ASSOCIO-SYMMETRIC ALGEBRAS

BY

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Abstract. Let A be an algebra over a field F satisfying $(x, x, x) = 0$ with a function $g: A \times A \times A \rightarrow F$ such that $(xy)z = g(x, y, z)x(yz)$ for all x, y, z in A . If $g(x_1, x_2, x_3) = g(x_{1\pi}, x_{2\pi}, x_{3\pi})$ for all π in S_3 and all x_1, x_2, x_3 in A then A is called an associo-symmetric algebra. It is shown that a simple associo-symmetric algebra of degree > 2 or degree $= 1$ over a field of characteristic $\neq 2$ is associative. In addition a finite-dimensional semisimple algebra in this class has an identity and is a direct sum of simple algebras.

Throughout we shall let A denote an associo-symmetric algebra over a field F of characteristic $\neq 2$. In §1 we show that A is power-associative and has a vector space decomposition $A = A_{11} + A_{10} + A_{01} + A_{00}$ relative to any idempotent e . In §2 the multiplicative properties of the submodules are studied and as a consequence of these one obtains Theorem 3.2 that if A is simple and $e \neq 1$ is an idempotent then $A_{11}(e)$ and $A_{00}(e)$ are associative subalgebras. The decomposition of A relative to several orthogonal idempotents, derived in §4, is used to obtain Theorem 4.2 that if A is simple and has degree > 2 then A is associative. The main result of §5 is that if A is finite dimensional and semisimple then A has an identity and is a direct sum of simple algebras. Finally in §6 an argument is adopted from alternative rings to show that if A is simple and of degree one then it is a field.

1. Preliminaries.

THEOREM 1.1. *If A is an associo-symmetric algebra then A is power-associative.*

Proof. We show that $x^a x^b = x^{a+b}$ for any x in A by induction on $k = a + b$. The result holds if $k = 3$ by third power-associativity. Assume that the result holds for all $k < n$ and let $0 < s \leq n - 1$. Then $x^{n-1}x = (x^{n-s-1}x^s)x = g(x^{n-s-1}, x^s, x)x^{n-s-1}(x^{s+1})$. On the other hand $x^{n-s}x^s = (x^{n-s-1}x)x^s = g(x^{n-s-1}, x, x^s)x^{n-s-1}x^{s+1}$. By associo-symmetry, however, $g(x^{n-s-1}, x^s, x) = g(x^{n-s-1}, x, x^s)$. Therefore $x^{n-1}x = x^{n-s}x^s$ and if we let $a = n - s$, $b = s$ then $x^a x^b = x^n = x^{a+b}$. Thus, A is power-associative by finite induction.

It seems worthwhile to remark here that the assumption $(xy)z = g(x, y, z)x(yz)$ with $g: A \times A \times A \rightarrow F$ and $g(x_1, x_2, x_3) = g(x_{1\pi}, x_{2\pi}, x_{3\pi})$ is not in itself sufficient to

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guarantee power-associativity even in finite dimension, as the following example indicates. Let A have basis a, b, c, d, e, f over a field of characteristic $\neq 2$ with multiplication given by $ab=c, cd=e, bd=2f, af=e$, and all other products zero. Then $g(x, y, z) = \frac{1}{2}$ for all x, y, z in A . However

$$(a+b+d)^2(a+b+d) = (c+2f)(a+b+d) = e$$

and

$$(a+b+d)(a+b+d)^2 = (a+b+d)(c+2f) = 2e.$$

Therefore A is not power-associative.

LEMMA 1.1. *Let e be an idempotent of an associo-symmetric algebra A over a field of characteristic $\neq 2$. Then $(a, e, e) = (e, a, e) = (e, e, a) = 0$ for all a in A . (Here $(a, b, c) = (ab)c - a(bc)$.)*

Proof. Since A is power-associative, A has a vector space decomposition $A = A_1 + A_{1/2} + A_0$ relative to e where $A_i = \{a_i \mid ea_i + a_ie = 2ia_i\}$ for $i = 1, 0, \frac{1}{2}$ [1]. Since A satisfies $(x, x, x) = 0$ it can be shown (see [6, p. 137]) that

