TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 161, November 1971

*-TAMING SETS FOR CRUMPLED CUBES. III: HORIZONTAL SECTIONS IN 2-SPHERES(1)

BY
JAMES W. CANNON

Abstract. We prove that a 2-sphere S in E^3 is tame if each horizontal section of S has at most four components. Since there are wild spheres in E^3 whose horizontal sections have at most five components, this result is, in a sense, best possible. Much can nevertheless be said, however, even if certain sections have more than five components; and we show that the wildness of a 2-sphere S in E^3 is severely restricted by the requirement that each of the horizontal sections of S have at most finitely many components that separate S.

1. **The main theorem.** Suppose S is a 2-sphere in E^3. We shall study the restrictions imposed on the embedding of S in E^3 by the requirement that each horizontal section of S have only finitely many components that separate S. Surprisingly, the restrictions are severe enough to allow us to catalogue fairly completely the possible wildness of S. Our results are summarized in Theorem 1 below. Our main tool will be the following theorem from the preceding paper of this series [6, Theorem 3].

THEOREM 0. *If X is a closed subset of E^3 and no horizontal section of X has a degenerate component, then X is a *-taming set.*

C. E. Burgess indicated in conversation that Theorem 0 could be used to prove that a 2-sphere in E^3 is tame if each of its horizontal sections has at most three components. He conjectured further that a sphere is tame if each of its horizontal sections has at most four components. This paper is the result of our (successful) attempt to verify Burgess' theorem and conjecture. We again express our indebtedness to Burgess for his instruction, friendship, and encouragement. These results generalize earlier sphere-slicing theorems of Eaton [7], Hosay [10], Loveland [12], and Jensen [11].

THEOREM 1. *Let S denote a 2-sphere in E^3.*

(1) *If each horizontal section of S has at most finitely many components that separate S, then the wild set $W(S)$ of S lies in a closed 0-dimensional set of horizontal levels, hence is tame* (*by* [6, *Theorem* 2]) *and therefore at most 1-dimensional.*

Received by the editors November 20, 1970.

AMS 1970 *subject classifications.* Primary 57A10.

Key words and phrases. Taming sets, *-taming sets, crumpled cubes, slices in 2-spheres, surfaces in 3-manifolds, 2-spheres in E^3.

(1) This research was supported by an NSF postdoctoral fellowship.

(2) *If, in addition, each horizontal section of S has at most countably many degenerate components, then the set Y of nonpiercing points of S is countable and the closure* cl Y *of Y is equal to $W(S)$. In particular, if $Y=\varnothing$ then S is tame.*

(3) *If the hypothesis of* (2) *is satisfied and S has local horizontal-separation index less than* 5 *at each degenerate component of each horizontal section of S* (*see* §2 *for a definition*), *then the set Y of nonpiercing points of S is empty and S is tame.*

(4) *If there is a positive integer n such that each horizontal section of S has at most n components that separate S, then each horizontal section of S has at most finitely many components and* (2) *applies.*

(5) *If, in* (4), $n<5$, *then* (3) *applies and S is tame.*

(6) *If, in* (4), $n=5$, *then $S \cup$* Int *S is a* 3*-cell and S is tame modulo two points.*

The proof of Theorem 1 will be given in this section but will follow some definitions and a remark. The proof will rely on the theorem from §2 as well as on the results of [3], [4], [5], [6]. §3 contains counterexamples to most of the obvious extensions of Theorem 1.

In general we rely on the terminology and notation of [5], [6]. If $r \in R$ ($R=$reals), then $P(r)$ denotes the horizontal plane $\{(x, y, z) \in E^3 \mid z=r\}$. If $X \subset E^3$ is any set, then $X(r)$ denotes the set $P(r) \cap X$ and is called the horizontal section of X at level r. If I is an interval of real numbers (finite, infinite, open, closed, or half-open), then XI denotes the set $\bigcup \{X(r) \mid r \in I\}$ (e.g., $X[a, b)=\bigcup \{X(r) \mid a \leqq r<b\}$). If $D \subset P(r)$ (for some r) and I is a set of real numbers, then $D \times I$ denotes the set $\{(x, y, z) \in E^3 \mid (x, y, r) \in D \text{ and } z \in I\}$.

