STRONG CONVERGENCE OF FUNCTIONS ON KÖTHE SPACES

BY
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Abstract. Let Λ be a rearrangement invariant Köthe space over a nondiscrete group G with Haar measure μ . For a function $f \in \Lambda$ and relatively compact 0-neighborhood U in G the function

$$T_U f(x) = \frac{1}{\mu(U)} \int_{U+x} f d\mu$$

is continuous and also belongs to Λ . The convergence $T_U f \rightarrow f$ (as $U \rightarrow 0$) for the strong Köthe topology on Λ is involved in establishing compactness criteria for subsets of a Köthe space. The main result of this paper is a necessary and sufficient condition for convergence $T_U f \rightarrow f$ in the strong topology on Λ .

1. In [3], [4] and [5] Köthe studied pairs of subspaces of real sequences that were in weak duality. Dieudonné later generalized the theory to subspaces of locally integrable functions over a locally compact measure space E with Radon measure μ . If E is a σ -compact, locally compact Hausdorff space with regular Radon measure μ , we let Ω be the space of all functions which are integrable on each compact set in E . For a subset Γ of Ω , the Köthe space associated with Γ is $\Lambda = \Lambda(\Gamma) = \{f \in \Omega : \int_E |fg| d\mu < \infty \text{ for all } g \in \Gamma\}$ and the Köthe dual is $\Lambda^* = \Lambda^*(\Gamma) = \Lambda(\Lambda(\Gamma))$. The pair (Λ, Λ^*) is in weak duality; an example of such a pair is (L^p, L^q) . The set Λ can be made into a complete locally convex topological vector space under the strong topology $S(\Lambda, \Lambda^*) = S$ defined by the seminorms

$$S_H(f) = \sup_{g \in H} \int |fg| d\mu$$

as H runs through the weakly bounded subsets of the Köthe dual Λ^* .

For $E=G$ an additive topological group, $f \in \Omega$, and U a relatively compact 0-neighborhood in G , we define the continuous function

$$T_U f(x) = \frac{1}{\mu(U)} \int_{U+x} f d\mu.$$

In [2] these functions were used in giving compactness criteria for subsets of Köthe spaces over G . The importance of the convergence of $T_U f$ to f (for the strong

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topology) as U runs through the relatively compact neighborhoods of 0 in G for each function f in a Köthe space will appear in papers by Welland and Goes which are as yet unpublished. In this paper a necessary and sufficient condition is established for the convergence of $T_U f$ to f for the strong topology on a rearrangement invariant Köthe space.

2. Two functions f and g are said to be equimeasurable or rearrangement invariant if

$$\mu(\{x : |f(x)| > r\}) = \mu(\{x : |g(x)| > r\})$$

for all nonnegative r . A Köthe space Λ is rearrangement invariant if $f' \in \Lambda$ whenever f' is equimeasurable with some $f \in \Lambda$. It is known that such a Λ is contained in the direct sum of $L^1(E, \mu)$ and $L^\infty(E, \mu)$, and that the Köthe dual Λ^* is also rearrangement invariant (proved in [7]).

A set of functions H is normal if $g \in H$ and $|h| \leq |g|$ implies $h \in H$. In [6] it was proven that if E has no atoms (i.e. a set $S \subset E$ of positive measure such that $S_1 \subset S$ implies $\mu(S_1) = 0$ or $\mu(S_1) = \mu(S)$) there is a fundamental system of normal, rearrangement invariant, weakly bounded subsets $\{H\}$ of Λ^* for which the seminorms S_H , which generate the strong topology of Λ , have the property that $S_H(f) = S_H(f')$ for f equimeasurable with f' .

In this paper G will be a σ -compact, locally compact, Hausdorff, nondiscrete topological group; μ will denote invariant Haar measure on G . The family of relatively compact neighborhoods of 0 in G will be denoted by $\mathcal{U}$. In addition, we will often write $\int_G f(x) dx$ to mean $\int_G f d\mu$, and f_y to be the function $f_y(x) = f(y+x)$ for $f \in \Omega$ and $y \in G$.

3. The following lemma will enable us to use the information we know about rearrangement invariant Köthe spaces over a nonatomic space G .

