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# DEGREE OF SYMMETRY OF A HOMOTOPY REAL PROJECTIVE SPACE

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**Abstract.** The degree of symmetry $N(M)$ of a compact connected differentiable manifold $M$ is the maximum of the dimensions of the compact Lie groups which can act differentiably and effectively on it. It is well known that $N(M) \leqq \dim SO(m+1)$, for an $m$-dimensional manifold, and that equality holds only for the standard $m$-sphere and the standard real projective $m$-space. W. Y. Hsiang has shown that for a high dimensional exotic $m$-sphere $M$, $N(M) < m^2/8+1 < (\frac{1}{4}) \dim SO(m+1)$, and that $N(M) = m^2/8+7/8$ for some exotic $m$-spheres. It is shown here that the same results are true for exotic real projective spaces.

0. **Introduction.** The *degree of symmetry* $N(M)$ of a compact connected differentiable $m$-manifold $M^m$ is the maximum of the dimensions of the compact Lie groups which can act effectively and differentiably on $M$. It is well known that

$$N(M^m) \leqq m(m+1)/2,$$

and that if $N(M^m) = m(m+1)/2$, then $M$ is diffeomorphic to the standard sphere $S^m$ or the standard real projective space $RP^m$ [3]. In [6] W. Y. Hsiang showed that if $\Sigma^m$ is an exotic sphere ($m \geqq 40$), then

$$N(\Sigma^m) < m^2/8+1.$$

Hsiang's result is best possible in the sense that

$$N(\Sigma_0^{8k+1}) = m^2/8+7/8$$

where $\Sigma_0^{8k+1}$ is the Kervaire sphere of dimension $m = 8k+1$.

In this paper we complement Hsiang's result by showing that if $M$ is a homotopy real projective $m$-space ($m \geqq 72$), not diffeomorphic to $RP^m$, then

$$N(M) < m^2/8+1.$$

Again this bound is best possible, as we exhibit an exotic homotopy real projective $m$-space (whose universal covering space is actually $\Sigma_0^{8k+1}$, $m = 8k+1$) with degree of symmetry $m^2/8+7/8$.

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Let us comment briefly on our approach. Suppose $M$ is a homotopy $RP^m$ with $N(M) \geqq m^2/8+1$ and let $G$ be a compact connected Lie group of dimension $N(M)$ acting effectively and differentiably on $M$. Then the action of $G$ on $M$ can be lifted to an effective and differentiable action of a covering group $\tilde{G}$ of $G$ on the universal covering space $\tilde{M}$ of $M$ [9]. Now $\tilde{M}$ is a homotopy $m$-sphere and

$$N(\tilde{M}) \geqq \dim \tilde{G} = \dim G = N(M) \geqq m^2/8+1.$$

Hence by the previously mentioned result of W. Y. Hsiang, $\tilde{M}$ must be a standard sphere. However, although $M$ is covered by the standard sphere $S^m$ the deck involution on $S^m$ could conceivably be exotic, and hence $M$ might be an exotic homotopy $RP^m$. Our approach, then, is to establish a structure theorem directly for actions of compact Lie groups on homotopy $RP^m$'s. We show that if $M^m$ is a homotopy $RP^m$ admitting an effective differentiable action of a compact connected Lie group with $S^k$ $(k \neq 1, 3)$ as principal orbit, then $M$ is diffeomorphic to $RP^m$. The proof of the structure theorem seems to hinge on the fact that the Whitehead group of the group $Z_2$ is trivial. Our structure theorem is analogous to one for homotopy spheres used by W. Y. Hsiang in [6] and with it we are able to establish our bound for the degree of symmetry of exotic $RP^m$'s.

For the sake of clarity we collect some preliminaries in the next two sections.