Remark. Suppose objects $S \subset E^3$, $r \in (a, b) \subset [a, b] \subset R$, $p \in S(r)$, and $D \subset P(r)$ given such that

(i) S is a 2-sphere and

(ii) D is a disk such that $p \in \operatorname{Int} D$ and $(\operatorname{Bd} D) \times [a, b] \subset E^3 - S$.

Define $C=D \times [a, b]$. Then C is said to be a cylinder at p which respects S. We say that C is an ε-cylinder if Diam $C<\varepsilon$. If $\{p\}$ is a degenerate component of $S(r)$ and $\varepsilon>0$, then there is an ε-cylinder C at p which respects S. (This is a simple exercise in plane topology.)

Proof of Theorem 1(1). Let $B=\{b \in R \mid S(b)$ has a degenerate component$\}$. We proceed in two steps.

Step 1. The set B is countable and nowhere dense in R. For suppose not. Then there is an infinite subset B' of B, each point of which is the limit of both an increasing and a decreasing sequence from B'. Fix a number $b \in B'$, an open interval (α, β) which contains b, and a positive number ε. We claim that there is a $b' \in B'$, an open interval (α', β') which contains b', with $[\alpha', \beta'] \subset (\alpha, \beta)$, and an $\varepsilon'>0$, such that each section $S(r)$ $(r \in [\alpha', \beta'])$ has a component K which separates S and satisfies $\varepsilon'<\operatorname{Diam} K<\varepsilon$. Suppose that we can establish this claim. Then an iteration yields sequences $\varepsilon_1>\varepsilon_2>\ldots>0$ and $[\alpha_1, \beta_1] \supset [\alpha_2, \beta_2] \supset \cdots$ such that for any $r \in \bigcap_{i=1}^{\infty} [\alpha_i, \beta_i]$, $S(r)$ has components $K_1, K_2, \ldots$ which separate S and satisfy

$\varepsilon_1 > \text{Diam } K_1 > \varepsilon_2 > \text{Diam } K_2 > \varepsilon_3 > \cdots$, an obvious contradiction to the hypothesis of our theorem. Hence, to complete Step 1 we need only establish our claim. Let $\{p\}$ be a degenerate component of $S(b)$. By the remark, there is an ε-cylinder $C = D \times [x, y]$ at p which respects S and satisfies $\alpha < x < y < \beta$. If U and V denote the components of $E^3 - S$ (with notation chosen so that $U \supset (\text{Bd } D) \times [x, y]$), then there is an arc A in $(V \cup \{p\}) \cap C$ which is irreducible from the set $P(x) \cup P(y)$ to the plane $P(b)$, say $A \subset P[x, b]$. Since $b \in B'$, there is a $b' \in B' \cap (x, b)$. Choose α' and β' such that $x < \alpha' < b' < \beta' < b$. Note that $A[\alpha', \beta'] \subset V$, hence that $\rho(A[\alpha', \beta'], S) > 0$. Choose $\varepsilon' > 0$, $\varepsilon' < \rho(A[\alpha', \beta'], S)$. Since S separates $A[\alpha', \beta']$ from $(\text{Bd } D) \times [x, y]$, one easily verifies that b', (α', β'), and ε' satisfy the requirements of the claim. This completes Step 1.

Step 2. By Step 1, cl B is 0-dimensional. By Theorem 0 and [5, Corollary 3.8], the wild set $W(S)$ of S lies in the 0-dimensional set cl B of levels. Hence $W(S)$ is tame by [6, Theorem 2] and is therefore obviously 1-dimensional at most. This completes Step 2 and the proof of Theorem 1(1).

Proof of Theorem 1(2). Let Y be the set of nonpiercing points of S and let D be a disk in $S - \text{cl } Y$. We show that if U is a component of $E^3 - S$, then $(\text{cl } U) - D$ is 1-ULC. It will then follow from any of a number of theorems in the literature that D is tame, hence that S is tame modulo cl Y. (See [4], for example, for a discussion of the 1-ULC property.) Let $S(r)_i$ be the union of those components of $S(r)$ having diameter at least $1/i$. Let $S_i = \bigcup \{S(r)_i \mid r \in R\}$. Then S_i is a *-taming set by Theorem 0. Hence $(\text{cl } U) - S_i$ is 1-ULC by [5, Theorem 3.1]. Let D_i denote the set $S_i \cap D$. It follows that $(\text{cl } U) - D_i$ is 1-ULC. Let $X(r)$ be the set of degenerate components of $S(r)$. Then $X(r)$ is countable for each r by hypothesis and empty for all but countably many r by Step 1 of the preceding proof. Thus $D \cap (\bigcup X(r))$ is a countable subset of $S - \text{cl } Y$. Let $p_1, p_2, \ldots$ be the points of $D \cap (\bigcup X(r))$. Then each p_i is a piercing point of S, and $\text{cl } U - \{p_i\}$ is therefore 1-ULC ([13, Theorem 1] or [4, (0.2)]). Thus $(\text{cl } U) - [\bigcup_{i=1}^{\infty} D_i \cup \{p_1, p_2, \ldots\}]$ is 1-ULC by [4, Lemma 2.3]. We conclude that D is tame (see, for example the introduction to [4] or see [3, Corollary 5.4]). The set Y is countable because $Y \subset \bigcup X(r)$ (cf. [3, Corollary 4.7]). This completes the proof of Theorem 1(2).

