QUOTIENT SHEAVES AND VALUATION RINGS

BY

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Abstract. In this paper a construction of a quotient sheaf of a sheaf of rings is given. This construction is analogous to the Utumi ring of quotients of a ring. For a valuation ring V , a sheaf of rings corresponding to V is introduced and its quotient sheaf is computed. It is shown that this quotient sheaf corresponds to the completion of V in case V is discrete rank one and that V is maximal if and only if its associated sheaf of rings is its own quotient sheaf.

Introduction. The first section of this paper defines and discusses dense extensions of modules over a sheaf of rings and the quotient sheaf of a sheaf of rings. Given a sheaf A of rings, a maximal dense extension of A as an A -Module is seen to be naturally a sheaf of rings extending A . We call this new sheaf the quotient sheaf of A and denote it as $Q(A)$. This idea of the quotient sheaf is thus analogous to the Utumi (maximal) ring of quotients of a ring. Gabriel's thesis [2] is used to give other characterizations of $Q(A)$.

In the second section we start with an arbitrary valuation ring V and form a corresponding sheaf of rings A_V . This is not the usual structure sheaf for V ; in fact its underlying topological space is the set of all divisorial ideals of V rather than the set of prime ideals. We compute the quotient sheaf $Q(A_V)$. From this computation we see that if V is discrete rank one and W is the completion of V then $Q(A_V) = A_W$, so that in this case the quotient sheaf gives exactly the same information as the classical completion. On the other hand if V is any valuation ring, $A_V = Q(A_V)$ if and only if V has no proper immediate extensions.

Throughout this paper ring means associative ring with identity and all ring homomorphisms and modules are unital.

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1. Dense extensions and the quotient sheaf. Throughout this section we let A be a sheaf of rings over a topological space X and let $\mathcal{M}$ be the category of (right-) A -Modules. Thus $\mathcal{M}$ is an abelian category with injective envelopes. For M and N in $\mathcal{M}$, $M \leq N$ will denote that M is a subobject of N .

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DEFINITION. If $M \leq N$ in $\mathcal{M}$ we say that M is dense in N and write $M \blacktriangle N$ in case for all L in $\mathcal{M}$ such that $M \leq L \leq N$, $\text{Hom}_{\mathcal{M}}(L/M, N) = 0$.

This notion of dense extensions corresponds to the notion of rational extensions of modules over a ring (e.g. [1, p. 58]), and the following two results are exact analogues of the ring situation. However we do not have a characterization of density in terms of elements, so proofs somewhat different than those in [1] are needed.

PROPOSITION 1.1. Let L, M and N in $\mathcal{M}$. (a) If $M \blacktriangle N$ then M is essential in N .

(b) If $M \blacktriangle N$ and $L \leq N$ then $(M \cap L) \blacktriangle L$.

(c) If $L \blacktriangle M$ and $M \blacktriangle N$ then $L \blacktriangle N$.

(d) If $M \blacktriangle N$ and $L \blacktriangle N$ then $(M \cap L) \blacktriangle N$.

(e) If $M \blacktriangle N$ and $M \leq L \leq N$ then $M \blacktriangle L$ and $L \blacktriangle N$.

(f) If $M \leq N \leq I$ where I is an injective envelope of N then $M \blacktriangle N$ if and only if for any $f \in \text{Hom}(I, I)$, $f(M) = 0$ implies $f(N) = 0$.

Proof. (a) If $0 \neq K \leq N$ and $(K \cap M) = 0$ then $\text{Hom}((M+K)/M, N) \cong \text{Hom}(K, N) \neq 0$ which contradicts $M \blacktriangle N$.

(b) Let $(M \cap L) \leq K \leq L$. Then $(M \cap L) = (K \cap M)$. So

$$\begin{aligned} \text{Hom}(K/(M \cap L), L) &= \text{Hom}(K/(K \cap M), L) \cong \text{Hom}((K+M)/M, L) \\ &\leq \text{Hom}((K+M)/M, N) = 0. \end{aligned}$$

(e) is easily checked.

(f) Suppose $M \blacktriangle N$. Let $f \in \text{Hom}(I, I)$ such that $f(M) = 0$. If $L = f^{-1}(N) \cap N$ then $M \leq L \leq N$ and $f|_L \in \text{Hom}(L, N)$ such that $f|_L(M) = 0$. So $f|_L = 0$, i.e. $0 = f(L) = N \cap f(N)$. Hence since N is essential in I , $f(N) = 0$.

For the converse let $M \leq L \leq N$ and $f \in \text{Hom}(L, N)$ such that $f(M) = 0$. Since I is injective there exists $g \in \text{Hom}(I, I)$ extending f . Thus $g(M) = 0$. It follows from our hypothesis that $g(N) = 0$, so $f(L) = 0$. Hence $M \blacktriangle N$.

(c) follows easily from (f) since $\mathcal{M}$ has injective envelopes.

(d) follows immediately from (b) and (c). $\square$

THEOREM 1.2. Every M in $\mathcal{M}$ has a maximal dense extension $Q(M)$ in $\mathcal{M}$, i.e. an extension $Q(M)$ satisfying the following two equivalent conditions:

(1) $M \blacktriangle Q(M)$ and if $Q(M) \blacktriangle N$ then $Q(M) = N$.

(2) $M \blacktriangle Q(M)$ and if $M \blacktriangle N$ then there exists a unique monomorphism from N to $Q(M)$ extending the identity on M .

Thus $Q(M)$ is unique up to unique isomorphism extending the identity of M . Moreover, if I is injective envelope of M such that $M \leq Q(M) \leq I$ and if $M \leq N \leq I$ then $M \blacktriangle N$ if and only if $N \leq Q(M)$.

Proof. Let I be an injective envelope for M .

Let $\Omega = \{f \in \text{Hom}(I, I) : f(M) = 0\}$.

Let $Q(M) = \bigcap \{\text{Kernel}(f) : f \in \Omega\}$.

Claim 1. $M \blacktriangle Q(M)$ and if $N \leq I$ such that $M \blacktriangle N$ then $N \leq Q(M)$: This follows from 1.1(f).