LEMMA 1. *If G does not have the discrete topology, then Haar measure μ is nonatomic; that is G has no atoms.*

Proof. We first show that if G contains an atom S , then $\mu(S)$ must be finite. If not, and $\{K_n\}_{n=1}^\infty$ is the increasing sequence of compact sets whose union is G , then $\mu(K_n \cap S)$ is finite ($n=1, 2, \dots$) and strictly less than $\mu(S) = \infty$. It follows that $\mu(K_n \cap S) = 0$ for each n , and $\mu(S) = 0$; this is a contradiction. Therefore we must assume that $\mu(S) < \infty$. Now since $\mu(S) = \sup \{\mu(K) : K \subset S, K \text{ is compact}\}$ and S is an atom, there is a compact set K which is an atom satisfying $\mu(K) = \mu(S)$. By the nondiscreteness of G , there is a nonempty open set U containing 0 such that $\mu(U) < \mu(K)$. Since $\{U+x : x \in K\}$ is an open cover of the compact K , there must be a finite number of elements $\{x_1, x_2, \dots, x_n\}$ in K such that $\bigcup_{i=1}^n U+x_i \supset K$.

But $\mu(U + x_i \cap K) \leq \mu(U) < \mu(K)$ ($i = 1, 2, \dots, n$) implies $\mu(U + x_i \cap K) = 0$. It then follows that $\mu(S) = \mu(K) = 0$.

LEMMA 2. If Λ is a rearrangement invariant Köthe space over G , U is a compact neighborhood of 0 in G and f is a function in Λ , then

- (i) $T_U f \in \Lambda$;
- (ii) $T_U f$ is a uniformly continuous function on G ;
- (iii) $\rho(T_U f) \leq \rho(f)$ as ρ runs through a certain family of seminorms that generate the strong topology of Λ .

Proof. If $f \in \Lambda$ and U is compact in G , we show that $T_U f \cdot g$ is integrable for every $g \in \Lambda^*$; that is, $T_U f \in \Lambda$. As Λ^* is rearrangement invariant, there is a normal, rearrangement invariant, weakly bounded subset H of Λ^* which g belongs to, and satisfying $S_H(f') = S_H(f)$ whenever f' is equimeasurable with f . Since f_y is equimeasurable with f for each $y \in G$, we have

$$\begin{aligned} \int_G |T_U f(x) \cdot g(x)| \, dx &= \int_G \frac{1}{\mu(U)} \cdot \left| \int_U f(x+y) \cdot g(x) \cdot dy \right| \cdot dx \\ &\leq \frac{1}{\mu(U)} \cdot \int_G \int_U |f(x+y) \cdot g(x)| \cdot dy \cdot dx \\ &= \frac{1}{\mu(U)} \cdot \int_U \int_G |f(x+y) \cdot g(x)| \cdot dx \cdot dy \\ &\leq \frac{1}{\mu(U)} \cdot \int_U \sup_{h \in H} \int |f(x+y) \cdot h(x)| \, dx \cdot dy \\ &= \frac{1}{\mu(U)} \int_U S_H(f_y) \, dy \\ &= \frac{1}{\mu(U)} \int_U S_H(f) \, d\mu = S_H(f) < \infty. \end{aligned}$$

Thus $T_U f \in \Lambda$. Furthermore, it is clear that $\int_G T_U f \cdot g \cdot d\mu \leq S_H(f)$ for all functions $g \in H$. Taking the supremum on all $g \in H$, we obtain $S_H(T_U f) \leq S_H(f)$. Since the seminorms S_H generate the strong topology of Λ , (iii) is proved. In addition, this also shows that $T_U: \Lambda \rightarrow \Lambda$ is a strongly continuous linear function.

In order to show that $T_U f$ is uniformly continuous for U compact and $f \in \Lambda$, we observe first that $f = h + g$ where $h \in L^1(G, \mu)$ and $g \in L^\infty(G, \mu)$ (since Λ is rearrangement invariant); we then have $T_U f = T_U h + T_U g$. We must show that for any $\varepsilon > 0$ there is a 0-neighborhood V in G such that $x - y \in V$ implies $|T_U f(x) - T_U f(y)| < \varepsilon$.

