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SLICING THEOREMS FOR n-SPHERES IN EUCLIDEAN $(n+1)$-SPACE

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Abstract. This paper describes conditions on the intersection of an n-sphere Σ in Euclidean $(n+1)$-space E^{n+1} with the horizontal hyperplanes of E^{n+1} sufficient to determine that the sphere be nicely embedded. The results generally are pointed towards showing that the complement of Σ is 1-ULC (uniformly locally 1-connected) rather than towards establishing the stronger property that Σ is locally flat. For instance, the main theorem indicates that $E^{n+1}-\Sigma$ is 1-ULC provided each nondegenerate intersection of Σ and a horizontal hyperplane be an $(n-1)$-sphere bicollared both in that hyperplane and in Σ itself ($n\neq 4$).

1. **Introduction.** Much of the literature that treats the problem of determining properties of the embedding of an object Σ in E^n from information about the intersection of Σ with the horizontal hyperplanes of E^n focuses on the case $n=3$. Such a problem first arose in this dimension when J. W. Alexander [1] suggested that a 2-sphere in E^3 might be embedded just as a round sphere if each of its intersections with the horizontal planes were either a point or a simple closed curve. Recently Eaton [11] and Hosay [12] showed this to be true. After generalizations by Loveland [15] and Jensen [13], Cannon proved that the same property is held by any 2-sphere Σ in E^3 such that no intersection of Σ with a horizontal plane has a degenerate component [7].

For higher dimensions Bryant has proved that a k-dimensional compact set X in E^n, where $n-k\geqq 3$, has a 1-ULC complement if the complement of X with respect to each member of some dense subset of the horizontal hyperplanes of E^n is 1-ULC [5].

The main results of this paper are found in §5, where it is shown that a closed n-manifold Σ topologically embedded in E^{n+1} ($n\neq 4$) is nice (meaning, $E^{n+1}-\Sigma$ is 1-ULC) if each nondegenerate intersection Σ_t of Σ and a horizontal, n-dimensional hyperplane in E^{n+1} is a PL $(n-1)$-manifold bicollared in that hyperplane and nice in Σ ($\Sigma-\Sigma_t$ is 1-ULC). Most of the techniques required are consigned to §4. In §6 other generalizations to the results mentioned in the first paragraph are given, the methods for which are taken from [7] and [12]. In addition, we describe

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in §3 some methods, similar to but weaker than those of the three-dimensional case (see [3]), for improving mappings of a disk into the closure of a complementary domain of an n-manifold in an $(n+1)$-manifold.

2. **Definitions and notation.** An *n-manifold* is a separable metric space which is locally homeomorphic to E^n; thus, the term manifold is reserved for manifolds without boundary. For simplicity we shall assume all manifolds to be connected, but they need not be compact or triangulated. A manifold that is compact (and without boundary) is said to be *closed.*

A subset S of a metric space is called an *ε-subset* if and only if the diameter of S, written diam S, is less than ε.

Suppose f and g are maps of a space X into a space Y that has a metric ρ. The symbol $\rho(f, g) < \varepsilon$ means that $\rho(f(x), g(x)) < \varepsilon$ for each x in X. The maps f and g are said to be *ε-homotopic* (*ε-isotopic*) if and only if there exists a homotopy (isotopy) h_t sending X into Y such that $h_0 = f$, $h_1 = g$ and $\rho(h_s, h_t) < \varepsilon$ for all s, t in $[0, 1]$.

A map f of the metric space Y into a subset A is an *ε-map* if and only if $\rho(y, f(y)) < \varepsilon$ for each $y \in Y$.

The symbol Δ^2 denotes a 2-simplex fixed throughout this paper and $\partial\Delta^2$ denotes its boundary. Given a triangulation R of Δ^2, we use $R^{(i)}$ to denote the i-skeleton of R $(i=0, 1)$.

For any point p in a metric space S and any positive number δ, $N_\delta(p)$ denotes the set of points in S whose distance from p is less than δ.

Let A denote a subset of a metric space X and p a limit point of A. We say that A is *locally simply connected at p*, written 1-LC at p, if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\partial\Delta^2$ into $A \cap N_\delta(p)$ can be extended to a map of Δ^2 into $A \cap N_\varepsilon(p)$. Furthermore, we say that A is *uniformly locally simply connected*, written 1-ULC, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\partial\Delta^2$ into a δ-subset of A can be extended to a map of Δ^2 into an ε-subset of A.

In the same context we say that A is *locally arcwise connected at p*, or 0-LC at p, if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that any map of ∂I (where $I = [0, 1]$) into $A \cap N_\delta(p)$ extends to a map of I into $A \cap N_\varepsilon(p)$. We define analogously the uniform condition 0-ULC.

For a subset U of a space S we use the symbol Cl U to denote the closure of U and Bd U to denote the (topological) boundary of U in S.