Claim 2. $Q(M)$ satisfies (2): Let $M \blacktriangle N$ and let J be an injective envelope for M such that $N \leq J$. Then there is an isomorphism ϕ from J to I which extends the identity on M . One can easily see that $M \blacktriangle \phi(N)$. So $\phi(N) \leq Q(M)$. Hence ϕ restricted gives a monomorphism τ from N into $Q(M)$ which extends the identity on M . Now suppose σ is another monomorphism from N to $Q(M)$ which extends the identity on M . Then using the fact that $M \blacktriangle Q(M)$ one sees that the monomorphism $\sigma\tau^{-1}: \tau(N) \rightarrow Q(M)$ must be the inclusion $\tau(N) \leq Q(M)$. Hence $\sigma = \tau$.

Claim 3. $Q(M)$ satisfies (1): Let $Q(M) \blacktriangle N$ and let J be an injective envelope for N . There exists an isomorphism ϕ from J to I which extends the identity on $Q(M)$. One can see easily that $M \blacktriangle \phi(N)$. So $\phi(N) \leq Q(M)$. Hence $N = Q(M)$.

Claim 4: (1) and (2) are equivalent. Let $Q \in \mathcal{M}$. We will say Q satisfies (1) (respectively (2)) if Q satisfies the conditions on $Q(M)$ in (1) (resp. (2)).

Suppose Q satisfies (1). Let J be an injective envelope of M such that $Q \leq J$. Then there exists an isomorphism ϕ from J to I such that ϕ extends the identity on M . One checks that $M \blacktriangle \phi(Q)$, so $\phi(Q) \blacktriangle Q(M)$ and so $Q = \phi^{-1}(Q(M))$ and ϕ restricted is an isomorphism from Q to $Q(M)$ which extends the identity on M . Since $Q(M)$ satisfies (2), so does Q .

Now suppose Q satisfies (2). If $Q \blacktriangle N$ then $M \blacktriangle N$ so there exists a unique monomorphism $\phi: N \rightarrow Q$ which extends the identity on M . If i is the inclusion $Q \leq N$ then by the uniqueness of ϕ , $\phi = \phi i \phi$. So since ϕ is a monomorphism, $i\phi$ is the identity on N . Hence i is an isomorphism, i.e. $Q = N$. Thus Q satisfies (1). So we have proved Claim 4.

It is routine to show that if Q satisfies (2) then there is a unique isomorphism from Q to $Q(M)$ which extends the identity on M .

The last sentence of the theorem follows from Claim 1 and 1.1(e). $\square$

For an open subset U of X we denote the restriction of A to U by $A|_U$. Likewise for M in $\mathcal{M}$, $M|_U$ denotes the restriction of M to U . Thus $M|_U$ is an $A|_U$ -Module. We denote by M_U the subsheaf of M which restricts to $M|_U$ on U and 0 on the complement of U in X (cf. [3, pp. 139–140]).

The notation $\text{Hom}(\ , \)$ will be used both to denote A -Module homomorphisms and to denote $A|_U$ -Module homomorphisms depending on the context. Likewise depending on context $\leq$ and $\blacktriangle$ will be used to denote subobjects and dense subobjects in both the categories of A -Modules and $A|_U$ -Modules. For M and N in $\mathcal{M}$, $\mathcal{H}\text{om}(M, N)$ denotes the sheaf of abelian groups given by

$$\mathcal{H}\text{om}(M, N)(U) = \text{Hom}(M|_U, N|_U)$$

with restrictions as connecting maps (cf. [3, p. 127]).

PROPOSITION 1.3. *For M in $\mathcal{M}$ and U an open subset of X , $Q(M)|_U = Q(M|_U)$, i.e. $Q(M)|_U$ is a maximal dense extension of $M|_U$ in the category of $A|_U$ -Modules.*

Proof. It is well known that the functor $(\)|_U$ is exact, preserves essential extensions, injectives, and arbitrary intersections.

Let I be an injective envelope of M . Then $I|_U$ is an injective envelope for $M|_U$ as an $A|_U$ -Module. Let

$$\Omega = \{f \in \text{Hom}(I, I) : f(M) = 0\}$$

and

$$\Psi = \{h \in \text{Hom}(I|_U, I|_U) : h(M|_U) = 0\}.$$

Clearly if $f \in \Omega$ then $f|_U \in \Psi$. We now show that all the elements of Ψ are of this form. Let $h \in \Psi$. Extending h by zero gives a homomorphism from I_U to I ; this homomorphism annihilates M_U so induces a homomorphism h' from I_U/M_U to I . Since $M_U = (M \cap I_U)$, I_U/M_U is embedded in I/M . So since I is injective there exists $f^* \in \text{Hom}(I/M, I)$ such that f^* restricts to h' on I_U/M_U . Let $f \in \text{Hom}(I, I)$ be the composition of f^* with the natural map $I \rightarrow I/M$. Then clearly $f \in \Omega$ and one checks that $f|_U = h$. Thus $\Psi = \{f|_U : f \in \Omega\}$.

Hence

$$\begin{aligned} Q(M)|_U &= \left(\bigcap \{ \text{Kernel}(f) : f \in \Omega \} \right) \Big|_U = \bigcap \{ \text{Kernel}(f)|_U : f \in \Omega \} \\ &= \bigcap \{ \text{Kernel}(f|_U) : f \in \Omega \} = \bigcap \{ \text{Kernel}(h) : h \in \Psi \} = Q(M|_U). \quad \square \end{aligned}$$

COROLLARY 1.4. *If $M \blacktriangle N$ in $\mathcal{M}$ and U is an open subset of X then $M|_U \blacktriangle N|_U$ as $A|_U$ -Modules.*

Proof. This follows easily from 1.2 and 1.3. $\square$

As in the case of rings and modules our main interest is in $Q(A)$ which is called the quotient sheaf of A .