1. **Preliminaries.** Let $G$ be a compact connected Lie group acting smoothly on a compact connected manifold $M$ such that the fixed-point set $F(G, M)$ is non-empty. Choose a base point $x_0 \in F(G, M)$, let $\tilde{M}$ be the universal covering space of $M$ represented as homotopy classes of paths in $M$ starting at $x_0$. Then there is a natural smooth action of $G$ on $\tilde{M}$ so that the projection map $p: \tilde{M} \to M$ is equivariant. It follows that for $y \in \tilde{M}$ and $x = p(y) \in M$, we always have $G_y \subset G_x$. But we can say more. In fact, $G_y$ is normal in $G_x$ and $G_x/G_y$ is isomorphic to a subgroup of the fundamental group $\pi_1 = \pi_1(M, x_0)$ of $M$. For if $g \in G_x$, then $p(gy) = gp(y) = gx = x = p(y)$. Hence there is a unique $\alpha_g \in \pi_1$ such that $gy = y\alpha_g$. (We write the action of $\pi_1$ on $\tilde{M}$ on the right.) Then $\theta: g \mapsto \alpha_g$ is a homomorphism of $G_x$ into $\pi_1$ with Ker $\theta = G_y$. In particular, as $G_x$ is compact and $\pi_1$ discrete, $G_x/G_y$ is finite. We can decide when $G_x = G_y$ as follows. Since $G$ is assumed to be connected, it is easily seen that the lifted action of $G$ on $\tilde{M}$ commutes with the action of $\pi_1$. Hence $\pi_1$ acts on the orbit space $\tilde{M}/G$ and the orbit space of this action can be identified with $M/G$. Now let $y \in \tilde{M}$ and $[y] \in \tilde{M}/G$ its orbit. It is easily seen that

$$(\pi_1)_{[y]} = \theta(G_x/G_y).$$

Therefore $G_y = G_x$ iff $[y] \in \tilde{M}/G$ is *free* under the action of $\pi_1$, and this would always be the case if $G_x$ is *connected.* We will be interested in applying this in two extreme cases:

(1.1) *If the principal isotropy subgroup type* $(H)$ *of* $(G, M)$ *is connected, then the action* $(G, \tilde{M})$ *also has* $(H)$ *as principal isotropy subgroup type.*

(1.2) *Since* $G$ *is connected,* $F(G, \tilde{M}) = p^{-1}(F(G, M))$.

2. **More preliminaries.** Let $(Z_2, X)$ be a smooth involution where $X$ is a contractible $n$-manifold with boundary $\partial X$ such that $(Z_2, \partial X)$ is free. Then

(2.1) *The fixed-point set $F(Z_2, X)$ is precisely a single point in the interior of $X$.*

(2.2) *The orbit space $X/Z_2$ is contractible.*

(2.3) *Let $D^n \subset \operatorname{Int} X$ be a disk around the fixed-point. Then the inclusion*

$$RP^{n-1} = \partial D^n/Z_2 \to X - \operatorname{Int} D^n/Z_2$$

*is a homotopy equivalence.*

(2.4) *$\partial X/Z_2$ is a manifold having the same integral cohomology as $RP^{n-1}$. Moreover, if $\partial X$ is simply connected and $n \geqq 6$, then the action $(Z_2, \partial X)$ is equivalent to the standard antipodal involution $(Z_2, S^{n-1})$, and hence $\partial X/Z_2$ is diffeomorphic to $RP^{n-1}$.*

**Proof.** By P. A. Smith theory $F = F(Z_2, X)$ is acyclic over $Z_2$. Since $(Z_2, \partial X)$ is free, $F$ is a manifold without boundary. It follows that $F$ can only be a single point, say $x_0 \in \operatorname{Int} X$. Let $D^n \subset \operatorname{Int} X$ be a disk around it. Consider the inclusion $\partial D^n \to X - \operatorname{Int} D^n$. From the exact triangle

$$\begin{tikzcd}
H_*(\partial D^n) \arrow[rr] & & H_*(X - \operatorname{Int} D^n) \arrow[dl] \\
& H_*(X - \operatorname{Int} D^n, \partial D^n) \arrow[ul] \arrow[d, equal] & \\
& H_*(X, D^n) = 0 &
\end{tikzcd}$$

we see that $H_*(\partial D^n) \to H_*(X - \operatorname{Int} D^n)$ is an isomorphism. Also $X - \operatorname{Int} D^n$ is simply connected by the Van Kampen theorem. It follows (e.g. by a spectral sequence argument) that both

$$H_*(\partial D^n/Z_2) \to H_*(X - \operatorname{Int} D^n/Z_2)$$

and

$$\pi_1(\partial D^n/Z_2) \to \pi_1(X - \operatorname{Int} D^n/Z_2)$$

are isomorphisms. Therefore by Whitehead's theorem,

$$RP^{n-1} = \partial D^n/Z_2 \to X - \operatorname{Int} D^n/Z_2$$

is a homotopy equivalence.