Let Σ be an n-manifold embedded in the interior of an $(n+1)$-manifold M as a closed subset, and let U be an open subset of $M - \Sigma$. Then Σ is *collared from U* if and only if there exists a homeomorphism g of $\Sigma \times I$ into Cl U such that $g(s, 0) = s$ for each s in Σ. Similarly, Σ is *bicollared in M* if and only if there exists a homeomorphism h of $\Sigma \times [-1, 1]$ into M such that $h(s, 0) = s$ for each s in Σ. In addition, Σ is *locally flat in M* if and only if each point s of Σ has a neighborhood N (relative to M) such that $N \cap \Sigma$ is bicollared in N.

Let n be a positive integer. For each real number t define E_t, the hyperplane of E^{n+1} at the t-level, as $\{(x_1, \ldots, x_{n+1}) \in E^{n+1} \mid x_{n+1} = t\}$. For any subset Σ of E^{n+1} we define Σ_t as $\Sigma \cap E_t$, and for any set C of real numbers we define $\Sigma(C)$ as $\bigcup \{\Sigma_t \mid t \in C\}$; however, for an interval (a, b) we simplify $\Sigma((a, b))$ to $\Sigma(a, b)$.

3. Altering maps of a disk.

LEMMA 3.1. *Suppose Σ is an n-manifold embedded in the interior of an $(n+1)$-manifold M as a closed separating subset, U a component of $M-\Sigma$, X a closed subset of Σ such that* Cl $U-X$ *is* 1-*LC at each point of X, R a triangulation of Δ^2, T a subdivision of R, and F a map of Δ^2 into* Cl U *such that $F(|R^{(1)}|) \subset U$.*

Then for each $\varepsilon > 0$ there exists a map G of Δ^2 into Cl U *such that*

(a) $G \mid |R^{(1)}| = F \mid |R^{(1)}|$,

(b) $\rho(F, G) < \varepsilon$,

(c) $G(|T^{(1)}|) \subset U$,

(d) $G(\Delta^2) \cap X = \varnothing$.

Proof. Note that, since $F(\Delta^2)$ is compact, there exist a neighborhood V of $F(\Delta^2)$ (relative to Cl U) and a positive number δ such that any map of $\partial\Delta^2$ into a δ-subset of $V-X$ can be extended so as to send Δ^2 into an $(\varepsilon/3)$-subset of Cl $U-X$.

Now we simply modify F, beginning with the 0-skeleton of T and working up. In case $v \in T^{(0)}$ and $F(v) \in \Sigma$, define $G(v)$ as a point of $U \cap V$ very close to $F(v)$; when $F(v) \notin \Sigma$, define $G(v) = F(v)$. Since U is 0-LC at each point of Σ [17, Theorem II.5.35], then, for each 1-simplex σ of T, G can be extended along σ in such a way that $G(\sigma) \subset U \cap V$ and $\rho(G|\sigma, F|\sigma) < \varepsilon/3$; in case $\sigma \subset |R^{(1)}|$, define $G|\sigma = F|\sigma$. Because first we could have subdivided T (if necessary), we can assume that, for each 2-simplex τ of T, Diam $F(\tau) < \varepsilon/3$ and Diam $G(\partial\tau) < \delta$. According to the previous paragraph, G can be extended over τ into an $\varepsilon/3$-subset of Cl $U-X$. It follows easily that $\rho(F, G) < \varepsilon$.

THEOREM 3.2. *Suppose Σ is an n-manifold embedded in the interior of an $(n+1)$-manifold M as a closed separating subset, U a component of $M-\Sigma$, and f a map of Δ^2 into* Cl U *with $f(\partial\Delta^2) \subset U$. Suppose $\{X^i\}$ is a countable collection of closed subsets of Σ such that* Cl $U-X^i$ *is* 1-*LC at each point of X^i $(i = 1, 2, \ldots)$. Then for each $\varepsilon > 0$ there exists a map g of Δ^2 into* Cl U *such that*

(1) $g|\partial\Delta^2 = f|\partial\Delta^2$,

(2) $\rho(f, g) < \varepsilon$,

(3) $g^{-1}(\Sigma \cap g(\Delta^2))$ *is* 0-*dimensional*,

(4) $g(\Delta^2) \cap X^i = \varnothing$ *for* $i = 1, 2, \ldots$.

The proof follows from routine applications of Lemma 3.1.

This result yields the following corollary, which can be regarded as a very weak version of [3, Theorem 4.2].

COROLLARY 3.3. *Suppose Σ is an n-manifold embedded in the interior of an $(n+1)$-manifold M as a closed separating subset, U a component of $M-\Sigma$, and f a map of Δ^2 into* Cl U *such that $f(\partial\Delta^2)\subset U$. Then for each $\varepsilon>0$ there exists a map g of Δ^2 into* Cl U *such that* (i) $g|\partial\Delta^2=f|\partial\Delta^2$, (ii) $\rho(f, g)<\varepsilon$, *and* (iii) $g^{-1}(\Sigma\cap g(\Delta^2))$ *is* 0-*dimensional.*

4. **Embedding Cartesian products in E^{n+1}.** Let n denote a fixed positive integer. Obviously there exists a countable collection $\mathscr{M}$ of closed, PL $(n-1)$-manifolds such that any closed, PL $(n-1)$-manifold is homeomorphic to some member of $\mathscr{M}$. (In fact, this holds even without the PL hypothesis [8].)