LEMMA 1.5. *As sheaves of abelian groups $\mathcal{H}\text{om}(A, Q(A))$ and $\mathcal{H}\text{om}(Q(A), Q(A))$ are isomorphic via an isomorphism ϕ where for U an open subset of X and $g \in \text{Hom}(A|_U, Q(A)|_U)$, $\phi_U(g)$ is the unique element of $\text{Hom}(Q(A)|_U, Q(A)|_U)$ which restricts to g on $A|_U$.*

Proof. Let U be an open subset of X and $g \in \text{Hom}(A|_U, Q(A)|_U)$. By 1.3 and 1.2 we may assume $A|_U \leq Q(A)|_U \leq I$ where I is an $A|_U$ -injective envelope of $A|_U$ and $Q(A)|_U = \bigcap \{ \text{Kernel}(f) : f \in \text{Hom}(I, I) \text{ and } f(A|_U) = 0 \}$. Since I is injective we can extend g to g' in $\text{Hom}(I, I)$. Let $f \in \text{Hom}(I, I)$ such that $f(A|_U) = 0$, then $fg'(A|_U) = fg(A|_U) \leq f(Q(A)|_U) = 0$. Hence $g'(Q(A)|_U) \leq Q(A)|_U$. So restricting g' we have $g'' \in \text{Hom}(Q(A)|_U, Q(A)|_U)$ such that g'' restricts to g on $A|_U$. Moreover, since $A|_U \blacktriangle Q(A)|_U$, g'' is the unique homomorphism in $\text{Hom}(Q(A)|_U, Q(A)|_U)$ which extends g . So the map $g \mapsto g''$ gives a bijection $\phi_U : \text{Hom}(A|_U, Q(A)|_U) \rightarrow \text{Hom}(Q(A)|_U, Q(A)|_U)$. Using the uniqueness property it is easy to see that ϕ_U is a group homomorphism and that $\phi = \{ \phi_U : U \text{ is an open subset of } X \}$ is a sheaf map. Hence ϕ is an isomorphism from $\mathcal{H}\text{om}(A, Q(A))$ to $\mathcal{H}\text{om}(Q(A), Q(A))$. $\square$

Recall that for any M in $\mathcal{M}$, $\mathcal{H}\text{om}(A, M)$ is an A -Module and M is A -isomorphic to $\mathcal{H}\text{om}(A, M)$ via the isomorphism $\mu = \{\mu_U\}$ where for U an open subset of X and $m \in M(U)$, $\mu_U(m)$ is the multiplication-like element of $\text{Hom}(A|_U, M|_U)$ corresponding to m , i.e. for W an open subset of U and $a \in A(W)$, $(\mu_U(m))_W(a) = (\rho_{WU}(m))a$ where ρ_{WU} is the connecting map from $M(U)$ to $M(W)$.

THEOREM 1.6. *There is a unique ring-sheaf structure for $Q(A)$ which extends the A -Module structure on $Q(A)$. With this structure $Q(A)$ is clearly a sheaf of rings extending A .*

Note. To say that $Q(A)$ has a ring-sheaf structure extending the A -Module structure on $Q(A)$ means that for each open subset U of X , $Q(A)(U)$ is a ring with multiplication $Q(A)(U) \times Q(A)(U) \rightarrow Q(A)(U)$ extending the $A(U)$ -module operation $Q(A)(U) \times A(U) \rightarrow Q(A)(U)$ and with the same addition as $Q(A)(U)$ has as an $A(U)$ -module; and also that the connecting maps of $Q(A)$ as an A -Module are ring homomorphisms.

Proof of 1.6. We have that as A -Modules $Q(A) \cong \mathcal{H}\text{om}(A, Q(A))$ and by 1.5 that as sheaves of abelian groups $\mathcal{H}\text{om}(A, Q(A)) \cong \mathcal{H}\text{om}(Q(A), Q(A))$. But $\mathcal{H}\text{om}(Q(A), Q(A))$ with composition as its multiplicative operations is a sheaf of rings. If we pull this ring-sheaf structure back to $Q(A)$ it requires only a routine check of identifications to see that this ring-sheaf structure extends the A -Module structure on $Q(A)$. To see that this ring-sheaf structure is unique we need the following:

LEMMA 1.7. *If B is a sheaf of rings extending A such that with the induced A -Module structure on B , $A \blacktriangle B$, then there exists a unique ring-sheaf embedding of B into $Q(A)$ which extends the identity on A .*

Proof. We may assume $A \leq B \leq Q(A)$ in $\mathcal{M}$. For existence it suffices to show that the ring-sheaf operations on B are just the restriction of those on $Q(A)$. Let U be an open subset of X and let $r \in B(U)$. Define μ and $\nu: B|_U \rightarrow Q(A)|_U$ as follows: for W open in U and $s \in B(W)$, $\mu_W(s) = s \circ \rho_{WU}(r)$ and $\nu_W(s) = s * \rho_{WU}(r)$ where $\circ$ is the multiplication on $B(W)$ and $*$ is the multiplication on $Q(A)(W)$ and ρ_{WU} is the connecting map from $Q(A)(U)$ to $Q(A)(W)$. Then one checks that both μ and ν are $A|_U$ -homomorphisms and that they agree on $A|_U$. So since $A|_U \blacktriangle Q(A)|_U$ we have that $\mu = \nu$. Hence if $s \in B(U)$ we have $s \circ r = s * r$. Thus the A -Module monomorphism $B \leq Q(A)$ is also a ring-sheaf homomorphism; so we have existence. If σ and τ are two ring-sheaf homomorphisms from B to $Q(A)$ which extend the identity on A then $\sigma - \tau$ is an A -homomorphism from B to $Q(A)$ and is zero on A , so, since $A \blacktriangle Q(A)$, $\sigma = \tau$. $\square$

Now to complete the Proof of 1.6 we only have to note that if $Q(A)$ had another ring-sheaf structure which extended the A -Module structure on $Q(A)$ then it would follow from 1.7 that there exists an A -Module homomorphism from $Q(A)$ to $Q(A)$

which extends the identity on A but is different from the identity on $Q(A)$; this would contradict $A \blacktriangle Q(A)$. $\square$

DEFINITION. We call the sheaf of rings $Q(A)$ given in 1.6 the quotient sheaf of A .