Now consider the other end $\partial X$ of $X - \operatorname{Int} D^n$. We have the exact triangle

$$\begin{tikzcd}
H_*(\partial X) \arrow[rr] & & H_*(X - \operatorname{Int} D^n) \arrow[dl] \\
& H_*(X - \operatorname{Int} D^n, \partial X) \arrow[ul] &
\end{tikzcd}$$

But Poincaré duality gives

$$H_*(X-\mathrm{Int}\ D^n, \partial X) = H^*(X-\mathrm{Int}\ D^n, \partial D^n) = H^*(X, D^n) = 0.$$

Thus again $H_*(\partial X) \to H_*(X-\mathrm{Int}\ D^n)$ and hence $H_*(\partial X/Z_2) \to H_*(X-\mathrm{Int}\ D^n/Z_2)$ are isomorphisms. Thus $H_*(\partial X/Z_2) = H_*(RP^{n-1})$. If $\partial X$ is simply connected, then

$$\pi_1(\partial X/Z_2) \to \pi_1(X-\mathrm{Int}\ D^n/Z_2)$$

would also be an isomorphism and hence the inclusion $\partial X/Z_2 \to X-\mathrm{Int}\ D^n/Z_2$ is also a homotopy equivalence. But this means $X-\mathrm{Int}\ D^n/Z_2$ is an $h$-cobordism between $\partial X/Z_2$ and $RP^{n-1}$. As $\mathrm{Wh}\,(Z_2)=0$, we conclude by the $s$-cobordism theorem that $X-\mathrm{Int}\ D^n/Z_2$ is diffeomorphic to $RP^{n-1}\times I$ when $n \geqq 6$. This completes the proof of (2.1) through (2.4).

3. **The Structure Theorem.** Let $(G, M)$ be a smooth action of a compact connected Lie group on a compact connected manifold. Assume that all the orbits of the action are of uniform dimension. Then the connectedness of $M$ implies that all the identity components $G_x^0$ of the isotropy subgroups $G_x$ are conjugate, say of the same type $(H)$. Let $P=F(H, M)$ be the fixed-point set of $H$, $N=N(H, G)$ the normalizer of $H$ in $G$ and $K=N/H$ the quotient group. Then $P$ is a submanifold of $M$ invariant under $N$, so we have a natural action of $K$ on $P$. On the other hand, there is the natural right translation of $K$ on the homogeneous space $G/H$. Therefore we can form the space $G/H \times_K P$. This is a smooth manifold (since $K$ acts freely on $G/H$) on which $G$ acts naturally via left translations on $G/H$. It is easily seen that

$$G/H\times_K P \to M, \qquad [gH, x] \mapsto gx$$

is a diffeomorphism identifying $M$ with $G/H \times_K P$ as $G$-manifolds. With this representation of $M$, for a point $m=gx$ with $g \in G$, $x \in P$, let $K_x$ be the isotropy subgroup of $x$ in the action $(K, P)$, $\tilde{K}_x \subset N$ the pull-back of $K_x$ under the projection $N \to K$. Then

$$G_{gx} = g\tilde{K}_x g^{-1}.$$

Thus $G_{gx}$ is of type $H$ iff $x \in P$ is free under $K$. For example if all orbits of $G$ on $M$ are actually of the same type $H$, then $K$ acts freely on $P$ and we get $M$ fibered over $P/K$ with fiber $G/H$ and group $K$, an observation due to A. Borel. The above situation, where each $K_x$ is permitted to be a finite group, is discussed in [2].