Let $\varepsilon > 0$ be given. Since h is integrable, there is a $\delta > 0$ such that A measurable and $\mu(A) < \delta$ implies $\int_A |h| \, d\mu < \varepsilon/4$. By the compactness of U in G and the regularity of μ , there is a symmetrical 0-neighborhood V such that $\mu(V + U \setminus U)$

$< \min \{ \frac{1}{4} \cdot \varepsilon / \|g\|_\infty; \delta \}$. Then for $x-y \in V$, we will have

$$\begin{aligned}
 & \mu(U) \cdot |T_U f(x) - T_U f(y)| \\
 &= \mu(U) \cdot |T_U h(x) - T_U h(y) + T_U g(x) - T_U g(y)| \\
 &\leq \left| \int_{U+x} h(t) dt - \int_{U+y} h(t) dt \right| + \left| \int_{U+x} g(t) dt - \int_{U+y} g(t) dt \right| \\
 &\leq \int_{x+U \setminus y+U} |h(t)| dt + \int_{y+U \setminus x+U} |h(t)| dt + \int_{x+U \setminus y+U} |g(t)| dt + \int_{y+U \setminus x+U} |g(t)| dt \\
 &= \int_{x-y+U \setminus U} |h(t+y)| dt + \int_{y-x+U \setminus U} |h(t+x)| dt + \int_{x-y+U \setminus U} |g(t+y)| dt \\
 &\quad + \int_{y-x+U \setminus U} |g(t+x)| dt \\
 &\leq \int_{V+U \setminus U} |h(t+y)| dt + \int_{V+U \setminus U} |h(t+x)| dt + \int_{V+U \setminus U} |g(t+y)| dt \\
 &\quad + \int_{V+U \setminus U} |g(t+x)| dt \\
 &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2 \cdot \|g\|_\infty \cdot \frac{\varepsilon}{\|g\|_\infty} \cdot \frac{1}{4} = \varepsilon.
 \end{aligned}$$

Thus $T_U f$ is uniformly continuous.

REMARK. On the real line such functions $T_U f$ are of the form

$$T^h f(x) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t) dt \quad \text{for } h > 0.$$

The T^h operation takes a function f and smoothes it out to a function that approximates the original function; in fact, $\lim_{h \rightarrow \infty} T^h f(x) = f(x)$ a.e. whenever f is locally integrable. As an example let us consider $f = \chi_{[a,b]}$ where a and b are real numbers with $a < b$. Then

$$\begin{aligned}
 T^h f(x) &= (x - (a - h/2))/h \quad \text{for } a - h/2 \leq x \leq a + h/2; \\
 &= 1 \quad \text{for } a + h/2 \leq x \leq b - h/2; \\
 &= (b + h/2 - x)/h \quad \text{for } b - h/2 \leq x \leq b + h/2.
 \end{aligned}$$

Obviously, $\lim_{h \rightarrow 0} T^h f(x) = 1$ for $x \in (a, b)$.

We now give an example of two locally integrable functions f and g on R^1 such that $\int_{R^1} f(t) \cdot g(t) dt = 0$, but $\int_{R^1} T^h f(t) \cdot g(t) dt = \infty$ for all $h > 0$. This example will show that there is a Köthe space over R^1 , $\Lambda = L_g^1 = \{f \in \Omega : \int_{R^1} |fg| d\mu < \infty\}$, such that $f \in \Lambda$, but $T^h f \notin \Lambda$ for any h .

For each integer $n \geq 5$ we choose numbers a_n, b_n, c_n and d_n such that $n < a_n < b_n < c_n < d_n$, $b_n - a_n = c_n - b_n = d_n - c_n = 1/n$, and $n+1 - d_n = a_n - n < \frac{1}{8}$. Set $A_n = (a_n, b_n) \cup (c_n, d_n)$, $B_n = (b_n, c_n)$ ($n \geq 5$), $f = \sum_{n=5}^{\infty} n \cdot \chi_{A_n}$ and $g = \sum_{n=5}^{\infty} n \cdot \chi_{B_n}$. Clearly,

$\int_{\mathbb{R}^1} f(t) \cdot g(t) dt = 0$. However, choosing $h \leq \frac{1}{8}$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} T^h f(x) \cdot g(x) dx &= \int_{-\infty}^{\infty} \frac{1}{h} \cdot \int_{-h/2}^{h/2} f(x+t) \cdot g(x) dt dx \\ &= \frac{1}{h} \cdot \int_{-h/2}^{h/2} \int_{-\infty}^{\infty} f(x+t) \cdot g(x) dx dt \\ &= \frac{1}{h} \cdot \int_{-h/2}^{h/2} \sum_{n=5}^{\infty} \int_n^{n+1} n^2 \cdot \chi_{A_n}(x+t) \cdot \chi_{B_n}(x) dx dt \\ &= \sum_{n=5}^{\infty} \frac{1}{h} \cdot \int_{-h/2}^{h/2} n^2 \cdot \mu(A_n - t \cap B_n) dt \\ &\geq \frac{1}{h} \sum_{n \geq 2/h} n^2 \cdot \int_0^{1/n} \mu(A_n + t \cap B_n) dt \\ &= \frac{1}{h} \sum_{n \geq 2/h} n^2 \cdot \int_0^{1/n} t \cdot dt = \frac{1}{h} \cdot \sum_{n \geq 2/h} n^2 \cdot \frac{1}{n^2} \cdot \frac{1}{2} = \infty. \end{aligned}$$