$$A_i = \{a_i \mid ea_i = a_ie = ia_i\} \quad \text{for } i = 1, 0.$$

Now let $a \neq 0$ be in A . From $(x, x, x) = 0$ we have $(e, e, a) + (e, a, e) + (a, e, e) = 0$. If we let $\alpha = g(e, e, a) = g(a, e, e) = g(e, a, e)$ then by the associo-symmetric identity, $(\alpha - 1)e(ea) + (\alpha - 1)e(ae) + (\alpha - 1)ae = 0$. If $\alpha = 1$ the lemma follows. Otherwise $e(ea) + e(ae) + ae = 0$. Let $a = a_1 + a_{1/2} + a_0$. Then $ae = a_1 + a_{1/2}e$, $e(ae) = a_1 + e(a_{1/2}e)$, and $e(ea) = a_1 + e(ea_{1/2})$. Thus we have $3a_1 + a_{1/2}e + e(a_{1/2}e + ea_{1/2}) = 0$. But $a_{1/2}e + ea_{1/2} = a_{1/2}$. Therefore we get $3a_1 + a_{1/2} = 0$, $a_{1/2} = 0$ and $a = a_1 + a_0$. (In fact if characteristic $F \neq 3$ then $a = a_0$.) Thus $(e, a, e) = (a, e, e) = (e, e, a) = 0$.

It is a well-known fact that in any algebra A , the results of Lemma 1.1 imply that A has a Peirce decomposition $A = A_{11} + A_{10} + A_{01} + A_{00}$. Therefore we have

THEOREM 1.2. *If A is an associo-symmetric algebra over a field of characteristic $\neq 2$ and if e is an idempotent of A then $A = A_{11}(e) + A_{10}(e) + A_{01}(e) + A_{00}(e)$ where $A_{ij}(e) = \{x_{ij} \mid ex_{ij} = ix_{ij} \text{ and } x_{ij}e = jx_{ij}\}$.*

It is clear that if $a = a_{11} + a_{10} + a_{01} + a_{00}$ then $a_{11} = eae$, $a_{10} = ea - eae$, $a_{01} = ae - eae$, and $a_{00} = a - ae - ea + eae$.

2. Multiplication of the modules.

LEMMA 2.1. $(A_{11} + A_{01})(A_{00} + A_{01}) = 0$.

Proof. Let $x \in A_{11} + A_{01}$, $y \in A_{00} + A_{01}$. Then $xy = (xe)y = g(x, e, y)x(ey) = 0$.

LEMMA 2.2. $A_{11}A_{11} \subseteq A_{11}$, $A_{11}A_{10} \subseteq A_{10}$.

Proof. Let $x, y \in A_{11}$ and $g(e, x, y) = \alpha$. Then $xy = (ex)y = \alpha e(xy)$. If $\alpha = 0$ then $xy = 0 \in A_{11}$. Suppose $\alpha \neq 0$ and $xy = a_{11} + a_{10} + a_{01} + a_{00}$. Then $xy = \alpha(a_{11} + a_{10})$ or $a_{11} + a_{10} + a_{01} + a_{00} = \alpha(a_{11} + a_{10})$. The vector space direct sum then forces $\alpha = 1$

and $a_{01}=a_{00}=0$. Therefore $xy \in A_{11} + A_{10}$. On the other hand $g(x, y, e)=\alpha=1$, $(xy)e=xy$. Therefore $xy \in A_{11} + A_{01}$. Thus, $xy \in (A_{11} + A_{10}) \cap (A_{11} + A_{01}) = A_{11}$.

Now, let $x \in A_{11}$, $y \in A_{10}$ and $\alpha=g(x, e, y)$. Then $xy=(xe)y=\alpha x(ey)=\alpha xy$. If $\alpha=0$ then $xy=0 \in A_{10}$. Otherwise $\alpha=1=g(e, x, y)$. Thus $xy=(ex)y=e(xy)$ and $xy \in A_{10} + A_{11}$. However, $(xy)e=x(ye)=0$. Therefore $xy \in A_{10} + A_{00}$. Thus, $xy \in A_{10}$.

LEMMA 2.3. $A_{01}A_{11} \subseteq A_{01}$, $A_{01}A_{10} \subseteq A_{00}$.

Proof. Let $x \in A_{01}$, $y \in A_{11}$, $\alpha=g(x, e, y)$. Then $xy=(xe)y=\alpha x(ey)=\alpha xy$. Therefore $\alpha=0$ or 1 . If $\alpha=0$ then $xy=0 \in A_{01}$. Otherwise $\alpha=1=g(e, x, y)=g(x, y, e)$. Therefore $0=(ex)y=e(xy)$ and $(xy)e=x(ye)=xy$. Thus, $xy \in (A_{01} + A_{00}) \cap (A_{01} + A_{11}) = A_{01}$.

Next let $x \in A_{01}$, $y \in A_{10}$. Clearly $xy \in A_{10} + A_{00}$ since $(xy)e=g(x, y, e)x(ye)=0$. Let $\alpha=g(x, e, y)=g(e, x, y)$. If $\alpha=0$ then $(xe)y=0$ and $xy=0 \in A_{00}$. Otherwise $g(e, x, y) \neq 0$. Then $0=(ex)y=g(e, x, y)e(xy)$ and $e(xy)=0$. Thus $xy \in A_{01} + A_{00}$. But $xy \in A_{10} + A_{00}$. Therefore $xy \in A_{00}$.