We now turn to Gabriel's thesis [2, p. 365 ff.] to get another characterization of $Q(A)$. Let $\mathcal{C}$ be the full subcategory of $\mathcal{M}$ composed of all M in $\mathcal{M}$ having $\text{Hom}(N, A) = 0$ for all subobjects N of M . It is easy to see that $\mathcal{C}$ is a Serre subcategory of $\mathcal{M}$, i.e. if $0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0$ is an exact sequence in $\mathcal{M}$, then L is in $\mathcal{C}$ if and only if both M and N are in $\mathcal{C}$. So we can form the quotient category $\mathcal{M}/\mathcal{C}$. One recalls that $\mathcal{M}/\mathcal{C}$ has the same objects as $\mathcal{M}$ and that for M and N in $\mathcal{M}/\mathcal{C}$, $\text{Hom}_{\mathcal{M}/\mathcal{C}}(M, N) = \text{dir lim Hom}_{\mathcal{M}}(M', N/N')$ where the direct limit is over all subobjects M' of M and N' of N such that M/M' and N' are in $\mathcal{C}$. The canonical functor $T: \mathcal{M} \rightarrow \mathcal{M}/\mathcal{C}$ is given by $T(M) = M$ for all M in $\mathcal{M}$ and, for $f \in \text{Hom}_{\mathcal{M}}(M, N)$, $T(f)$ is the image of f in $\text{Hom}_{\mathcal{M}/\mathcal{C}}(M, N)$. It follows easily from [2, Corollary 1, p. 375] that T has an adjoint functor S , i.e. that $\mathcal{C}$ is a localizing subcategory of $\mathcal{M}$.

Gabriel shows that S may be defined on objects as follows: If M is in $\mathcal{M}/\mathcal{C}$ then there exists a unique subobject N of M which is maximal with respect to being in $\mathcal{C}$. If I is the injective envelope of M/N in $\mathcal{M}$ then $S(M)$ is the unique subobject of I which is an overobject of M/N and is maximal with respect to having $S(M)/(M/N)$ in $\mathcal{C}$. With such an S we have

PROPOSITION 1.8. $Q(A) = ST(A)$.

Proof. Let I be a fixed injective envelope of A in $\mathcal{M}$. We may assume $A \leq Q(A) \leq I$ as in the proof of 1.2 and $A \leq ST(A) \leq I$ as above. Since A has no nonzero subobjects in $\mathcal{C}$, $ST(A)$ is the subobject of I which is maximal such that $ST(A)/A$ is in $\mathcal{C}$. Suppose $f \in \text{Hom}_{\mathcal{M}}(I, I)$ with $f(A) = 0$. If $f(ST(A)) \neq 0$ then $f(ST(A)) \cap A \neq 0$. Hence f would induce a nonzero element of $\text{Hom}((ST(A) \cap f^{-1}(A)), A)$ which restricts to zero on A . This would contradict the fact that $ST(A)/A$ is in $\mathcal{C}$. Thus $ST(A) \leq Q(A)$.

On the other hand it is easy to see that since $A \blacktriangle Q(A)$, $Q(A)/A$ is in $\mathcal{C}$. So by the maximality of $ST(A)$, $ST(A) = Q(A)$. $\square$

We now define a new sheaf of rings $R = R(A)$ over X as follows: for an open subset U of X ,

$$R(U) = \text{dir lim Hom}_{A|_U}(D, A|_U)$$

where the direct limit is over all dense subobjects D of $A|_U$, and for open subsets $U \subseteq W$ of X the connecting map from $R(W)$ to $R(U)$ is the map induced by the restrictions $f \mapsto f|_U$. It is easy to see that via $[g][f] = [g \circ (f|_{f^{-1}(\text{domain } g)})]$, $R(U)$ is a ring and the connecting maps are ring homomorphisms. Also A is naturally embedded in R as a subsheaf of rings via the isomorphism $A \cong \mathcal{H}\text{om}(A, A)$.

THEOREM 1.9. $Q(A)$ and $R(A)$ are isomorphic as sheaves of rings via an isomorphism which extends the identity on A .

Proof. Let U be an open subset of X . Let $\mathcal{M}'$ be the category of $A|_U$ -Modules, $\mathcal{C}'$ the full subcategory of all M in $\mathcal{M}'$ with $\text{Hom}_{\mathcal{M}'}(N, A|_U) = 0$ for all subobjects N of M . Let $T': \mathcal{M}' \rightarrow \mathcal{M}'/\mathcal{C}'$ and $S': \mathcal{M}'/\mathcal{C}' \rightarrow \mathcal{M}'$ be the corresponding adjoint functors of Gabriel. Then as abelian groups

$$\begin{aligned}
 Q(A)(U) &\cong \text{Hom}_{\mathcal{M}'}(A|_U, Q(A)|_U) \\
 &\cong \text{Hom}_{\mathcal{M}'}(Q(A)|_U, Q(A)|_U) \quad (1.5) \\
 &\cong \text{Hom}_{\mathcal{M}'}(S'T'(A|_U), S'T'(A|_U)) \quad (1.3, 1.8) \\
 &\cong \text{Hom}_{\mathcal{M}'/\mathcal{C}'}(T'S'T'(A|_U), T'S'T'(A|_U)) \quad [2, \text{III-2, Lemmas 1, 2}] \\
 &\cong \text{Hom}_{\mathcal{M}'/\mathcal{C}'}(T'(A|_U), T'(A|_U)) \quad [2, \text{III-2, Proposition 3}] \\
 &\cong \text{dir lim Hom}_{\mathcal{M}'}(D, A|_U) \quad \text{where the limit is over all } D \blacktriangle A|_U \\
 &= R(U).
 \end{aligned}$$

Let η_U be the composition $Q(A)(U) \rightarrow R(U)$. Following Gabriel (in particular the construction of S in the proof of Proposition 4 on p. 372 in [2]) one can check that for q in $Q(A)(U)$, if λ_q is the multiplication-like element of $\text{Hom}_{\mathcal{M}'}(A|_U, Q(A)|_U)$ corresponding to q and λ_q^* is the element of $\text{Hom}_{\mathcal{M}'}(\lambda_q^{-1}(A|_U), A|_U)$ gotten by restricting λ_q , then $\eta_U(q) = [\lambda_q^*] \in R(U)$. With this it is easy to see that for each open subset U of X , η_U is a ring homomorphism and the appropriate connecting diagrams commute, so that $\eta = \{\eta_U\}$ is a ring-sheaf isomorphism from $Q(A)$ to R . Also if U is open in X and $a \in A(U) \leq Q(A)(U)$ then $\eta_U(a) = [\lambda_a: A|_U \rightarrow A|_U]$ which is the way $A(U)$ is embedded in $R(U)$. So we have that η extends the identity on A . $\square$

2. The sheaves associated with a valuation ring. In this section we start with a valuation ring V which is not a field. We form a sheaf of rings A_V corresponding to V and we compute the quotient sheaf $Q(A_V)$. We will see that this quotient sheaf corresponds to the completion of V if V is discrete rank one and that A_V is its own quotient sheaf if and only if V is maximal.