Proof of Theorem 1(3). Each point of S is a piercing point of S by Theorem 2. Hence $Y = \varnothing$ and S is tame by 1(2).

Proof of Theorem 1(4), 1(5), and 1(6). The conclusion of Theorem 1(4) follows easily in the manner of Step 1 of the proof of Theorem 1(1). Theorem 1(3) clearly applies if $n < 5$ to show S tame. Suppose finally that $n = 5$. Let p be a nonpiercing point of S. Suppose that p is neither in the uppermost nor the lowermost section of S which is nonempty. Note that p is in a degenerate component of $S(r)$ where $P(r)$ is the level containing p [3, Corollary 4.7]. Let D be a disk in $P(r)$ containing p in its interior such that $\text{Bd } D \subset P(r) - S$ and such that D contains no separating component of $S(r)$. Then arbitrarily close to the level r there is a level r' such that $D(r') = \{(x, y, r') \mid (x, y, r) \in D\}$ contains at least 5 separating components of $S(r')$

(Theorem 2). It follows easily from the fact that r is neither the uppermost nor the lowermost section in S that for r' sufficiently close to r there is a separating component of $S(r')$ which does not intersect $D(r')$. Thus $S(r')$ has at least 6 separating components, a contradiction. Similarly, if p is a nonpiercing point of S in the uppermost or lowermost level r of S and $S(r)$ has at least two components, we obtain a contradiction. We conclude that the set Y of nonpiercing points of S consists of at most two points p and q, p in the uppermost level of S, q in the lowermost. Clearly p and q are accessible by tame arcs from Ext S. Since Y contains at most two points, S is tame modulo p and q by Theorem 1(2). Since S is tame modulo two points, each of which is accessible by a tame arc (which is a *-taming set [5, Theorem 3.7]) from $S \cup \text{Ext } S$, $S \cup \text{Int } S$ is a 3-cell by definition of *-taming set. This completes the proof.

2. **Local horizontal-separation index.** Let S denote a 2-sphere in E^3. Suppose that some horizontal section $S(\alpha)$ of S has a degenerate component $\{p\}$. The sphere S is said to have local horizontal-separation index $\leqq n$ at p if there are (i) a disk D in $P(\alpha)$ with $p \in \text{Int } D$ and $\text{Bd } D \subset P(\alpha) - S(\alpha)$ and (ii) a closed interval $[a, b]$ with α in the open interval (a, b), such that if $r \in (a, b)$ and $D(r) = \{(x, y, r) \mid (x, y, \alpha) \in D\}$ then (iii) $\text{Bd } D(r) \subset E^3 - S$, and (iv) $D(r) \cap S(r)$ has at most n components that separate S.

THEOREM 2. *Let S and p be as described in the first paragraph of* §2. *If S has local horizontal-separation index $\leqq 4$ at p, then p is a piercing point of S.*

Proof. Let U and V be the components of $E^3 - S$. We shall show that p is accessible from both U and V by a tame arc. This will imply that p is a piercing point of S by [13, Theorem 3].

Let $D = D(\alpha)$ and $[a, b]$ be as promised by the definition of local horizontal-separation index $\leqq 4$ at p. By cutting D and $[a, b]$ down in size and choosing notation properly, we may require that $D - S(\alpha)$ be connected and lie in U. A tame arc to p from U can easily be constructed in $(D - S(\alpha)) \cup \{p\}$; we leave this exercise in plane topology to the reader.

A tame arc to p from V requires a delicate construction. We first choose an arc A in $V \cup \{p\}$ which has p as an endpoint and satisfies two conditions:

(i) A is locally polyhedral except possibly at p.