LEMMA 2.4. $A_{10}(A_{10} + A_{11})=0$, $A_{10}A_{00} \subseteq A_{10}$, $A_{10}A_{01} \subseteq A_{11}$.

Proof. Let $x \in A_{10}$, $y \in A_{1i}$ for $i=0, 1$ and $\alpha=g(x, e, y)=g(e, x, y)$. Then $0=(xe)y=\alpha x(ey)=\alpha xy$. If $\alpha \neq 0$ then $xy=0$. Otherwise $\alpha=g(e, x, y)=0$. Then $xy=(ex)y=0e(xy)=0$. Therefore $xy=0$. Next let $x \in A_{10}$, $y \in A_{00}$. Then $xy=(ex)y=\alpha e(xy)$. As in the proof of Lemma 2.2 this forces $\alpha=0$ or 1 . If $\alpha=0$ we are done. If $\alpha=1$ then $xy \in A_{10} + A_{11}$. But $(xy)e=0$. Therefore $xy \in A_{10} + A_{00}$. Hence, $xy \in A_{10}$.

Finally, let $x \in A_{10}$, $y \in A_{01}$. Then $xy=(ex)y=\alpha e(xy)$. Again, $\alpha=0$ or 1 . If $\alpha=0$ then $xy=0 \in A_{11}$. Otherwise $\alpha=1$, $(xy)e=x(ye)=xy$ and $xy=(ex)y=e(xy)$. Thus $xy \in A_{11}$.

LEMMA 2.5. $A_{00}(A_{10} + A_{11})=0$, $A_{00}A_{01} \subseteq A_{01}$, $A_{00}A_{00} \subseteq A_{00} + A_{10}$.

Proof. Let $x \in A_{00}$, $y \in A_{1i}$ for $i=0, 1$. If $\alpha=1$ then $0=(xe)y=x(ey)=xy$. Otherwise $\alpha \neq 1$. Linearization of third power-associativity gives $(x, e, y) + (x, y, e) + (y, x, e) + (y, e, x) + (e, x, y) + (e, y, x) = 0$ or

$$(\alpha-1)[x(ey)+x(ye)+y(xe)+y(ex)+e(xy)+e(yx)] = 0.$$

Since $\alpha \neq 1$ and by the definition of the modules, we have

$$(1) \quad xy + i(xy) + e(xy) + e(yx) = 0.$$

If $i=1$ then $yx \in A_{11}A_{00}=0$ by Lemma 2.1. Therefore $-\frac{1}{2}e(xy)=xy$, which forces $xy=0$. If $i=0$ then reconsider α . If $\alpha=g(x, e, y) \neq 0$ then $0=(xe)y=\alpha x(ey)=\alpha xy$. Therefore $xy=0$. Otherwise $\alpha=0=g(e, y, x)$ and $yx=(ey)x=0$. Therefore (1) reduces to $xy+e(xy)=0$. This again forces $xy=0$ to show that $A_{00}(A_{10} + A_{11})=0$.

Now let $x \in A_{00}$, $y \in A_{01}$. If $\alpha = 1$ then $(xy)e = x(ye) = xy$ and $e(xy) = (ex)y = 0$. Therefore $xy \in A_{01}$. If $\alpha \neq 1$ then by a linearization of third power-associativity as in Lemma 2.5 we have $x(ey) + x(ye) + y(xe) + y(ex) + e(xy) + e(yx) = 0$ which reduces to $xy + e(xy) + e(yx) = 0$ since $x \in A_{00}$, $y \in A_{01}$. Now $yx \in A_{01}A_{00} = 0$ by Lemma 2.1. Therefore $xy + e(xy) = 0$ which forces $xy = 0 \in A_{01}$. Therefore $A_{00}A_{01} \subseteq A_{01}$. Finally, the last statement of Lemma 2.5 is immediate.

The results of Lemmas 2.1–2.5 give the following.

THEOREM 2.1. *If e is an idempotent of an associo-symmetric algebra A over a field of characteristic $\neq 2$ then the modules $A_{ij}(e)$ have the multiplicative relations*

- (2) $A_{11}A_{11} \subseteq A_{11}$,
- (3) $A_{00}A_{00} \subseteq A_{00} + A_{10}$,
- (4) $A_{ij}A_{kl} = 0$ if $j \neq k$,
- (5) $A_{ij}A_{jl} \subseteq A_{il}$ unless $i = j = l = 0$.

It should be noted that if A has an identity 1 and $e \neq 1$ then (3) can be strengthened to $A_{00}(e)^2 \subseteq A_{00}(e)$. For $A_{00}(e)^2 = A_{11}(1 - e)^2 \subseteq A_{11}(1 - e) = A_{00}(e)$.