We shall need a slightly more general case than the above. Namely we assume $G$ has fixed-point set $F=F(G, M)$ but all other orbits are of the same dimension. Let $(H)$ be the principal orbit type. We assume that $G/H=S^k$ is a sphere of dimension $k\neq 1, 3$. Let $U$ be an equivariant tubular neighborhood of $F$. Then $G$ acts on $M-\mathrm{Int}\ U$ with uniform dimensional orbits and we can apply the above to represent $M-\mathrm{Int}\ U$ by $G/H\times_K P$, where $P=F(H, M-\mathrm{Int}\ U)$. Now we put $U$ back in. We have $U \to F$ a $D^p$ bundle over $F$ with group $G$ acting on $D^p$ via normal representation, where $p$ is the codimension of $F$ in $M$. Thus $(G, D^p)$ is a linear action with principal orbit type $G/H=S^k$. According to [4, Lemma 1, p. 425], if $k\neq 1, 3$, then $p=k+1$ and the linear action of $G$ on the unit sphere $S^{p-1}=S^k$ is just the natural translation of $G$ on $G/H=S^k$. Now on $\partial U$, the diagram

$$\begin{tikzcd} G/H\times_K\partial P = S^k\times_K\partial P \arrow[r] \arrow[d] & \partial U \arrow[d] \\ \partial P/K \arrow[r] & F \end{tikzcd}$$

shows that the sphere bundle $\partial U\to F$ has $\partial P\to\partial P/K$ as its associated principal bundle. It follows that the disk bundle $U\to F$ can be identified with

$$D^{k+1}\times_K\partial P\to\partial P/K,$$

and we may represent $M$ as

$$M = S^k\times_K P\cup D^{k+1}\times_K\partial P,$$

identified along $S^k\times_K\partial P$. Notice that from this representation, the orbit space can be given by

$$M/G = P/K\cup[0, 1]\times\partial P/K,$$

with $(1)\times\partial P/K\subset[0, 1]\times\partial P/K$ attached to $\partial P/K\subset P/K$. (For more details, see [8].)

We are ready to prove the structure theorem.

STRUCTURE THEOREM. *Let $G$ be a compact connected Lie group acting smoothly on a homotopy real projective $m$-space $M$. Suppose the principal orbit of the action is a $k$-sphere $S^k$ with $k\neq 1, 3$. Then*

(i) *There are precisely three types of orbits. Namely the sphere $S^k$, the real projective $k$-space $RP^k$, and fixed-points. Moreover there is exactly one orbit of type $RP^k$.*

(ii) *The fixed-point set $F=F(G, M)$ has the integral homology of $RP^n$, where $n=m-k-1$. If $\pi_1(F)=Z_2$ and $n\geqq 5$, then $F$ is actually diffeomorphic to $RP^n$.*

(iii) *Let $Y=M/G$ be the orbit space of the action $(G, M)$, $y_0\in M/G$ the orbit of type $RP^k$ (see (i)) and $V$ a neighborhood of $y_0$ in $M/G$. Then $Y$ is a contractible space and $(Y, y_0)$ a relative manifold of dimension $n+1=m-k$. More precisely, $Y-\operatorname{Int} V$ is a smooth manifold with boundary $\partial(Y-\operatorname{Int} V)=F\cup RP^n$, the inclusion $RP^n\to Y-\operatorname{Int} V$ is a homotopy equivalence and $V$ is a cone over $RP^n$ with $y_0$ as vertex.*

(iv) *If $m\geqq 5$, then $M$ is diffeomorphic to $RP^m$.*

**Proof.** (i), (ii), and (iii) are relatively simple. First of all, there must be a fixed-point. For if not, then since $M$ has no rational cohomology, Theorem 4 of [4] applies to assert that the orbits of the action have uniform dimension. As mentioned above, we can write $M$ as

$$M = G/H\times_K P = S^k\times_K P.$$

Consider the homotopy exact sequence

$$\cdots \to \pi_{k+1}(M) \to \pi_k(K) \to \pi_k(S^k \times P) \to \pi_k(M) \to \cdots.$$

Since $K$ acts freely on $S^k$, it has rank 1 and so $\pi_k(K)$ is finite, for $k \neq 1, 3$ (the identity component $K^0$ is either $S^1$ or $S^3$). As $\pi_k(S^k \times P)$ is infinite, we have $\pi_k(M) \neq 0$. This is impossible since $k < m$ and $M$ is a homotopy $RP^m$.