(ii) If r is a real number and x and y are points of A which lie in the same component of $V(r)$, then the subarc A_{xy} of A from x to y lies either in $P(-\infty, r]$ or in $P[r, \infty)$.

Such an arc can be obtained by trimming excess vertical folds from an arc A' which satisfies all requirements made of A except (ii). (Similar adjustments will be made in more detail as we proceed.) If there is such an arc A which is vertically monotone, then A is tame and we are finished. Otherwise, we proceed to adjust A.

We identify as follows subarcs $A_1, A_2, \ldots$ of $A - \{p\}$ on which there is no vertical folding. Assume A ordered with p as last point. The maximal straight line segments of which $A-\{p\}$ is the union inherit thereby a direction which we describe as upward, downward, or horizontal. We choose a sequence $A_1', A_2', \ldots$ of such segments directed alternately upward and downward iteratively. Let A_1' be the first nonhorizontal segment of $A-\{p\}$. Let A_i' $(i>1)$ be the first segment of $A-\{p\}$ which follows A_{i-1}' in the order on A and is directed oppositely to A_{i-1}'. Let A_i $(i\geqq 1)$ be the union of A_i' and the segments of $A-\{p\}$ between A_i' and A_{i+1}'. We call $A_1, A_2, \ldots$ the component arcs of A and observe that each has a natural designation as increasing or decreasing. We call A_i and A_{i+1} adjacent.

We assume A normalized so that, for $i\neq j$, $A_i \cap A_{i+1}$ and $A_j \cap A_{j+1}$ lie in different horizontal levels. We further assume the interval $[a, b]$ and the arc A shortened so that A is irreducible from $P(a) \cup P(b)$ to $P(\alpha)$ and lies, except for p, entirely above or entirely below Int D, say below Int D (i.e., $A\subset(\text{Int } D)\times[a, \alpha]$). (This is possible since $D\subset \text{cl } U$ while $A\subset V\cup\{p\}$.) As a consequence, A_1 is increasing, A_2 is decreasing, and so forth.

To each level $r\in(a, \alpha)$ we assign the integer $n(r)$ equal to the number of components of $A\cap P(r)$. We ignore the countably many levels which contain the sets $A_i\cap A_{i+1}$ and note that the set of remaining levels falls naturally into a null sequence $I_1, I_2, \ldots$ of disjoint open intervals, each one vertically above the preceding, and on each of which the function n is constant. We call the constant value of n on I_i, n_i, and note that $n_{i+1}=n_i\pm 2$ for each i since no more than one of the sets $A_j\cap A_{j+1}$ lies in any one level. We are now forced to consider cases, all of which may actually occur (see Figures 1 and 2).

Case 1. There are infinitely many integers i for which $n_i=1$. In this case, let $\varepsilon>0$ be given. Let $C=D\times[x, y]$ be an ε-cylinder at p which respects S (§1, Remark). Choose an integer i such that $n_i=1$ and such that $I_i\subset[x, \alpha]$. Choose a level $r\in I_i$ such that $A\cap P(r)$ is a single point. Then $S_0=\text{Bd }(D\times[r, y])$ is a 2-sphere in E^3 such that $p\in\text{Int } S_0$, Diam $S_0<\varepsilon$, and $S_0\cap A$ is a single point. Thus, it is easily seen that A is locally peripherally unknotted (see [9] or [5, §4]). This arc also lies on a 2-sphere since it is tame modulo the point p. We conclude in Case 1 that A is tame ([9, Theorem VI] or [5, Corollary 4.3]).

Case 2. There are infinitely many integers i for which $n_i\geqq 7$. We show in this case that A may be adjusted so as to satisfy the hypothesis of Case 1. Choose an increasing sequence $J=\{j_1, j_2, \ldots\}$ of positive integers such that for each $j\in J$, $n_j\geqq 7$. Require further that, for $j, k\in J$ and $j<k$, each component arc of A which intersects PI_j precedes (in the order on A) each of those which intersects PI_k. For each $j\in J$, choose a level $r_j\in I_j$ such that $P(r_j)$ contains no one of the countably many horizontal segments of A and also contains no one of those subcontinua of S which separate S into at least three components. (Such subcontinua can occur in at most countably many horizontal sections of S.) Then $D(r_j)\cap S$ has at most four components which separate $D(r_j)$ and each of those separates $D(r_j)$ into precisely two components. Since Bd $D(r_j)\subset U$, $D(r_j)-S$ has at most four components in V. Since (ii) above is satisfied, only adjacent component arcs A_k and A_{k+1} can intersect