For the rest of this section let V be a fixed valuation ring with quotient field F and let P be the maximal ideal of V .

Let X be the set of divisors of V , i.e. the collection of all ideals of V which are the intersection of principal ideals. For each $x \in X$, let $U_x = \{y \in X : x \subseteq y\}$. Topologize X by letting $\mathcal{B} = \{U_x : x \in X\}$ be a basis of open sets for X . Since X is linearly ordered by inclusion it is easy to check (cf. [5, p. 28]) that every presheaf of rings or modules over the basis $\mathcal{B}$ gives rise to a unique sheaf over X ; the structure at the nonbasic open sets is gotten by taking inverse, i.e. projective, limits. We will apply this remark repeatedly without further comment by defining sheaves over X and sheaf homomorphisms by only describing their action on the basis $\mathcal{B}$.

We form a sheaf A_V over X via $A_V(U_x) = V/x$ for all $x \in X$ and for $U_x \subseteq U_y$ (i.e. $y \subseteq x$), the connecting map $\rho_{U_x U_y}: A_V(U_y) \rightarrow A_V(U_x)$ is the natural map $V/y \rightarrow V/x$.

The idea of looking at this sheaf A_V is due to Harrison. For some idea of why

A_V comes up one might like to look now at 2.7. For our purposes in this paper there are other possibilities for X , for example we could have let X be only the principal ideals. However Hoyt Warner's unpublished work on extensions of these sheaves makes the divisorial ideals seem the correct choice.

We will denote A_V simply by A when no confusion is likely.

For $x \in X$, we let $W_x = \{y \in X : x \subseteq y\}$; $\bigcap W_x$ denotes the intersection of all the ideals in W_x . Thus $\bigcap W_x \in X$.

We will see in 2.5 that $Q(A)$ is given by, for $x \in X$,

$$\begin{aligned} Q(A)(U_x) &= A(U_x) & \text{if } x \neq \bigcap W_x, \\ &= A(W_x) & \text{if } x = \bigcap W_x, \end{aligned}$$

with connecting maps the appropriate ones from A .

It seems certain that there is a simple direct proof that the sheaf of rings defined above is a maximal dense extension of A , but I have not been able to find it. Instead the proof given here depends on 1.9 and requires that we calculate the dense subobjects of the various $A|_{U_x}$.

Before launching into a series of results which lead up to the calculation of $Q(A)$ in 2.5, we need to make some easy but useful observations about X .

Note 2.1. (a) If U is an open subset of X then either $U = U_x$ for some x in X or $U = W_x$ for some x in X such that $x = \bigcap W_x$.

(b) If P is principal and $x \in X$ and $x \neq V$ then $x = \bigcap W_x$ if and only if x is zero or nonprincipal.

(c) If P is not principal then for every x in X with $x \neq V$, $x = \bigcap W_x$.

(d) For any x in X , $\bigcap W_x = \bigcap \{y \in W_x : y \text{ is principal}\}$.

Proof. One checks (a) and (d).

(b) If $x=0$ then it is easy to see that $x = \bigcap W_x$, so we may assume $x \neq 0$. Let $P = pV$. If $x = dV$ then it is easy to see that $x \subseteq p^{-1}dV = \bigcap W_x$. Conversely if x is nonprincipal then clearly $x = \bigcap W_x$.

(c) Suppose P is not principal. Let $x \in X$ such that $x \neq V$. Just suppose $x \subseteq \bigcap W_x$. Then x must be principal, say $x = dV$. Choose $a \in (\bigcap W_x) \setminus x$. Then $x \subseteq aV \subseteq \bigcap W_x$ which implies that $x \subseteq aP \subseteq aV$. If $x \subseteq aP$ then choosing $b \in (aP \setminus x)$ would give $x \subseteq bV \subseteq \bigcap W_x$ which is impossible. So $x = aP$. Hence $P = a^{-1}dV$. Thus P is principal, a contradiction. $\square$

We now show that for any x in X , $A|_{U_x}$ has a unique minimal dense subobject. The description of this minimal dense subobject depends on whether P and/or x is principal. Once we have these minimal dense subobjects, 1.9 will allow us to compute $Q(A)$. Much of the following calculation depends on methods developed by Hoyt Warner.

We consider first the case where P is principal.

PROPOSITION 2.2. *Let P be principal and x in X .*

(a) *If x is principal and nonzero then $A|_{U_x}$ has no proper dense subobjects.*

(b) If x is nonprincipal or zero and $D = (A|_{U_x})_{W_x}$, i.e. D is the subobject of $A|_{U_x}$ given by, for $y \in U_x$,

$$\begin{aligned} D(U_y) &= 0 & \text{if } y = x, \\ &= V/y & \text{if } y \in W_x, \end{aligned}$$

then $D \blacktriangle A|_{U_x}$ and if D' is any other dense subobject of $A|_{U_x}$ then $D \leq D'$.

Proof. (a) This is trivial if $x = V$, so suppose $x \neq V$. Let $P = pV$ and $x = dV$ with $d \neq 0$. Let E be a proper subobject of $A|_{U_x}$. It is easy to see that this implies $E(U_x) \subsetneq V/x$. Hence $E(U_x) \subseteq P/x$. Define $f: A|_{U_x} \rightarrow A|_{U_x}$ via: for $y \in U_x$ and $a + y$ in $A(U_y)$, $f_{U_y}(a + y) = adp^{-1} + y$. Clearly this gives an $A|_{U_x}$ -homomorphism f and, for $y \in W_x$, $f_{U_y} = 0$ since $Pdp^{-1} = x \subsetneq y$ implies $Vdp^{-1} \subseteq y$. And $f(E) = 0$ since $f_{U_x}(P/x) = 0$. But $f \neq 0$, since $f_{U_x}(1 + x) = dp^{-1} + x \neq 0$. So E is not dense in $A|_{U_x}$.