3. Simple algebras. In an associative algebra the set $B = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$ is an ideal. We prove the same result for associo-symmetric algebras and use it to characterize the simple algebras. According to convention “simple” means “simple but not nil”.

LEMMA 3.1. $A_{11}(A_{10}A_{01}) \subseteq A_{10}A_{01}$.

Proof. Let $x \in A_{11}$, $y \in A_{10}$, $z \in A_{01}$. If $\alpha = g(x, y, z) = g(x\pi, y\pi, z\pi) = 1$ then $x(yz) = (xy)z \in A_{10}A_{01}$ by (5). Otherwise the linearization of $(a, a, a) = 0$ gives

$$(6) \quad x(yz) + x(zy) + y(xz) + y(zx) + z(xy) + z(yx) = 0.$$

Now $x(zy) \in A_{11}A_{00} = 0$, $xz \in A_{11}A_{01} = 0$, and $yx \in A_{10}A_{11} = 0$ by (4). Therefore (6) reduces to $x(yz) + y(zx) + z(xy) = 0$. But $x(yz) \in A_{11}$, $y(zx) \in A_{11}$, and $z(xy) \in A_{00}$. Therefore $z(xy) = 0$ and (6) reduces to $x(yz) = -y(zx)$. But $y(zx) \in A_{10}(A_{01}A_{11}) \subseteq A_{10}A_{01}$ by (5). Therefore $x(yz) \in A_{10}A_{01}$.

LEMMA 3.2. $A_{00}(A_{01}A_{10}) \subseteq A_{01}A_{10}$.

Proof. Let $x \in A_{00}$, $y \in A_{01}$, $z \in A_{10}$. If $g(x, y, z) = 1$ then $x(yz) = (xy)z \in A_{01}A_{10}$. Otherwise we have (6). But $x(zy) \in A_{00}A_{11} = 0$, $xz \in A_{00}A_{10} = 0$ and so $y(xz) = 0$, and $z(yx) \in A_{10}(A_{01}A_{00}) = 0$. Therefore (6) reduces to $x(yz) + z(xy) + y(zx) = 0$. But $x(yz) \in A_{00}(A_{01}A_{10}) \subseteq A_{00}^2 \subseteq A_{00} + A_{10}$, $y(zx) \in A_{01}A_{10} \subseteq A_{00}$, and $z(xy) \in A_{10}A_{01} \subseteq A_{11}$. Therefore $z(xy) = 0$ and $x(yz) = -y(zx) \in A_{01}A_{10}$ to prove the lemma.

THEOREM 3.1. *In any associo-symmetric algebra A with idempotent e , $B = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$ is an ideal of A .*

Proof. Since $A = \sum_{i,j=0,1} A_{ij}$ it is sufficient to show that $A_{ij}B + BA_{ij} \subseteq B$ for $i, j = 0, 1$. Now the multiplicative properties in Theorem 2.1 and associo-symmetry

immediately imply that $BA_{ij} \subseteq B$ for $i, j = 0, 1$. Similarly, Theorem 2.1 implies that $A_{10}B + A_{01}B \subseteq B$. Consider $A_{11}B = A_{11}(A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10})$. Now $A_{11}A_{01} = A_{11}(A_{01}A_{10}) = 0$, $A_{11}A_{10} \subseteq A_{10} \subseteq B$ by Theorem 2.1 and $A_{11}(A_{10}A_{01}) \subseteq A_{10}A_{01} \subseteq B$ by Lemma 3.1. We similarly use Lemma 3.2 to show that $A_{00}B \subseteq B$. Thus B is an ideal of A .

COROLLARY 1. *If $e \neq 1$ is an idempotent of a simple associo-symmetric algebra A then $A_{11}(e) = A_{10}(e)A_{01}(e)$ and $A_{00}(e) = A_{01}(e)A_{10}(e)$.*

Proof. Since B is an ideal either $B = A$ or $B = 0$. If $B = 0$ then $A_{10} = A_{01} = 0$. Hence $A_{00}A_{00} \subseteq A_{00}$. Thus $A = A_{11}(e) \oplus A_{00}(e)$ and $A_{11}(e)$, $A_{00}(e)$ are ideals of A . Since $e \notin A_{00}(e)$, $A_{00}(e) \neq A$. Therefore $A_{00}(e) = 0$ and $A = A_{11}(e)$. But this contradicts the assumption that $e \neq 1$. Therefore $B = A$, $A_{11}(e) = A_{10}(e)A_{01}(e)$ and $A_{00}(e) = A_{01}(e)A_{10}(e)$.