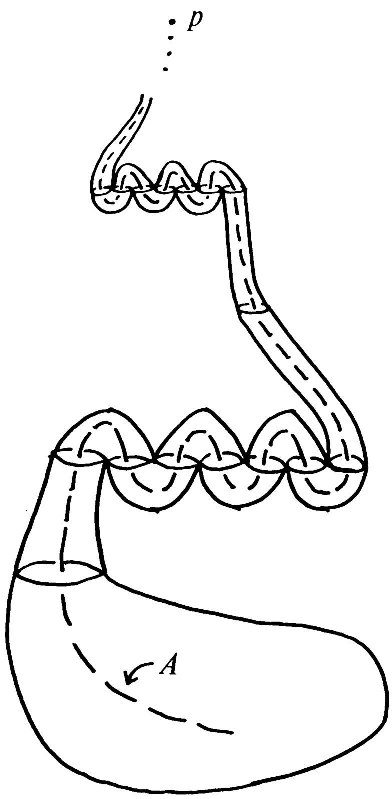

FIGURE 1

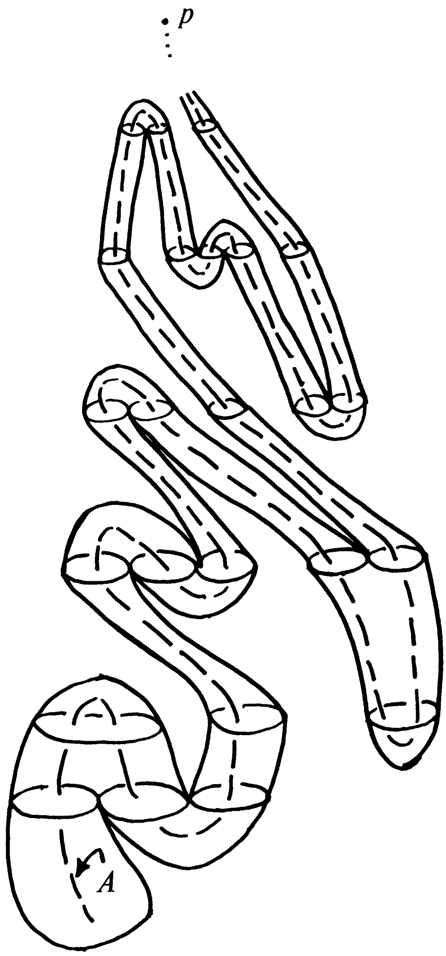

FIGURE 2

the same component of $D(r_j)-S$. It follows that $n_j=7$, that $D(r_j)-S$ has exactly four components V_1, V_2, V_3, V_4 in V, and that, after suitable numbering of the V_i and of the seven component arcs $A_{j_1}, \ldots, A_{j_7}$ of A which intersect $D(r_j)$, A_{j_1} and A_{j_2} are adjacent component arcs of A and intersect V_1, A_{j_3} and A_{j_4} are adjacent component arcs and intersect V_2, A_{j_5} and A_{j_6} are adjacent and intersect V_3, and A_{j_7} intersects V_4. Remove the arc in $A_{j_1} \cup A_{j_2}$ from $A_{j_1} \cap V_1$ to $A_{j_2} \cap V_1$ and replace it by an arc from $A_{j_1} \cap V_1$ to $A_{j_2} \cap V_1$ in V_1. Make similar replacements in $A_{j_3} \cup A_{j_4}$ and $A_{j_5} \cup A_{j_6}$. Push the resulting new arcs $A_{j_1} \cup A_{j_2}$, $A_{j_3} \cup A_{j_4}$, and $A_{j_5} \cup A_{j_6}$ slightly below or above level r_j to obtain a new arc A which intersects level r_j in only one point. The procedure, repeated for all $j \in J$, automatically generates a null sequence of changes in A and yields an arc A satisfying the hypothesis of Case 1.