(b) Again let $P = pV$. Let $D \leq E \leq A|_{U_x}$. Let $f: E \rightarrow A|_{U_x}$ such that $f(D) = 0$. Then $f_{U_y} = 0$ for all $y \in W_x$ and if $(a + x) \in E(U_x)$ with say $f_{U_x}(a + x) = b + x$, then $b + y = f_{U_y}(b + y) = 0$ for all y in W_x . So $b \in \bigcap W_x = x$, 2.1(b). Thus $f = 0$. Hence $D \blacktriangle A|_{U_x}$.

Now let $D' \blacktriangle A|_{U_x}$. Just suppose D is not a subobject of D' . Then there exists $z \in W_x$ such that $D(U_z) \not\subseteq D'(U_z)$, i.e. such that $V/z \not\subseteq D'(U_z)$. By 2.1(b), (d) we may assume z is principal, say $z = eV$. Now $V/z \not\subseteq D'(U_z)$ implies that $D'(U_z) \subseteq P/z$. Let E be the subobject of $A|_{U_x}$ given by, for y in U_x ,

$$\begin{aligned} E(U_y) &= P/y & \text{if } x \subseteq y \subsetneq z, \\ &= V/y & \text{if } z \subseteq y. \end{aligned}$$

Then $D' \leq E \leq A|_{U_x}$. Let $f: E \rightarrow A|_{U_x}$ be given by, for y in U_x and $(a + y)$ in $E(U_y)$,

$$\begin{aligned} f_{U_y}(a + y) &= 0 & \text{if } x \subseteq y \subsetneq z, \\ &= (aep^{-1} + y) & \text{if } z \subseteq y. \end{aligned}$$

Then it is easy to see that f is an $A|_{U_x}$ -homomorphism and that $f(D') = 0$. But f is nonzero since $f_{U_z}(1 + z) = (ed^{-1} + z) \neq 0$. This contradicts $D' \blacktriangle A|_{U_x}$. $\square$

PROPOSITION 2.3. If P is nonprincipal, $x \in X$ and if D is the subsheaf of $A|_{U_x}$ given by, for $y \in U_x$,

$$\begin{aligned} D(U_y) &= 0 & \text{if } y = x, \\ &= P/y & \text{if } y \in W_x, \end{aligned}$$

then $D \blacktriangle A|_{U_x}$ and if D' is any other dense subobject of $A|_{U_x}$ then $D \leq D'$. We will denote this D as $(PA|_{U_x})_{W_x}$.

Proof. For $x = V$ the result is trivial so we assume $x \neq V$. To see that $D \blacktriangle A|_{U_x}$ let $D \leq E \leq A|_{U_x}$ and $f \in \text{Hom}(E, A|_{U_x})$ such that $f(D) = 0$. Just suppose $f \neq 0$. If $f_{U_y} = 0$ for all $y \in W_x$ then, by 2.1(c), $f_{U_x} = 0$, hence $f = 0$, a contradiction. So choose $y \in W_x$ such that $f_{U_y} \neq 0$. Then $E(U_y) = V/y$ so there exists $c \in V$ such that, for all $a \in V$, $f_{U_y}(a + y) = ca + y$. But $f_{U_y}(P/y) = 0$. So we have $cP \subseteq y \subsetneq cV$. Since y is a

divisor we can choose $b \in V$ such that $y \subseteq bV \subsetneq cV$. Hence $P \subseteq c^{-1}bV \subsetneq V$. Thus P is principal, a contradiction. Hence $f=0$ and $D \blacktriangle A|_{U_x}$.

Now let $D' \blacktriangle A|_{U_x}$. Just suppose D is not a subobject of D' . Then there exists some w in W_x such that $P/w \not\subseteq D'(U_w)$. Thus $D'(U_w) = \alpha/w$ where α is an ideal of V such that $\alpha \subsetneq P$. Choose $d \in (P \setminus \alpha)$ and e in $(P \setminus dV)$ (which is possible since P is not principal). So we have $w \subsetneq \alpha \subsetneq dV \subsetneq eV \subsetneq P$. By 2.1(c), (d) there exists a principal ideal z of V such that $x \subsetneq z \subsetneq w$. Note that this implies $e^{-1}z \subsetneq e^{-1}w$ and since z is principal and nonzero, $e^{-1}z \subsetneq d^{-1}z$. So $e^{-1}z \subsetneq (e^{-1}w \cap d^{-1}z)$. Choose c in $(e^{-1}w \cap d^{-1}z)$ but not in $e^{-1}z$.

Let E be the subobject of $A|_{U_x}$ given by, for y in U_x ,

$$\begin{aligned} E(U_y) &= D'(U_y) & \text{if } x \subseteq y \subsetneq z, \\ &= eV/y & \text{if } z \subseteq y \subseteq w, \\ &= V/y & \text{if } w \subsetneq y. \end{aligned}$$

Note that $D' \leq E \leq A|_{U_x}$.

Let $f: E \rightarrow A|_{U_x}$ be the $A|_{U_x}$ -homomorphism given by, for y in U_x and $(a+y)$ in $E(U_y)$,

$$\begin{aligned} f_{U_y}(a+y) &= 0 & \text{if } x \subseteq y \subsetneq z, \\ &= (ca+y) & \text{if } z \subseteq y \subseteq w, \\ &= 0 & \text{if } w \subsetneq y. \end{aligned}$$

Then one checks that $f(D')=0$. But $f_{U_z}(e+z) \neq 0$, so $f \neq 0$. Hence D' is not dense in $A|_{U_x}$, a contradiction. $\square$

LEMMA 2.4. *Let P be nonprincipal, x in X , and $D = (PA|_{U_x})_{W_x}$ as in 2.3. Any f in $\text{Hom}(D, A|_{U_x})$ can be extended uniquely to f^* in $\text{Hom}((A|_{U_x})_{W_x}, A|_{U_x})$. This gives a bijection $(\)^*$ from $\text{Hom}(D, A|_{U_x})$ to $\text{Hom}((A|_{U_x})_{W_x}, A|_{U_x})$ for which the following equation makes sense and holds for all f and g in $\text{Hom}(D, A|_{U_x})$: $(f \circ g)^* = f^* \circ g^*$.*

Proof. Let $f \in \text{Hom}(D, A|_{U_x})$. Let $w \in W_x$. By 2.1(c), (d) we know there is a principal ideal $z = dV$ such that $x \subsetneq z \subsetneq w$. Choose an element e of P such that $z \subseteq ew$. (One such e is gotten by choosing $h \in V$ such that $z \subsetneq hV \subsetneq w$ and letting $e = h^{-1}d$.)