COROLLARY 2. *If e is an idempotent of a simple associo-symmetric algebra A then $A_{00}(e)^2 \subseteq A_{00}(e)$.*

Proof. Let $x, y \in A_{00}(e)$. Then by Corollary 1, $x = x_{01}x_{10}$ and $xy = (x_{01}x_{10})y = g(x_{01}, x_{10}, y)x_{01}(x_{10}y)$. The right-hand side is clearly in $A_{00}(e)$. Therefore the result follows.

LEMMA 3.3. *If e is an idempotent of an associo-symmetric algebra A and $A_{ij} = A_{ij}(e)$ then*

- (a) $(A_{10}, A_{01}, A_{11}) = 0$,
- (b) $(A_{11}, A_{10}, A_{01}) = 0$,
- (c) $(A_{01}, A_{11}, A_{10}) = 0$.

Proof. The linearization of fourth power-associativity, $(x, x, x^2) = 0$, gives

$$\begin{aligned}
 (7) \quad & (x, y, zw + wz) + (z, y, xw + wx) + (w, y, xz + zx) + (y, x, zw + wz) \\
 & + (z, x, wy + yw) + (w, x, yz + zy) + (z, w, xy + yx) \\
 & + (x, w, yz + zy) + (y, w, xz + zx) + (w, z, xy + yx) \\
 & + (x, z, yw + wy) + (y, z, wx + xw) = 0 \quad [6, p. 129].
 \end{aligned}$$

Let $x \in A_{10}$, $y \in A_{01}$, $z \in A_{11}$, and $w = e$. Then $zw + wz = 2z$, $xw + wx = x$, and $wy + yw = y$. Also, by Theorem 2.1, $xz = zy = 0$. Therefore for these specializations (7) reduces to

$$\begin{aligned}
 (8) \quad & 2(x, y, z) + (z, y, x) + (e, y, zx) + 2(y, x, z) + (z, x, y) + (e, x, yz) + (z, e, xy) \\
 & + (z, e, yx) + (x, e, yz) + (y, e, zx) + (e, z, xy) + (e, z, yx) \\
 & + (x, z, y) + (y, z, x) = 0.
 \end{aligned}$$

Now $(z, y, x) = 0$ since $A_{11}A_{01} = 0$, $(y, x, z) = 0 = (x, z, y)$ since $A_{10}A_{11} = 0$. Also $ey = e[y(zx)] = 0$, $xe = e(yz) = 0$. Therefore $(e, yz, x) = (x, e, yz) = 0$. Also $ex = x$ and $e[x(yz)] = x(yz)$ since $x(yz) \in A_{11}$. Therefore $(e, x, yz) = 0$. Similarly $(z, e, xy) = (z, e, yx) = (y, e, zx) = (e, z, xy) = (e, z, yx) = 0$. Therefore (8) reduces to

$$(9) \quad 2(x, y, z) + (z, x, y) + (y, z, x) = 0.$$

But third power-associativity gives

$$(10) \quad (x, y, z) + (z, x, y) + (y, z, x) = 0.$$

Therefore $(x, y, z) = 0$ proving that $(A_{10}, A_{01}, A_{11}) = 0$. In (10) again $(z, x, y) + (y, z, x) = 0$. But $(z, x, y) \in A_{11}$ and $(y, z, x) \in A_{00}$. Therefore $(z, x, y) = (y, z, x) = 0$. Hence $(A_{11}, A_{10}, A_{01}) = (A_{01}, A_{11}, A_{10}) = 0$.

LEMMA 3.4. *Under the same hypothesis as the previous lemma, $(A_{10}, A_{00}, A_{01}) = (A_{11}, A_{10}, A_{00}) = 0$.*

Proof. Let $x \in A_{10}$, $y \in A_{00}$, and $z \in A_{01}$. Then third power-associativity reduces to $(x, y, z) + (y, z, x) + (z, x, y) = 0$. But $(x, y, z) \in A_{11}$, $(y, z, x) + (z, x, y) \in A_{00}$. Therefore $(x, y, z) = 0$. Similarly if $x \in A_{11}$, $y \in A_{10}$, and $z \in A_{00}$ we immediately obtain $(x, y, z) = 0$.

THEOREM 3.2. *Let $e \neq 1$ be an idempotent of a simple associo-symmetric algebra A over a field of characteristic $\neq 2$. Then $A_{ii}(e)$ is associative for $i = 0, 1$.*

Proof. Let $x, y, z \in A_{11} = A_{11}(e)$. By the corollary to Lemma 3.1, $y = y_{10}y_{01}$ for some $y_{ij} \in A_{ij}$. Then $(xy)z = [x(y_{10}y_{01})]z$. By (b) of the previous lemma $x(y_{10}y_{01}) = (xy_{10})y_{01}$. Therefore $(xy)z = [(xy_{10})y_{01}]z$. Since $xy_{10} \in A_{10}$ and by (a), $[(xy_{10})y_{01}]z = (xy_{10})(y_{01}z)$. Therefore $(xy)z = (xy_{10})(y_{01}z)$. By (b) again and since $y_{01}z \in A_{01}$, $(xy_{10})(y_{01}z) = x[y_{10}(y_{01}z)]$. Finally, by (a), $y_{10}(y_{01}z) = (y_{10}y_{01})z$. Therefore $(xy)z = x[(y_{10}y_{01})z] = x(yz)$ and A_{11} is associative. Since $A_{00}(e) = A_{11}(1 - e)$, A_{00} is also associative.

