taming set by Theorem 5 and [5, Theorem 3.7]. We conclude that C is a 3-cell by the definition of *-taming set.

We have, of course, looked at *horizontal* sections for convenience only. We could look at "horizontal" sections with respect to curvilinear or rectilinear Cartesian coordinate systems on E^3 other than the standard one. Using the ideas we have illustrated above and the general properties of *-taming sets [5], the reader can produce any number of theorems concerning spheres which have appropriate types of cross sections. We include three further examples here. More difficult and interesting examples will appear in the third paper of this series [6].

EXAMPLE 1. A 2-sphere S in E^3 is tame if there are countably many planes $P_1, P_2, \ldots$ in E^3 such that, given any point p of S, there is a plane P in E^3 parallel to some P_i such that p is in a nondegenerate component of $P \cap S$.

EXAMPLE 2. The Alexander Horned Sphere S [1] can be described in E^3 in such a manner that each horizontal section of $S \cup \text{Ext } S$ is connected. It follows from either Theorem 1 or 3 that $S \cup \text{Ext } S$ is a *-taming set. Hence $S \cup \text{Int } S$ is a 3-cell by the definition of *-taming set.

EXAMPLE 3. (This example generalizes Example 2.) A crumpled cube C in E^3 is a 3-cell if it is bounded and if, for each real number r, $P(r)-C$ is connected. For if $P(r) \cap C^*$ has a degenerate component for some real number r, then Int C separates points of (Int C^*) $\cap P(s)$ for some real number s near r, a contradiction. Hence C^* is a *-taming set by Theorem 3 and $C=(C^*)^*$ is a 3-cell.

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