Claim 1. For any $p \in P$ there exists an $a \in V$ such that $f_{U_z}(p+z) = (a+z)(p+z)$.

This claim is clear if $p \in z$. If $p \notin z$ then $p^{-1}d \in V$. Let $b \in V$ such that $f_{U_z}(p+z) = b+z$. Then $0 = f_{U_z}(d+z) = (p^{-1}d+z)(b+z)$. So $p^{-1}db \in dV$. Hence $b \in pV$, which gives the claim.

Claim 2. There is a unique element $(a+w)$ of V/w such that, for all p in P , $f_{U_w}(p+w) = (a+w)(p+w)$.

Let $a \in V$ such that $f_{U_z}(e+z) = (a+z)(e+z)$. Let $p \in P$. If $pV \subseteq eV$ then it is easy to check that $f_{U_z}(p+z) = (a+z)(p+z)$, hence $f_{U_w}(p+w) = (a+w)(p+w)$. So suppose $eV \not\subseteq pV$. Let $c \in V$ such that $f_{U_z}(p+z) = (c+z)(p+z)$. Then since $e \in pV$ it is easy to check that $f_{U_z}(e+z) = (c+z)(e+z)$. Hence $(a-c)e$ is in z . This implies that $a-c$

is in w , for if not then $w \subsetneq (a-c)V$ and so $z \in ew \subsetneq (a-c)eV$, a contradiction. Hence $f_{U_w}(p+w) = (c+w)(p+w) = (a+w)(p+w)$. So we have existence. Now suppose that $a' \in V$ such that $f_{U_w}(p+w) = (a'+w)(p+w)$ for all $p \in P$. If $a' + w \neq a + w$ then $w \subsetneq (a' - a)V$ and so (by 2.1(c), (d)) there exists b in V such that $w \subsetneq bV \subsetneq (a' - a)V$. Hence $b(a' - a)^{-1} \in P$ and $(a' + w)(b(a' - a)^{-1} + w) = f_{U_w}(b(a' - a)^{-1} + w) = (a + w) \cdot (b(a' - a)^{-1} + w)$. Subtracting, we get that $b + w = 0$. Thus $bV \subseteq w$, a contradiction. Hence $a' + w = a + w$ and we have uniqueness, which completes the claim.

Now for each w in W_x let a_w in V such that $a_w + w$ is the unique element of V/w given in Claim 2. It is easy to see that if $y \subseteq w$ in W_x then $a_w - a_y$ is in w . Thus if $f^*: (A|_{U_x})_{W_x} \rightarrow A|_{U_x}$ is defined via for $y \in U_x$ and $(r+y) \in (A|_{U_x})_{W_x}(U_y)$

$$\begin{aligned} f_{U_y}^*(r+y) &= 0 && \text{if } x = y, \\ &= (a_y + y)(r+y) && \text{if } y \in W_x, \end{aligned}$$

then f^* is an $A|_{U_x}$ -homomorphism which clearly extends f . It is unique since D is dense in $A|_{U_x}$.

Finally if f and g are in $\text{Hom}(D, A|_{U_x})$, $f \circ g$ makes sense since $g^{-1}(D) \blacktriangle A|_{U_x}$ so, by 2.3, $D \subseteq g^{-1}(D)$. It is obvious that $g^*((A|_{U_x})_{W_x}) \subseteq (A|_{U_x})_{W_x}$, so $f^* \circ g^*$ makes sense and using the uniqueness condition it is easy to see that $(f \circ g)^* = f^* \circ g^*$. $\square$

Let S be the sheaf of rings extending A given by, for x in X ,

$$\begin{aligned} S(U_x) &= A(U_x) && \text{if } x \neq \bigcap W_x, \\ &= A(W_x) && \text{if } x = \bigcap W_x, \end{aligned}$$

with connecting maps the corresponding ones for A . Thus

$$\begin{aligned} S(U_x) &= V/x && \text{if } P \text{ is principal and } x \text{ is nonzero and principal,} \\ &= \text{proj lim } V/y && \text{where the limit is over all } y \text{ in } W_x, \text{ otherwise.} \end{aligned}$$

The embedding of A into S is the natural one composed of the identities on V/x when $x \neq \bigcap W_x$ and the natural maps $V/x \rightarrow \text{proj lim } V/y$ when $x = \bigcap W_x$.

We now see that $S = Q(A)$ which completes our computation of $Q(A)$.

THEOREM 2.5. *S and $Q(A)$ are isomorphic as sheaves of rings via an isomorphism which extends the identity on A .*

Proof. By 1.9 it will suffice to give an isomorphism from $R(A)$ to S which extends the identity on A . We let $R = R(A)$.

Consider first the case where P is not principal. In this case it follows from 2.3 that for any x in X , $R(U_x) = \text{Hom}((PA|_{U_x})_{W_x}, A|_{U_x})$ and the ring operation is just composition. Using 2.4 it is easy to see that the map $f \mapsto f^*$ is a ring isomorphism from $R(U_x)$ to $\text{Hom}((A|_{U_x})_{W_x}, A|_{U_x})$. And by elementary sheaf theory (cf. [3, 7.3.1]), $\text{Hom}((A|_{U_x})_{W_x}, A|_{U_x}) \cong \text{Hom}(A|_{W_x}, A|_{W_x}) \cong A(W_x) = S(U_x)$ as groups, and it is clear that these are also ring homomorphisms. So composing we have a ring isomorphism $\mu_{U_x}: R(U_x) \rightarrow S(U_x)$. And it is easy to check that this gives a ring-sheaf

isomorphism μ from R to S . Recalling the way A is embedded in $R(A)$ and S , it is easy to see that μ extends the identity on A . So we are done in case P is not principal.