4. **Decomposition relative to several idempotents.** If A is an alternative or Jordan algebra and $e_1, e_2, \dots, e_t$ are orthogonal idempotents of A , then one has a vector space decomposition $A = \sum A_{ij}$ ($i, j = 0, 1, \dots, t$) with $A_{ij} = \{x \mid e_k x = \delta_{ki} x \text{ and } x e_l = \delta_{jl} x\}$ with δ the Kronecker delta. We show that the same decomposition is obtained for associo-symmetric algebras.

LEMMA 4.1. *Let e, e' be orthogonal idempotents of an associo-symmetric algebra A with 1. Then $e(e'x) = (xe')e = 0$ and $(e, x, e') = 0$.*

Proof. Let $A = A_{11} + A_{10} + A_{01} + A_{00}$ be the decomposition relative to the idempotent e . Then since e and e' are orthogonal, $e' \in A_{00}$. Thus if $x \in A$ then $x = x_{11} + x_{10} + x_{01} + x_{00}$ and $e'x \in A_{00}(A_{11} + A_{10} + A_{01} + A_{00}) \subseteq A_{01} + A_{00}$. (Here we are using the stronger form of (3) in an algebra with 1; namely, $A_{ij}^2 \subseteq A_{00}$.) Therefore $e(e'x) = 0$. Similarly $xe' \in A_{10} + A_{00}$ and so $(xe')e = 0$. Now from third power-associativity either $g(e, x, e') = 1$ or $e(xe' + e'x) + e'(xe + ex) + x(ee' + e'e) = 0$ which reduces to $e(xe') + e'(xe) = 0$. But $e(xe') \in A_{10}$, $e'(xe) \in A_{01}$. Therefore $e(xe') = 0$ and $(ex)e' = g(e, x, e')e(xe') = 0$ and in this case also $(e, x, e') = 0$.

In routine fashion the previous lemma gives

THEOREM 4.1. *Let $e_1, e_2, \dots, e_t$ be orthogonal idempotents of an associo-symmetric algebra A with 1. Then $A = \sum A_{ij}$ ($i, j = 0, 1, \dots, t$) is a vector space decomposition of A with $A_{ij} = \{x \mid e_k x = \delta_{ik} x \text{ and } x e_l = \delta_{jl} x\}$.*

LEMMA 4.2. $A_{ij}A_{kl}=0$ if $j \neq k$ ($i, j, k, l=0, 1, 2, \dots, t$).

Proof. Either $j \neq 0$ or $k \neq 0$. If $j \neq 0$ then $A_{ij} \subseteq A_{11}(e_j) + A_{01}(e_j)$. But $A_{kl} \subseteq A_{01}(e_j) + A_{00}(e_j)$. Therefore $A_{ij}A_{kl}=0$. Similarly if $k \neq 0$.

We now prove the following fundamental theorem on associo-symmetric algebras.

THEOREM 4.2. *Let A be a simple associo-symmetric algebra over a field of characteristic $\neq 2$ and let $1=e_1+e_2+\dots+e_t$ for pairwise orthogonal idempotents e_i . Then if $t > 2$, A is associative.*