Thus we can apply the results of §1. Let $\tilde{M}$ be the universal covering of $M$ and $(G, \tilde{M})$ the lifted action. Since the principal orbit $G/H = S^k$ is simply connected, $H$ is connected, and (1.1) implies that $(G, \tilde{M})$ also has principal orbit $S^k$. Of course $\tilde{M}$ is a homotopy sphere and actions of this kind are completely understood by the work of W. C. Hsiang and W. Y. Hsiang [4]. For example there are only orbits of type $(H)$ and fixed-points. Thus the only possible orbit types of $(G, M)$ are $(H)$, $(G)$ and $(G_x)$ with $G_x/H = \pi_1(M) = Z_2$, i.e. the only possible orbits are spheres $S^k$, fixed-points, and projective spaces $RP^k$.

Let $X = \tilde{M}/G$ be the orbit space of $(G, \tilde{M})$. We know that $\pi_1(M) = Z_2$ acts on $X$ with orbit space $X/Z_2 = M/G = Y$, and a point $[x] = y_0 \in Y$ has orbit $RP^k$ iff $x \in X$ is a fixed-point of $Z_2$. Now by [4], $X$ is a contractible manifold of dimension $m-k = n+1$ with $\partial X = F(G, \tilde{M})$. By (1.2), $F(G, \tilde{M}) = p^{-1}(F(G, M))$, hence $Z_2$ acts freely on $\partial X$. We can then apply (2.1) to conclude that $F(Z_2, X)$ is precisely a single point $x_0 \in X$. Hence there is exactly one orbit of type $RP^k$.

Let $D^{n+1}$ be a disk around $x_0$. Then $V = D^{n+1}/Z_2 \subset Y$ is a neighborhood of $y_0$ in $Y$ and $V$ is a cone over $RP^n = \partial D^{n+1}/Z_2$ with $y_0$ as vertex. By (2.2), $Y$ is a contractible space and $Y - \operatorname{Int} V$ is a manifold with boundary $\partial(Y - \operatorname{Int} V) = \partial X/Z_2 \cup RP^n = F \cup RP^n$. Statements (ii) and (iii) now follow from (2.2) through (2.4).

Now we proceed to prove (iv), which is our main concern. Let $U$ be an equivariant tubular neighborhood of $F$. As we have seen above, we have

$$M - \operatorname{Int} U = S^k \times_K P, \qquad U = D^{k+1} \times_K \partial P,$$

and

$$M = S^k \times_K P \cup D^{k+1} \times_K \partial P.$$

We know the orbit of type $RP^k$ is determined by a point $x_0 \in P$ such that $K_{x_0} = Z_2$, and this point projects, of course, to the point $y_0 \in Y$ in the orbit space. Now by the slice theorem, the orbit of $x_0$ in $P$ has a tubular neighborhood $W$ in $P$ given by $W = K \times_{Z_2} D^{n+1}$. The singular orbit $RP^k$ has therefore a tubular neighborhood $\hat{W}$ in $M$ given by

$$\hat{W} = S^k \times_K (K \times_{Z_2} D^{n+1}) = S^k \times_{Z_2} D^{n+1}$$

and $\hat{W}$ projects to $\hat{W}/G = D^{n+1}/Z_2 = V$, the cone neighborhood of $y_0 \in Y$. Hence $Y - \operatorname{Int} V$ is covered by

$$M - \operatorname{Int} \hat{W} = S^k \times_K (P - \operatorname{Int} W) \cup D^{k+1} \times_K \partial P.$$

Now consider the principal $K$-bundle

$$\xi \colon P - \operatorname{Int} W \to (P - \operatorname{Int} W/K) \sim (Y - \operatorname{Int} V).$$