In the following theorem $C_0 = C_0(G)$ will denote the continuous functions of compact support on G and $S(\Lambda, \Lambda^*) = S$ will denote the strong Köthe topology on Λ generated by the rearrangement invariant seminorms

$$S_B(f) = \sup_{g \in B} \int_G fg d\mu \quad (\text{for } f \in \Lambda)$$

where B runs through the normal, rearrangement invariant, weakly bounded subsets of Λ^* . Recall that $\lim_{U \in \mathcal{U}} T_U f = f$ for the topology S means that for each weakly bounded $B \subset \Lambda^*$ there is a relatively compact 0-neighborhood U_0 in G such that $S_B(T_U f - f) \leq 1$ for all 0-neighborhoods $U \subset U_0$.

THEOREM. *If Λ is a rearrangement invariant Köthe space over G with the strong topology S , then the following are equivalent:*

- (1) C_0 is dense in Λ ;
- (2) $\lim_{U \in \mathcal{U}} T_U f = f$ for each $f \in \Lambda$; $\lim_{n \rightarrow \infty} \varphi \cdot \chi_{K_n} = \varphi$ for each nonnegative uniformly continuous φ in Λ .

Proof. The implication (1) implies (2) will be proven first. We begin by showing that $\lim_{U \in \mathcal{U}} T_U \varphi = \varphi$ for each $\varphi \in C_0$. Let $\varphi \in C_0$ have its support on K compact, and let B be a normal, weakly bounded subset of Λ^* . If U is a compact and symmetric 0-neighborhood in G , then $U + K = \{u + k : u \in U, k \in K\}$ is compact, $d = S_B(\chi_{U+K})$ is a finite number, and the support of $T_U \varphi$ is contained in $U + K$. In order to see this last statement we observe that for $x \in E \setminus U + K$ we will have $U + x \cap K = \emptyset$; for if there was a $u \in U$ and $k \in K$ satisfying $u + x = k$, we would have $x = k - u \in K + U$ (as U is symmetrical), a contradiction. Thus for $x \in E \setminus U + K$, we obtain

$$T_U \varphi(x) = \frac{1}{\mu(U)} \cdot \int_{U+x} \varphi(y) dy = \frac{1}{\mu(U)} \cdot \int_{U+x \cap K} \varphi(y) dy = 0.$$

If $d=0$ the above argument shows that $S_B(T_V\varphi - \varphi) = 0$ for all 0-neighborhoods $V \subset U$. We can then assume that $d > 0$. Now since φ is uniformly continuous, there is a symmetric 0-neighborhood V with $V + V \subset U$ such that $x - y \in V$ implies $|\varphi(x) - \varphi(y)| < 1/d$. The 0-neighborhood V is relatively compact, and for any 0-neighborhood $V' \subset V$ we have

$$\begin{aligned} S_B(T_V\varphi - \varphi) &= S_B(\chi_{K+U} \cdot (T_V\varphi - \varphi)) \\ &= \sup_{g \in B} \int_G |\chi_{K+U}(x) \cdot (T_V\varphi(x) - \varphi(x)) \cdot g(x)| dx \\ &\leq \sup_{g \in B} \int_G \chi_{K+U}(x) \cdot \left\{ \frac{1}{\mu(V')} \cdot \int_{V'} |\varphi(x) - \varphi(x+y)| dy \right\} \cdot g(x) dx \\ &< \sup_{g \in B} \int_G \chi_{K+U} \cdot \frac{1}{d} \cdot g \cdot d\mu = \frac{1}{d} \cdot S_B(\chi_{K+U}) = 1. \end{aligned}$$

Thus $\lim_{U \in \mathcal{U}} T_U\varphi = \varphi$ for each $\varphi \in C_0$. Fix $f \in \Lambda$. We now show $T_U f \rightarrow f$ in Λ . Let B be a normal, rearrangement invariant, weakly bounded subset of Λ^* whose associated seminorm S_B is rearrangement invariant. Since C_0 is strongly dense in Λ , there is a $\varphi \in C_0$ such that $S_B(\varphi - f) < \frac{1}{3}$.