Proof. We shall be considering the Peirce decomposition $A = \sum A_{ij}$ relative to $e_1, e_2, \dots, e_t$. Let $e = e_1 + e_i$. Then $A_{11}(e) = eAe = A_{11} + A_{1i} + A_{i1} + A_{ii}$ is associative by Theorem 3.2. Therefore $(A_{1i}, A_{i1}, A_{1i}) = 0$. But $A_{10}(e_1) = \sum_{j=2}^t A_{1j}$ and $A_{01}(e_1) = \sum_{j=2}^t A_{j1}$. Now let $a, c \in A_{10}(e_1)$ with $b \in A_{01}(e_1)$. Then $a = \sum_{j=2}^t a_{1j}$, $c = \sum_{j=2}^t c_{1j}$, and $b = \sum_{j=2}^t b_{j1}$. Then $(ab)c = \sum_{j,k,l=2}^t (a_{1j}b_{kl})c_{1l}$. By the previous remark if $j=k=l$ then $(a_{1j}b_{kl})c_{1l} = a_{ij}(b_{kl}c_{1l})$. If $j \neq k$ then $a_{1j}b_{kl} = 0$ by Lemma 4.2. Therefore $(ab)c = \sum_{j,l=2}^t (a_{1j}b_{j1})c_{1l} + \sum_{j=2}^t a_{1j}(b_{j1}c_{1j})$. But $a_{1j} \in A_{01}(e_j) = A_{10}(1-e_j)$, $b_{j1} \in A_{10}(e_j) = A_{01}(1-e_j)$, and $c_{1l} \in A_{00}(e_j) = A_{11}(1-e_j)$. Since, by Lemma 3.3, $(A_{10}, A_{01}, A_{11}) = 0$ we have $(a_{1j}b_{j1})c_{1l} = a_{1j}(b_{j1}c_{1l})$. Therefore $(ab)c = \sum_{j,l=2}^t a_{1j}(b_{j1}c_{1l})$. On the other hand $a(bc) = \sum_{j,k,l=2}^t a_{1j}(b_{kl}c_{1l})$. But $b_{kl}c_{1l} \in A_{10}(e_k)A_{00}(e_k) \subseteq A_{10}(e_k)$ and $a_{1j} \in A_{00}(e_k)$. Therefore $a_{1j}(b_{kl}c_{1l}) = 0$ if $j \neq k$ and $a(bc) = \sum_{j,l=2}^t a_{1j}(b_{j1}c_{1l})$ also. Thus $(ab)c = a(bc)$. A similar argument shows that $(A_{01}(e_1), A_{10}(e_1), A_{01}(e_1)) = 0$. Thus we have

LEMMA 4.3. $(A_{10}(e_1), A_{01}(e_1), A_{10}(e_1)) = (A_{01}(e_1), A_{10}(e_1), A_{01}(e_1)) = 0$.

Lemma 4.3 together with Lemmas 3.3 and 3.4 is sufficient to prove the associativity of A by showing that all associators $(A_{ij}(e_1), A_{kl}(e_1), A_{rs}(e_1)) = 0$. We show this for several cases which indicate the method to be used in general. Clearly $(A_{ij}, A_{kl}, A_{rs}) = 0$ if $j \neq k$ or $l \neq r$. By Lemmas 3.3, 3.4 and Theorem 3.2, $(A_{11}, A_{10}, A_{01}) = (A_{11}, A_{10}, A_{00}) = (A_{11}, A_{11}, A_{11}) = 0$. We show that $(A_{11}, A_{11}, A_{10}) = 0$. (A_{ij} indicates $A_{ij}(e_1)$.) Let $x, y \in A_{11}$, $z \in A_{10}$. Then by the corollary to Theorem 3.1 $y = y_{10}y_{01} \in A_{10}A_{01} = A_{11}$. Then $(xy)z = [x(y_{10}y_{01})]z$. Since $(A_{11}, A_{10}, A_{01}) = 0$, $(xy)z = [(xy_{10})y_{01}]z$. But $xy_{10} \in A_{10}$, $z \in A_{10}$ and, by Lemma 4.3, $(A_{10}, A_{01}, A_{10}) = 0$. Therefore $(xy)z = (xy_{10})(y_{01}z) = x[y_{10}(y_{01}z)]$ using $(A_{11}, A_{10}, A_{00}) = 0$. Finally $y_{10}(y_{01}z) = (y_{10}y_{01})z$ by Lemma 4.3. Therefore $(xy)z = x[(y_{10}y_{01})z] = x(yz)$. We have shown that all associators which have an element of A_{11} in the first place are zero.

We now consider associators having an element of A_{10} in the first entry. We know that $(A_{10}, A_{00}, A_{01}) = (A_{10}, A_{01}, A_{11}) = (A_{10}, A_{01}, A_{10}) = 0$ by Lemmas 3.3, 3.4, and 4.3. What remains is (A_{10}, A_{00}, A_{00}) . Let $x \in A_{10}$, $y, z \in A_{00}$. Then since $A_{00} = A_{01}A_{10}$, $y = y_{01}y_{10}$ and $(xy)z = [x(y_{01}y_{10})]z$. By Lemma 4.3, $x(y_{01}y_{10}) = (xy_{01})y_{10}$. Therefore $(xy)z = [(xy_{01})y_{10}]z$. But $(A_{11}, A_{10}, A_{00}) = 0$. Therefore $(xy)z = (xy_{01})(y_{10}z)$. Finally we get $(xy)z = x[y_{01}(y_{10}z)]$ from $(A_{10}, A_{01}, A_{10}) = 0$. But $y_{01} \in A_{10}(1-e)$, $y_{10} \in A_{01}(1-e)$ and $z \in A_{11}(1-e)$. Therefore $y_{01}(y_{10}z) = (y_{01}y_{10})z$

and $(xy)z = x[(y_{01}y_{10})z] = x(yz)$, the desired result. The same arguments are used to show that associators with elements of $A_{01}(e_1)$ or $A_{00}(e_1)$ in the first entry are zero. Therefore A is associative.