Case 3. For each sufficiently large integer i, $n_i=3$ or 5. Shortening A, we may assume that the pattern of n_i's is $(1, 3, 5, 3, 5, \ldots)$, exactly as is the case in Figure 3 (wild arc) (see below) and Figure 2 (tame arc). Let $I_i=(r_{i-1}, r_i)$ be one of the intervals for which $n_i=5$. Let $A_{i_1}, \ldots, A_{i_5}$ $(i_1<\cdots<i_5)$ be the five component arcs of A which intersect the levels between r_{i-1} and r_i. Then A_{i_1}, A_{i_3}, and A_{i_5} are increasing while A_{i_2} and A_{i_4} are decreasing. We see easily that either A_{i_1} and A_{i_2} are adjacent, with $A_{i_1} \cap A_{i_2} \subset D(r_i)$, or A_{i_3} and A_{i_4} are adjacent, with $A_{i_3} \cap A_{i_4} \subset D(r_i)$. Similarly, either A_{i_2} and A_{i_3} are adjacent, with $A_{i_2} \cap A_{i_3} \subset D(r_{i-1})$, or A_{i_4} and A_{i_5} are adjacent, with $A_{i_4} \cap A_{i_5} \subset D(r_{i-1})$. All of these possibilities can occur (Figure 1), but there is an infinite set J of integers such that, for each $i \in J$, $A_{i_1} \cap A_{i_2} \neq \varnothing$, $A_{i_1} \cap A_{i_2} \subset D(r_i)$, $A_{i_4} \cap A_{i_5} \neq \varnothing$, and $A_{i_4} \cap A_{i_5} \subset D(r_{i-1})$. For otherwise one could easily identify three disjoint open subarcs of A, each of which has p as a limit point, an absurdity. We require that for $j, k \in J$, $j<k$, $A_{j_1}, \ldots, A_{j_5}$ all precede $A_{k_1}, \ldots, A_{k_5}$ in the order on A. Fix some $j \in J$ for consideration. By a proof like that in Case 2, there is a dense set L of levels r in I_j, such that $D(r) \cap V$ has at most four components and $D(r) \cap A$ is a finite set. At such a level r, some two of the A_{j_i} must intersect the same component of $D(r) \cap V$ and, by (ii), be adjacent. We consider two subcases.

Case 3a. For some $r \in L$, A intersects at most three of the components of $D(r) \cap V$. Then one can proceed exactly as in Case 2 to adjust A so that its intersection with $D(r)$ is a single point.

Case 3b. For each $r \in L$, A intersects four components of $D(r) \cap V$ (i.e., intersects all four components).

We first claim in this case that for no $r \in I_j$ does A_{j_3} intersect the same component of $D(r) \cap V$ intersected by another A_{j_i} (necessarily A_{j_2} or A_{j_4} by (ii)). Suppose to the contrary, for example, that A_{j_3} and A_{j_4} intersect the same component of $D(r) \cap V$. Then choose $r' \in L$, $r<r'<r_j$, such that A_{j_1} and A_{j_2} intersect the same component of $D(r') \cap V$. We obtain a contradiction by showing that A_{j_3} and A_{j_4} intersect the same component of $D(r') \cap V$, hence that A intersects at most three components of $D(r') \cap V$. If A_{j_3} and A_{j_4} were separated by S in $D(r')$, then we could consider, first, a closed curve C formed by adding to the portion of $A_{j_3} \cup A_{j_4}$ above $P(r)$ an arc B in $D(r) \cap V$ from $A_{j_3} \cap D(r)$ to $A_{j_4} \cap D(r)$ and, second, a closed curve C' very near S in $D(r')$ which separates $A_{j_3} \cap D(r')$ from $A_{j_4} \cap D(r')$ in $D(r')$. Then C and C' would link although C lies in V and C' is homotopically close to U, a contradiction (see [2, Theorem 4.7.1]).

Our claim establishes the fact that for each $r \in L$, either A_{j_1} and A_{j_2} intersect the same component of $D(r) \cap V$ or A_{j_4} and A_{j_5} do. In the former case we say that $r \in L(1, 2)$, in the latter that $r \in L(4, 5)$. A proof like the proof of our claim together with the fact that we are in Case 3b rather than 3a also establishes that if $r \in L(1, 2)$ and $r' \in L(4, 5)$, then $r'<r$. Since L is dense in I_j and since both $L(1, 2)$ and $L(4, 5)$ are clearly nonempty, we conclude that there is a unique critical level r_j such that, for each $r \in L(1, 2)$ and $r' \in L(4, 5)$, $r'<r_j<r$. We may assume that each A_{j_i} has been adjusted so as to be vertical near level r_j.