The convergence of $T_U\varphi$ to φ implies there is a relatively compact 0-neighborhood U_0 such that $S_B(T_U\varphi - \varphi) < \frac{1}{3}$ for all 0-neighborhoods $U \subset U_0$. By Lemma 2 we have $S_B(T_U f - f) \leq S_B(T_U f - T_U\varphi) + S_B(T_U\varphi - \varphi) + S_B(\varphi - f) \leq 2 \cdot S_B(f - \varphi) + S_B(T_U\varphi - \varphi) < 1$ for $U \subset U_0$. Thus $\lim_{U \in \mathcal{U}} T_U f = f$ for each $f \in \Lambda$.

For the second part of (1) implies (2) we suppose that $\psi \in \Lambda$ is a nonnegative and uniformly continuous function. If $B \subset \Lambda^*$ is weakly bounded and normal, there is a function $\varphi \in C_0$ such that $S_B(\psi - \varphi) < 1$. Suppose the support of φ is contained in K_m for some integer m . If $n \geq m$ we have

$$\begin{aligned} 0 &= |\psi(x) - \chi_{K_n} \cdot \psi(x)| \leq |\varphi(x) - \psi(x)| \quad \text{for any } x \in K_n; \\ |\psi(x) - \chi_{K_n} \cdot \psi(x)| &= |\psi(x)| = |\psi(x) - \varphi(x)| \quad \text{for } x \in E \setminus K_n. \end{aligned}$$

Therefore, for $n \geq m$ we have

$$S_B(\psi - \chi_{K_n} \cdot \psi) \leq S_B(\psi - \varphi) < 1,$$

which was to be shown.

In order to prove (2) implies (1) we must first prove that if B is a normal, rearrangement invariant, weakly bounded subset of Λ^* whose associated seminorm S_B is rearrangement invariant, and $\varepsilon > 0$ is arbitrary, then there is a $\delta > 0$ (dependent upon ε) such that $S_B(\chi_A) < \varepsilon$ for any measurable set A satisfying $\mu(A) < \delta$. Since G is nondiscrete, there is a compact set K with nonempty interior K° for which the boundary of K , $\partial K = K \setminus K^\circ$, contains a point x_0 with the property that every open set about x_0 has a nonempty intersection with the open sets K° and $E \setminus K$. Since we have assumed $\lim_{U \in \mathcal{U}} T_U \chi_K = \chi_K$ for the strong topology of Λ , there is a compact, symmetric 0-neighborhood U_0 in G such that $S_B(T_U \chi_K - \chi_K) < \varepsilon/2$ for any 0-neighborhood $U \subset U_0$. Now we show that there is a 0-neighborhood $V_0 \subset U_0$ such

that $T_{V_0}\chi_K(x) > \frac{1}{2}$ for all x in some nonempty open set contained in $E \setminus K$. As $U_0 + x_0$ is an open neighborhood of x_0 meeting K° in a nonempty open set and μ is a regular measure, there is a symmetric 0-neighborhood $U' \subset U_0$ such that $0 < \mu(U') < \mu(U_0 + x_0 \cap K)$. The 0-neighborhood $V_0 = (U' + x_0 \cap E \setminus K) \cup (U_0 + x_0 \cap K) - x_0 \supset (U' + x_0 \cap E \setminus K) \cup (U' + x_0 \cap K) - x_0 = U'$ is contained in U_0 .

Since $\mu(U' + x_0 \cap E \setminus K) \leq \mu(U' + x_0) = \mu(U') < \mu(U_0 + x_0 \cap K)$, we have

$$\begin{aligned} T_{V_0}\chi_K(x_0) &= \frac{1}{\mu(V_0)} \cdot \int_{V_0+x_0} \chi_K d\mu = \frac{\mu(V_0+x_0 \cap K)}{\mu(V_0+x_0)} \\ &= \frac{\mu(V_0+x_0 \cap K)}{\mu(U'+x_0 \cap E \setminus K) + \mu(U_0+x_0 \cap K)} \\ &= \frac{\mu(U_0+x_0 \cap K)}{\mu(U'+x_0 \cap E \setminus K) + \mu(U_0+x_0 \cap K)} \\ &> \frac{\mu(U_0+x_0 \cap K)}{\mu(U_0+x_0 \cap K) + \mu(U_0+x_0 \cap K)} = \frac{1}{2}. \end{aligned}$$