5. Semisimple algebras. A power-associative algebra A is called semisimple if its nilradical = maximal nil ideal is zero. If A is a finite-dimensional nonnil algebra then a familiar argument (see [5, p. 39]) shows that A has a principal idempotent e . Clearly e is a principal idempotent of A^+ . In [3] Kokoris has shown that $A_{1/2}(e) + A_0(e) \subseteq \text{Rad } A^+$ (cf. proof of Lemma 1.1 for notation). But $A_{10}(e) + A_{01}(e) = A_{1/2}(e)$ and $A_{00}(e) = A_0(e)$. Therefore $A_{10} + A_{01} + A_{00} \subseteq \text{Rad } A^+$. Let $x \in A_{10}$, $y \in A_{01}$. Then $2x \cdot y = xy + yx \in \text{Rad } A^+$. Since $yx \in A_{01}A_{10} \subseteq A_{00} \subseteq \text{Rad } A^+$ we conclude that $xy \in \text{Rad } A^+$. Therefore $A_{10}A_{01} \subseteq \text{Rad } A^+$. Thus the ideal $B = A_{10}(e)A_{01}(e) + A_{10}(e) + A_{01}(e) + A_{01}(e)A_{10}(e)$ is a nil ideal. If we assume that A is semisimple then $B = 0$. Therefore $A_{10}(e) = A_{01}(e) = 0$ and $A = A_{11}(e) + A_{00}(e)$. Since $A_{10} = 0$, A_{00} is a subalgebra and the sum is a direct sum $A = A_{11} \oplus A_{00}$. Since e is principal A_{00} is nil. Therefore $A_{00} = 0$ and $A = A_{11}(e)$. Therefore e is an identity element of A . We have proved the first part of the following theorem.

THEOREM 5.1. *Let A be a finite-dimensional semisimple associo-symmetric algebra. Then A has an identity and is the direct sum of simple algebras.*

To complete the proof assume that D is an ideal of A . Since D is not nil it has principal idempotent e . Thus, as before, $D_{10} + D_{01} + D_{00} \subseteq \text{Rad } D^+$ and $D_{10}D_{01} + D_{10} + D_{01} + D_{01}D_{10}$ is a nil ideal of D . Note however that $D_{10} = A_{10}$, $D_{01} = A_{01}$ and $D_{11} = A_{11}$. For $D_{10} = D \cap A_{10} \subseteq A_{10}$. On the other hand if $x \in A_{10}$ then $x = ex \in D$. Thus $A_{10} \subseteq D$ and $A_{10} = D_{10}$. Similarly for the others. Therefore $B = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$ is a nil ideal of A . Since A is semisimple $B = 0$. Therefore $A = A_{11} \oplus A_{00} = D_{11} \oplus A_{00}$. Since any ideal of A_{ii} is automatically an ideal of A , A_{ii} ($i = 1, 2$) is semisimple. Therefore A is a direct sum of semisimple algebras of smaller dimension and an easy induction completes the theorem.

6. We close with a short discussion of the degree one case. If A is a finite-dimensional associo-symmetric algebra whose only idempotent is the identity 1 over an algebraically closed field F , then an argument of Albert's [2, p. 526] shows that every element $a \in A$ is of the form $a = \alpha 1 + n$ with $\alpha \in F$ and n a nilpotent element.

THEOREM 6.1. *A finite-dimensional, simple degree one algebra over a field of characteristic $\neq 2$ is a field.*

Proof. Assume that A is simple, degree one over F . We may assume without loss of generality that F is algebraically closed. Then every a in A is of the form $\alpha 1 + n$ and since A is power-associative, if $\alpha \neq 0$ then a has an inverse in A . Let $N = \{n \in A \mid n \text{ nilpotent}\}$. We show that N is a subalgebra, hence an ideal of A . By Albert [2] and Oehmke [4], N is a subspace of A . Let $x, y \in N$ with $y^n = 0$, $y^{n-1} \neq 0$. If xy is not nil-

potent then $(xy)^{-1}$ exists in A . Then $y^{n-1} = [(xy)^{-1}(xy)]y^{n-1} = g((xy)^{-1}, xy, y^{n-1}) \times (xy)^{-1}[(xy)y^{n-1}] = g((xy)^{-1}, xy, y^{n-1})g(x, y, y^{n-1})(xy)^{-1}[xy^n] = 0$. Therefore $y^{n-1} = 0$, a contradiction. Hence $xy \in N$ and N is an ideal of A . Since A is simple $N=0$ and $A = F1$.

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