Here "$\sim$" means of the same homotopy type, as $Y-\text{Int } V$ is obtained from $P-\text{Int } W/K$ by attaching a collar $[0, 1]\times\partial P/K$. Now $\partial V=RP^n\subset Y-\text{Int } V$ is a homotopy equivalence by (iii). Over $\partial V$, the bundle $\xi$ is given by

$$\xi|\partial V\colon \partial W = K\times_{Z_2}S^n \to S^n/Z_2 = RP^n = \partial V.$$

This means the bundle $\xi|\partial V$ has a $Z_2$-reduction. Since $\partial V\to Y-\text{Int } V$ is a homotopy equivalence, there is a unique $Z_2$-reduction of $\xi$ which extends the reduction of $\xi|\partial V$. That is, there is a principal $Z_2$-bundle $Q\to Y-\text{Int } V$, with $Q$ a smooth manifold with boundary $\partial Q=S^n\cup B$, over $\partial V=RP^n$ and $F\subset Y-\text{Int } V$ respectively, and an identification $K\times_{Z_2}Q=P-\text{Int } W$ compatible with the identification $K\times_{Z_2}S^n=\partial W$. We can therefore represent $P$ as

$$P = K\times_{Z_2}Q\cup K\times_{Z_2}D^{n+1} = K\times_{Z_2}R,$$

where $R=Q\cup D^{n+1}$ is obtained from $Q$ by adjoining a disk along $S^n$. Notice $R$ has a smooth action of $Z_2$ with exactly a single fixed-point. Now we can write $M$ as

$$\begin{aligned} M &= S^k\times_K P\cup D^{k+1}\times_K\partial P \\ &= S^k\times_K(K\times_{Z_2}R)\cup D^{k+1}\times_K(K\times_{Z_2}\partial R) \\ &= S^k\times_{Z_2}R\cup D^{k+1}\times_{Z_2}\partial R = \partial(D^{k+1}\times R)/Z_2. \end{aligned}$$

The diagram

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & Q \arrow[d] \\ RP^n \arrow[r] & Y-\text{Int } V \end{tikzcd}$$

and the fact that $RP^n\subset Y-\text{Int } V$ is a homotopy equivalence implies that $S^n\subset Q$ is a homotopy equivalence. Hence $R=Q\cup D^{n+1}$ is contractible and so is $D^{k+1}\times R$. Thus we have a smooth action of $Z_2$ on a contractible manifold, which is free on the boundary. Since $\partial(D^{k+1}\times R)$ is necessarily simply connected (e.g. by the Van Kampen theorem), we conclude from (2.4) that $M$ is diffeomorphic to $RP^m$ when $m\geqq 5$. This completes the proof of the Structure Theorem.

4. **Main results.** With the Structure Theorem we are now able to prove the following theorem:

THEOREM. *If $M^m$ is an exotic $HRP^m$ ($m\geqq 72$), then*

$$N(M) < m^2/8+1.$$

**Proof.** Suppose $M^m$ is a $HRP^m$ with $N(M)\geqq m^2/8+1$. We proceed to show that $M^m$ is standard. Let $G$ be a compact connected Lie group of dimension $\geqq m^2/8+1$ acting effectively and differentiably on $M$; let $H$ be a principal isotropy subgroup of the action. Let $\Sigma^m$ be the universal covering space of $M^m$. The action of $G$ on $M$ lifts to an effective differentiable action of a covering group $\tilde{G}$ of $G$ on $\Sigma^m$. Now $\dim \tilde{G}=\dim G$.

*Case* A. $\dim G/H = m$.

In this case the action of $G$ on $M$, and hence, $\tilde{G}$ on $\Sigma^m$ is transitive. However, all transitive effective actions of compact connected Lie groups on spheres have been classified [11], [1], [13]. In all cases $\Sigma^m$ is diffeomorphic to $S^m$. We obtain the principal $Z_2$-bundle

$$\begin{array}{ccc} S^m & = & \tilde{G}_1/\tilde{H}_1 \\ & & \Big\downarrow \tilde{H}_0/\tilde{H}_1 \\ M^m & = & \tilde{G}_1/\tilde{H}_0 \end{array}$$

where $\tilde{G}_1 = SO(m+1), SU((m+1)/2)$ or $Sp((m+1)/4)$ and $\tilde{H}_1 = SO(m), SU((m-1)/2)$ or $Sp((m-3)/4)$ (respectively) standardly imbedded. Now the free action of $\tilde{H}_0/\tilde{H}_1$ on $S^m$ is the restriction of the orthogonal action of $N(\tilde{H}_1, \tilde{G}_1)/\tilde{H}_1$ on $S^m$. It follows that $M^m$ is diffeomorphic to $RP^m$.