The fact that $T_{V_0}\chi_K$ is continuous and $T_{V_0}\chi_K(x_0) > \frac{1}{2}$ implies that there is an open set $V(x_0)$ about x_0 such that $T_{V_0}\chi_K(x) > \frac{1}{2}$ for all $x \in V(x_0)$. By the choice of x_0 , $V = V(x_0) \cap E \setminus K$ is a nonempty open set contained in $E \setminus K$ for which $x \in V$ implies $T_{V_0}\chi_K(x) > \frac{1}{2}$. It follows that $\frac{1}{2} \cdot \chi_V \leq T_{V_0}\chi_K \cdot \chi_{E \setminus K}$. Since $V_0 \subset U_0$, for any measurable set $A \subset V$ we have

$$\begin{aligned} \frac{1}{2} \cdot S_B(\chi_A) &\leq S_B(\tfrac{1}{2}\chi_V) \leq S_B(T_{V_0}\chi_K \cdot \chi_{E \setminus K}) \\ &= S_B(\chi_{E \setminus K} \cdot (T_{V_0}\chi_K - \chi_K)) \leq S_B(T_{V_0}\chi_K - \chi_K) < \varepsilon/2. \end{aligned}$$

Set $\delta = \frac{1}{2}\mu(V) > 0$, and let A be a measurable set with $\mu(A) < \delta$. Since G has no atoms, there is a measurable set A' contained in V satisfying $\mu(A') = \mu(A)$. The equimeasurability of $\chi_{A'}$ with χ_A implies

$$S_B(\chi_A) = S_B(\chi_{A'}) \leq S_B(\chi_V) < \varepsilon.$$

We now show that C_0 is dense in Λ . Let $f \in \Lambda$; we assume without loss of generality that $f \geq 0$. Given $B \subset \Lambda^*$ normal, rearrangement invariant and weakly bounded, there is a compact 0-neighborhood U_0 such that $S_B(T_{U_0}f - f) < \frac{1}{3}$. The function $\varphi = T_{U_0}f$ is nonnegative, uniformly continuous and contained in Λ . By the hypothesis of (2), there is an integer n for which $S_B(\varphi - \varphi\chi_{K_n}) < \frac{1}{3}$. Setting $d = \sup_{x \in K_{n+1}} \varphi(x)$, we can find a $\delta > 0$ such that $\mu(A) < \delta$ implies $S_B(\chi_A) < 1/3d$. Let U be an open set such that $K_n \subset U \subset K_{n+1}$ and $\mu(U \setminus K_n) < \delta$, and let g be a continuous Urysohn function with its support contained in U and having the properties that $g \equiv 1$ on K_n and $g(x) \leq 1$ for all $x \in G$. The function $g \cdot \varphi = \psi$ is a continuous function whose compact support is contained in U satisfying

$$\begin{aligned} S_B(\varphi \cdot \chi_{K_n} - \psi) &= S_B(\varphi \cdot \chi_{K_n} - \varphi \cdot g) = S_B(\varphi \cdot (g - \chi_{K_n})) = S_B(\varphi \cdot \chi_{U \setminus K_n} \cdot (g - \chi_{K_n})) \\ &= S_B(g \cdot \varphi \chi_{U \setminus K_n}) \leq d \cdot 1 \cdot S_B(\chi_{U \setminus K_n}) < \frac{d}{3d} = \frac{1}{3}. \end{aligned}$$

We finally have

$$S_B(f-\psi) \leq S_B(f-\varphi) + S_B(\varphi - \varphi\chi_{K_n}) + S_B(\varphi\chi_{K_n} - \psi) < 1.$$

Thus C_0 is dense in Λ for the strong topology. The Theorem is proved.