Let $\alpha_1, \alpha_2, \ldots$ be a strictly decreasing and $\beta_1, \beta_2, \ldots$ a strictly increasing sequence of real numbers from $L(1, 2)$ and $L(4, 5)$, respectively, each sequence converging to r_j. In level α_i, let X_i be a polygonal arc which misses S and joins the points of A_{j_1} and A_{j_2}. Let Y_i be a corresponding arc in level β_i. We may assume that the sequence $X_1, X_2, \ldots$ converges to a continuum X in $(S \cup V) \cap D(r_j)$ which intersects both A_{j_1} and A_{j_2}. Similarly, we obtain a continuum Y corresponding to the Y_i.

We next claim that $X \cap Y = \varnothing$. Suppose not. Then choose $r \in L$ and let W be the component of $D(r) \cap V$ which intersects A_{j_3}. Note that $W \cap A = W \cap A_{j_3}$. We know that W is simply connected, for otherwise there is a level $r' \in L$ near r such that $D(r') - S$ has at most five components and at least two of them lie in U. But this leaves at most three components in V. Since W is simply connected, there is a simple closed curve C' very near S in W which contains $W \cap A_{j_3}$ but no other points of A in its interior. Form a simple closed curve C by taking a subarc of A irreducible from $A_{j_2} \cap D(r)$ to $A_{j_4} \cap D(r)$ and joining the two by an arc very near $A_{j_2} \cup A_{j_4} \cup X \cup Y$ in $D(r) \cap (V \cup S)$. Then if care has been taken, C and C' can be moved by small homotopies into V and U respectively although they link by construction, a contradiction [2, Theorem 4.7.1]. Thus $X \cap Y = \varnothing$.

Finally, since $X \cap Y = \varnothing$, since $(X \cup Y) \cap A_{j_3} = \varnothing$ (a consequence of our first claim in the discussion of Case 3b), and since each A_{j_t} is vertical near level r_j, it is possible to choose an integer i so large that the three sets

$$P[\beta_i, \alpha_i] \cap [X_{i+1} \cup A_{j_1} \cup A_{j_2}],$$

$$P[\beta_i, \alpha_i] \cap [Y_{i+1} \cup A_{j_4} \cup A_{j_5}] \quad \text{and} \quad P[\beta_i, \alpha_i] \cap A_{j_3}$$

have vertical projections in $D(r_j)$ which are contained in disjoint disks D_1, D_2, and D_3 in $D(r_j)$. Replace the segment in $A_{j_1} \cup A_{j_2}$ which joins the ends of X_{i+1} by X_{i+1}. Similarly, insert Y_{i+1} in A. Consider the disk

$$\begin{aligned} E_j = {} & [D(r_j) - D_1 - D_2] \cup (\text{Bd } D_1 \times [r_j, \alpha_i]) \cup (D_1 \times \{\alpha_i\}) \\ & \cup (\text{Bd } D_2 \times [\beta_i, r_j]) \cup (D_2 \times \{\beta_i\}). \end{aligned}$$

Then E_j intersects the new A in just one point. It is easy to see that by a sequence of such changes in A one obtains a new arc A which, for essentially the same reason as given in Case 1, is tame. This concludes the proof of Theorem 2.

3. **Examples.** The first two figures show 2-spheres having local horizontal-separation index $\leqq 4$ at a point p. The first shows the situation which arises with Cases 1 and 2. The second isolates the problems of Case 3, which are the most difficult problems to handle.

Figure 3 shows a 3-cell wild at two points such that each horizontal section of the boundary has at most five components that separate the boundary. The example is borrowed, of course, from [8]. The example shows that the results of Theorem 1(3) and 1(4) are best possible. The example also gives an indication of why the