Now consider the case where P is principal. In this case we have by 2.2 that for any x in X , if $x \neq \bigcap W_x$,

$$R(U_x) = \text{Hom}(A|_{U_x}, A|_{U_x}) \cong A(U_x) = S(U_x),$$

where the isomorphism is by $A \cong \mathcal{H}\text{om}(A, A)$; and if $x = \bigcap W_x$,

$$\begin{aligned} R(U_x) &= \text{Hom}((A|_{U_x})_{W_x}, A|_{U_x}) \cong \text{Hom}(A|_{W_x}, A|_{W_x}) \\ &\cong A(W_x) = S(U_x), \end{aligned}$$

where the first isomorphism is by elementary sheaf theory (cf. [3, 7.3.1]) and the second is by $A \cong \mathcal{H}\text{om}(A, A)$. It is clear that the compositions $R(U_x) \rightarrow S(U_x)$ are ring isomorphisms and it is easy to check that the necessary diagrams commute so we have a ring-sheaf isomorphism $\mu: R \rightarrow S$. Again recalling how A is embedded in R and S it is easy to see that μ extends the identity on A . $\square$

Now that we have finally calculated $Q(A)$, we can pretty much read off the results claimed in the first paragraph of this section.

If V is a discrete rank one valuation ring then its completion is $\hat{V} = \text{proj lim } V/x$ where the limit is over all $x \in X \setminus \{0\}$. The map $x \rightarrow x\hat{V}$ is a homeomorphism from X to the space of divisors of $\hat{V}$, so we can identify the space of divisors of $\hat{V}$ with X . Forming the sheaf $A_{\hat{V}}$ over X we have that $A_{\hat{V}}$ is a sheaf of rings naturally extending $A = A_V$.

PROPOSITION 2.6. *If V is a discrete rank one valuation ring with completion $\hat{V}$, then $Q(A_V) = A_{\hat{V}}$.*

Proof. In this case P and all the elements of X are principal. So by 2.6, $Q(A)$ is given by, for x in X , if $x \neq 0$,

$$Q(A)(U_x) = A(W_x) = \text{proj lim } V/y \text{ (over } y \in W_0) \cong A_{\hat{V}}(U_x);$$

and if $x \neq 0$,

$$Q(A)(U_x) = A(U_x) = V/x \cong \hat{V}/x\hat{V} \cong A_{\hat{V}}(U_x).$$

And it is easy to see that the connecting maps correspond so $Q(A_V) = A_{\hat{V}}$. $\square$

Now we recall that a valuation ring V is called maximal in case V has no proper immediate extensions, i.e. in case any valuation ring properly lying over V has either its value group or its residue-class field properly extending that of V . In [6, Theorem 4] Zelinsky showed (with credit to Kaplansky) that a valuation ring V is maximal if and only if for any collection of pairs $(e_\gamma, E_\gamma)_{\gamma \in \Gamma}$ where the e_γ are elements of V and the E_γ are ideals of V and where $(e_\gamma - e_\beta) \in E_\beta$ when $E_\gamma \subseteq E_\beta$, there exists $e \in V$ with $(e - e_\gamma) \in E_\gamma$ for all $\gamma \in \Gamma$.

The following result is due to Harrison and is what led to the interest in A_V in the first place.

PROPOSITION 2.7. *A valuation ring V is maximal if and only if A_V is flabby (i.e. has epic connecting maps).*

Proof. Suppose $A = A_V$ is flabby. Let $\{(e_\gamma, E_\gamma)\}_{\gamma \in \Gamma}$ as above. We must show that there exists an e in V with $e - e_\gamma \in E_\gamma$ for all γ in Γ . If there exists τ in Γ such that $E_\tau \subseteq E_\gamma$ for all $\gamma \in \Gamma$ then $e = e_\tau$ works. So suppose no such τ exists. Let $U = \{x \in X: \text{there exists } \gamma \in \Gamma \text{ with } E_\gamma \subseteq x\}$, then U is an open subset of X . For each $x \in U$, choose $\beta \in \Gamma$ such that $E_\beta \subseteq x$ and $a_x = e_\beta$. It is easy to see that $a_x + x$ is independent of the choice of β and that, for $x \subseteq y$ in U , $a_x - a_y \in y$. Now $A(U) = \text{proj lim } V/x$ (over $x \in U$) and $\{a_x + x\}_{x \in U}$ is an element of $A(U)$. So since A is flabby there exists an a in V with $(a - a_x) \in x$ for all x in U . Now if $\gamma \in \Gamma$, choose $\delta \in \Gamma$ such that $E_\delta \subseteq E_\gamma$ and then choose a principal ideal x such that $E_\delta \subseteq x \subseteq E$. Then $a - e_\gamma = (a - a_x) + (a_x - e_\delta) + (e_\delta - e_\gamma) \in x + x + E_\gamma \subseteq E_\gamma$. Hence $e = a$ works and V is maximal.

Now suppose V is maximal. If U is an open subset of X then $A(U) = \text{proj lim } V/x$ (over $x \in U$) and it is easy to use Zelinsky's characterization of maximality to see that ρ_{UX} is onto. Hence A is flabby. $\square$

So finally we have

PROPOSITION 2.8. *A valuation ring V is maximal if and only if $A_V = Q(A_V)$.*

Proof. Using the characterization $Q(A) = S$ of 2.5 and recalling the way A is embedded in S , it is easy to see that $A = Q(A)$ if and only if the maps $\rho_{W_x U_x}: A(U_x) \rightarrow A(W_x)$ are epic for all $x \in X$ with $x = \bigcap W_x$. Thus if A is flabby then clearly $A = Q(A)$. Conversely suppose that for all $x \in X$ with $x = \bigcap W_x$ we have that $\rho_{W_x U_x}$ is onto. Then since $\rho_{U_x X}$ is onto for all x in X and by 2.1(a) any open subset of X is either of the form U_x or of the form W_x where $x = \bigcap W_x$, it is easy to see that for any open subset U of X , ρ_{UX} is onto, hence A is flabby. So we are done by 2.7. $\square$

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