COROLLARY. *If Λ is a rearrangement invariant Köthe space over G which contains all the constant functions, then $T_U f \rightarrow f$ ($U \in \mathcal{U}$) strongly for each $f \in \Lambda$ if and only if the uniformly continuous functions are strongly dense in Λ .*

Proof. To show sufficiency we observe that $S_B(\chi_G) < \infty$ for $B \subset \Lambda^*$ weakly bounded. From this we can show, as in the Theorem, that $T_U \varphi \rightarrow \varphi$ for any uniformly continuous $\varphi \in \Lambda$. We again use the denseness of the uniformly continuous functions and the fact that $S_B(T_U f) \leq S_B(f)$ for seminorms S_B generating the topology on Λ to show $T_U f \rightarrow f$ ($U \in \mathcal{U}$) strongly. The necessity part of the Corollary is obvious since $T_U f$ is uniformly continuous for U compact.

REMARK. The L^p spaces for $1 \leq p < +\infty$ are spaces in which $T_U f \rightarrow f$ ($U \in \mathcal{U}$). L^∞ is a space which does not have this property; the continuous functions are not dense for the strong (norm) topology. We now give an example of a Köthe space that is not rearrangement invariant, in which the continuous functions of compact support are dense, and for which $f \in \Lambda$ implies $T_U f \in \Lambda$, but $T_U f$ does not converge to f for the strong topology.

Let $G = R^1$ and let μ be Lebesgue measure on R^1 . Construct sequences of positive numbers $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, and $\{c_n\}_{n=1}^\infty$ such that the following is true:

$$a_n < b_n < c_n < a_{n+1} \dots;$$

$$c_n - b_n = b_n - a_n = 1/n^2 \text{ for each } n;$$

if $A_n = (a_n, b_n)$ and $B_n = (b_n, c_n)$ ($n = 1, 2, \dots$), then $\bigcup_{n=1}^\infty A_n \cup \bigcup_{n=1}^\infty B_n$ is contained in a compact interval.

Set $f = \sum_{n=1}^\infty n^{3/4} \cdot \chi_{A_n}$ and $g = \sum_{n=1}^\infty n^{3/4} \cdot \chi_{B_n}$. Both f and g are integrable, of compact support, and $f \cdot g \equiv 0$. Thus $f \in L^1_g$ which is Köthe space in which the continuous functions of compact support are dense, and in which $T^h f' \in L^1 g$ whenever $f' \in L^1 g$ (as $L^1 g$ contains all the continuous functions). We show $\lim_{h \rightarrow 0} \int_{-\infty}^\infty T^h f(x) \cdot g(x) dx = \infty$. Choosing h sufficiently small, we will have

$$\begin{aligned} \int_{-\infty}^\infty T^h f(x) \cdot g(x) dx &= \sum_{n=1}^\infty \int_{B_n} \frac{1}{h} \cdot \int_{-h/2}^{h/2} f(x+t) \cdot g(x) dt dx \\ &\geq \sum_{n=1}^\infty \frac{1}{h} \cdot \int_{-h/2}^0 \int_{B_n} n^{3/2} \cdot \chi_{A_n}(x+t) dx dt \\ &\geq \sum_{n \leq n(h)} \frac{1}{h} \int_{-h/2}^0 \int_{b_n}^{b_n+t} \chi_{A_n-t}(x) dx dt \quad (\text{where } n(h) = [\sqrt{(2/h)}] + 1) \\ &= \sum_{n \leq n(h)} n^{3/2} \cdot \frac{1}{h} \int_{-h/2}^0 t \cdot dt = \frac{h}{8} \sum_{n \leq n(h)} n^{3/2} \geq \frac{1}{10} \frac{1}{h^{1/4}} \rightarrow \infty \quad \text{as } h \rightarrow 0. \end{aligned}$$

REFERENCES

1. J. Dieudonné, *Sur les espaces de Köthe*, J. Analyse Math. **1** (1951), 81–115. MR **12**, 834.
2. S. Goes, *Some compactness criteria for Köthe spaces*, Dissertation, Northwestern University, Evanston, Ill., 1967 (unpublished).
3. G. Köthe and O. Toeplitz, *Lineare Räume mit unendlichvielen Koordinaten und Ringe unendlicher Matrizen*, J. Reine Angew. Math. **171** (1934), 193–226.
4. G. Köthe, *Neubegründung der Theorie der vollkommenen Räume*, Math. Nachr. **4** (1951), 70–80. MR **12**, 615.
5. ———, *Die Stufenräume, eine einfache Klasse linearer vollkommener Räume*, Math. Z. **51** (1948), 317–345. MR **10**, 255.
6. G. Silverman, *Strong topology on a rearrangement invariant Köthe space* (unpublished).
7. ———, *Rearrangement invariant Köthe spaces*, Math. Ann. **189** (1970), 222–234.

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