*Case* B. $\dim G/H = m-1$.

Now $\tilde{G}$ acts on $\Sigma^m$ with principal orbit also of codimension one. H. C. Wang has classified the groups acting effectively on spheres with principal orbit of codimension one [14], [6, p. 355]. It follows that $\tilde{G}$ is locally isomorphic to a subgroup of one of the following:

(i) $SO(t) \times SO(2)$, $m = 2t-1$,

(ii) $U(t) \times U(2)$, $m = 4t-1$,

(iii) $Sp(t) \times Sp(2)$, $m = 8t-1$.

However in all cases,

$$\dim G = \dim \tilde{G} < m^2/8+1,$$

which is a contradiction.

*Case* C. $\dim G/H \leqq m-2$.

Now

$$\dim G \geqq m^2/8+1 \geqq ((m^2+8)/8(m-2)) \dim G/H.$$

Hence

$$\dim G \geqq r \dim G/H \tag{4.1}$$

where

$$r = (m^2+8)/8(m-2). \tag{4.2}$$

It follows from results of W. Y. Hsiang [7, Propositions 1 and 2] that there exists a simple connected normal subgroup $G_1$ of $G$ such that

$$\dim G_1 \geqq r[m(G_1)], \tag{4.3}$$

and

$$\dim G_1 + \dim N(H_1, G_1)/H_1 \geqq r \dim G_1/H_1, \tag{4.4}$$

where

(i) $m(G_1)$ = smallest positive codimension of proper subgroups of $G_1$,

(ii) $H_1 = G_1 \cap H$,

(iii) $N(H_1, G_1)$ = normalizer of $H_1$ in $G_1$.

Since $m$, and consequently $r$, are large it follows from (4.3) and a knowledge of the maximal dimensions of proper subgroups of the exceptional Lie groups [8] that $G_1$ is a classical Lie group. Therefore $G_1$ is locally isomorphic to $SO(n)$, $SU(n)$, or $Sp(n)$. We consider only the case where $G_1$ is locally isomorphic to $SO(n)$ as the other two cases are similar and less difficult.

Since $m(SO(n)) = n-1$, it follows from (4.3) that

$$n(n-1)/2 = \dim SO(n) \geqq ((m^2+8)/8(m-2))(n-1).$$

and

$$n > m/4 \tag{4.5}$$

after simplifying. Since we are assuming that $m \geqq 72$ it follows from (4.5) that

$$(n-1)^2 \geqq 4m. \tag{4.6}$$

Now $M$ has zero first rational Pontrjagin class and with (4.6) we may conclude that $H_1^0$, the identity component of $H_1$, is locally isomorphic to $SO(n-k)$, $1 \leqq k \leqq n$ [5, Theorem 2.1]. We show $k \leqq 4$. By the underlying hypothesis of Case C, $\dim G/H \leqq m-2$. Hence

$$m-2 \geqq \dim G_1/H_1 = \dim SO(n) - \dim SO(n-k)$$

and consequently

$$m \geqq nk - k(k+1)/2 + 2. \tag{4.7}$$

Combining (4.5) with (4.7) yields

$$4n > nk - k(k+1)/2 + 2 \tag{4.8}$$

and since $n$ is large it follows that $k \leqq 4$. For $k = 3, 4$ we may employ (4.4) directly to obtain an easy contradiction. Consider then $k = 2$. From (4.4) we obtain

$$\dim SO(n) + \dim SO(2) \geqq ((m^2+8)/8(m-2)) \dim G_1/H_1$$

or

$$n(n-1)/2 + 1 \geqq ((m^2+8)/8(m-2))(2n-3). \tag{4.9}$$

Now it follows from (4.5) and (4.7) that $m/4 < n \leqq (m+1)/2$. Let

$$f(n) = n(n-1)/2 + 1 - r(2n-3)$$

where, as we recall, $r = (m^2+8)/8(m-2)$. We proceed to reach a contradiction to (4.9) by showing that $f(n) < 0$ for $m/4 < n \leqq (m+1)/2$.