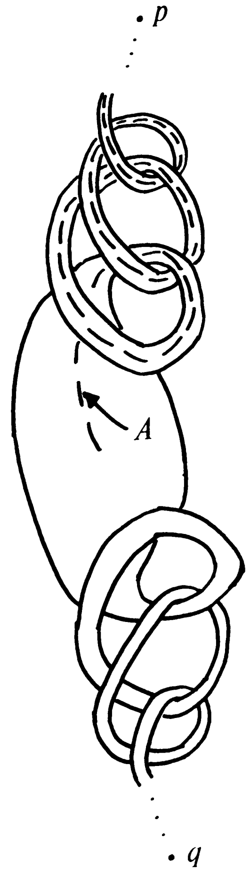

FIGURE 3

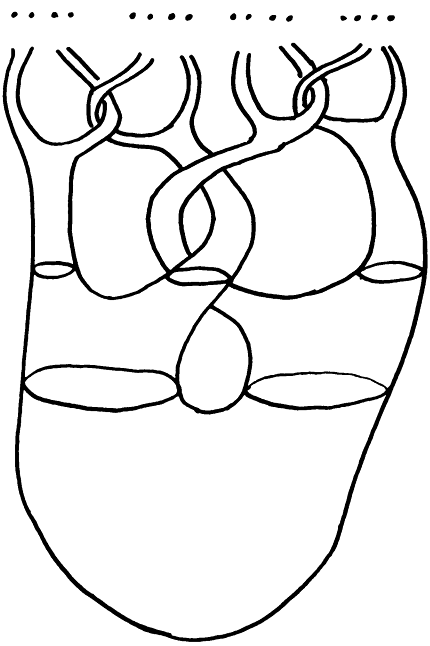

FIGURE 4

proof of Theorem 2 is so tedious; indeed, the arc A from Figure 2 and the arc A from Figure 3 follow precisely the same vertical folding pattern although the first is tame and the second wild.

Figure 4 shows the Alexander Horned Sphere S [1] embedded so as to satisfy the hypothesis of Theorem 1(1). This shows that the hypotheses of Theorem 1(2) are necessary since every point of S is a piercing point of S.

As a last example we indicate how a sphere S can be constructed in E^3 so as to satisfy the hypotheses of Theorem 1(2) and yet have a wild set of dimension >0. We leave it to the reader to produce more sophisticated examples. Each horizontal section of our example will have at most six components. Using cylindrical coordinates, start with the tame sphere

$$\{(\rho, \theta, z) \mid (\rho = 1 \text{ and } |z| \leqq 1) \text{ or } (\rho \leqq 1 \text{ and } |z| = 1)\}.$$

Consider the spiral

$$A = \{(\rho, \theta, z) \mid \rho = 1, 1 \leqq \theta < \infty, z = 1-(1/\theta)\}.$$

Pick a sequence of points $p_1, p_2, \ldots$ from A converging monotonically in z-coordinate to 1, and having closure which contains the circle

$$C = \{(\rho, \theta, z) \mid \rho = 1, z = 1\}.$$

Insert near each p_i a knotted feeler like the upper half of the sphere in Figure 3.

References

1. J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. U.S.A. **10** (1924), 8–10.

2. C. E. Burgess and J. W. Cannon, *Embeddings of surfaces in E^3*, Rocky Mt. J. Math. **1** (1971), 259–344.

3. J. W. Cannon, *Characterization of taming sets on 2-spheres*, Trans. Amer. Math. Soc. **147** (1970), 289–299. MR **41** #2644.

4. ———, *Sets which can be missed by side approximations to spheres*, Pacific J. Math. **34** (1970), 321–334.

5. ———, **-taming sets for crumpled cubes*. I: *Basic properties*, Trans. Amer. Math. Soc. **161** (1971), 429–440.

6. ———, **-taming sets for crumpled cubes*. II: *Horizontal sections in closed sets*, Trans. Amer. Math. Soc. **161** (1971), 441–446.

7. W. T. Eaton, *Cross sectionally simple spheres*, Bull. Amer. Math. Soc. **75** (1969), 375–378. MR **39** #957.

8. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. (2) **49** (1948), 979–990. MR **10**, 317.

9. O. G. Harrold, Jr., H. C. Griffith and E. E. Posey, *A characterization of tame curves in three-space*, Trans. Amer. Math. Soc. **79** (1955), 12–34. MR **19**, 972.

10. Norman Hosay, *A proof of the slicing theorem for 2-spheres*, Bull. Amer. Math. Soc. **75** (1969), 370–374. MR **39** #956.

11. R. A. Jensen, *Cross sectionally connected 2-spheres are tame*, Bull. Amer. Math. Soc. **76** (1970), 1036–1038.

12. L. D. Loveland, *Cross sectionally continuous spheres in E^3*, Bull. Amer. Math. Soc. **75** (1969), 396–397. MR **39** #958.

13. D. R. McMillan, Jr., *Some topological properties of piercing points*, Pacific J. Math. **22** (1967), 313–322. MR **35** #7319.

University of Wisconsin, Madison, Wisconsin 53706