It is easily checked that $f((m+1)/2)<0$ and $f(m/4)<0$. Since $f(n)$ is concave upward, the contention follows.

We are left with $k=1$. The almost effective action of $SO(n)$ on $M$ may be lifted to an almost effective action of $SO(n)$ on $\tilde{M}$, the universal covering space of $M$. Since $\tilde{M}$ is a homotopy sphere and since $(n-1)^2 \geqq 4m$, it follows from a result of the Hsiang brothers [5, Theorem 3.1] that $SO(n)$ acts on $\tilde{M}$ with principal orbit $S^{n-1}$ and with a nonempty fixed-point set. From our considerations in the last section we know now that $SO(n)$ acts on $M$ with a fixed-point. Now the principal orbit of the action of $SO(n)$ on $M$ is either $S^{n-1}$ or $RP^{n-1}$. However, since $SO(n)$ acts linearly in a neighborhood of a fixed-point of $M$, it follows that the principal orbit is actually $S^{n-1}$. Finally the Structure Theorem applies and $M$ is diffeomorphic to $RP^m$.

EXAMPLE. We employ the Hirzebruch-Brieskorn construction of homotopy spheres to show that this theorem is best possible. Let $\tilde{M}^{8k+1}$ be the intersection of the complex hypersurfaces

$$Z_1^2+Z_2^2+\cdots+Z_{4k+1}^2+Z_{4k+2}^2 = 0$$

and

$$|Z_1|^2+|Z_2|^2+\cdots+|Z_{4k+2}|^2 = 1$$

in $C^{4k+2}$. It is known that $\tilde{M}^{8k+1}$ is diffeomorphic to the Kervaire sphere $\Sigma_0^{8k+1}$ (see, for example, [12, pp. 55–56]). It is easily seen that a standard subgroup $O(4k+1)$ in $U(4k+2)$ leaves $\tilde{M}^{8k+1}$ invariant in $C^{4k+2}$. In addition there is an $S^1$ action on $\tilde{M}^{8k+1}$ which commutes with the $O(4k+1)$ action. Namely, if $w=e^{i\theta}\in S^1$,

$$w(Z_1, Z_2, \ldots, Z_{4k+1}, Z_{4k+2}) = (e^{3i\theta}Z_1, e^{3i\theta}Z_2, \ldots, e^{3i\theta}Z_{4k+1}, e^{2i\theta}Z_{4k+2}).$$

We obtain an almost effective action of $O(4k+1)\times S^1$ on $\tilde{M}$. Consider the involution $t=e^{i\pi}=-1$ in $S^1$. Clearly $t$ acts freely on $\tilde{M}$. Let $A$ be the group of order two generated by $t$ and let $M^{8k+1}=\tilde{M}^{8k+1}/A$. Now $M$ is a homotopy $RP^{8k+1}$ and, since its universal covering space $\tilde{M}$ is an exotic sphere, $M$ is not diffeomorphic to $RP^{8k+1}$. On the other hand, the almost effective action of $O(4k+1)\times S^1$ on $\tilde{M}$ can be pushed down to an almost effective action on $M$. Hence if $m=8k+1$,

$$N(M) \geqq \dim\,[O(4k+1)\times S^1] = 8k^2+2k+1 = m^2/8+7/8.$$

REMARKS. 1. It is possible, by a somewhat more careful argument, to lower the bound in the preceding theorem to $m \geqq 59$.

2. The statements and proofs of the above results and of the example can be modified so as to apply to $Z_3$ actions on spheres, since the Whitehead group of $Z_3$ is zero. In particular, if $M^m$ is an exotic lens space arising from such an action, $N(M^m)<m^2/8+1$. On the other hand, it follows from the main lemma in [10] that, for a standard simple lens space $L_p^m$, with $p\neq 2$, $N(L_p^m)=m^2/4+m/2+1/4$.

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