In this section Σ will denote an n-manifold embedded in E^{n+1} as a closed subset, U a component of $E^{n+1}-\Sigma$, and (a, b) an interval of real numbers such that, for each $t\in(a, b)$, Σ_t is homeomorphic to some member of $\mathscr{M}$ and is collared from U_t. Let $(a, b)_M$ be the set of all t in (a, b) such that Σ_t is homeomorphic to M, where $M\in\mathscr{M}$. For each such M and each $t\in(a, b)_M$ define an embedding of $M\times[-1, 1]$ into Cl U_t such that $\lambda_t(M\times\{-1\})=\Sigma_t$. Topologize the set $\{\lambda_t \mid t\in(a, b)_M\}$ by the sup-norm metric in E^{n+1}, producing a separable metric space.

As suggested by Bryant [4] (and in the unpublished work of Bing [2]), one can easily establish the following lemma.

LEMMA 4.1. *There exists a countable subset $\mathscr{D}$ of (a, b) such that to each t in $(a, b)-\mathscr{D}$ there correspond two sequences $\{s(i)\}$ and $\{u(i)\}$ of real numbers, with $a<s(i)<t<u(i)<b$, such that each of the associated sequences $\{\lambda_{s(i)}\}$ and $\{\lambda_{u(i)}\}$ converges (homeomorphically) to λ_t.*

Throughout the rest of §§4 and 5, $\mathscr{D}$ will denote the subset of (a, b) described in Lemma 4.1, and n will denote a fixed integer other than 4.

The following result can be established by adding simple epsilonics to the proof of Borsuk's theorem (see [10, Theorem 10.2]).

LEMMA 4.2. *Let A be an ANR embedded as a closed subset of a metric space X, $\varepsilon>0$, and f: $X\to A$ an ε-map such that $f|A$ is ε-homotopic (in A) to the identity map. Then there exists a 2ε-retraction of X onto A.*

LEMMA 4.3. A. *If $t\in(a, b)_M-\mathscr{D}$, then there exists a homeomorphism h of $M\times[-1, 1]$ onto a subset A_t of* Cl U *such that*

$$\Sigma\cap A_t = h(M\times\{-1\})\cup h(M\times\{1\}) = \Sigma_s\cup\Sigma_u,$$

where $a<s<t<u<b$ and $A_t\cap E_z=\varnothing$ for z not in $[s, u]$.

B. *For any such A_t, let X_t denote the closure of the component of* Cl $U-A_t$ *containing Σ_t. Then, for each $\varepsilon>0$, A_t can be obtained so that there exists an ε-retraction of X_t onto A_t.*

Proof. Fix a point t of $(a, b)_M-\mathscr{D}$. By restricting $[-1, 1]$ to a subinterval containing -1, if necessary, we may assume that diam $\lambda_t(\{p\}\times[-1, 1])$ is less than

$\varepsilon/18$ for each p in M. Since $\lambda_t(M\times\{0\})$ is an ANR, the obvious retraction of $\lambda_t(M\times[-1,1])$ onto $\lambda_t(M\times\{0\})$ can be extended over a neighborhood N of $\lambda_t(M\times[-1,1])$ to an $\varepsilon/18$-retraction R of N onto $\lambda_t(M\times\{0\})$. Furthermore, N can be chosen as a product $N'\times(s',u')\subset E^n\times E^1$ where N' is a bounded open subset of E^n and $N\cap\Sigma=\Sigma(s',u')$.

It is sufficient to describe the homeomorphism h of $M\times[-1,1]$ onto some (as yet undefined) A_t subject to the following conditions:

(1) $X_t\subset N$,

(2) diam $h(\{p\}\times[-1,1])<\varepsilon/6$ for each p in M,

(3) $h(p,0)=\lambda_t(p,0)$ for each p in M.

Let g denote the map of A_t to $h(M\times\{0\})$ sending each $h(\{p\}\times[-1,1])$ to $h(p,0)$. The product structure on A_t will provide the natural guide for defining an $\varepsilon/6$-homotopy $G_s\colon A_t\to A_t$ between g and the identity map. Note that condition (2) above and the definition of R imply that

$$\operatorname{diam} Rh(\{p\}\times[-1,1]) < \varepsilon/3$$

for each p in M. Thus, RG_s will be an $\varepsilon/3$-homotopy between $Rg=g$ and R. This means that $R|A_t$ will be $\varepsilon/2$-homotopic in A_t to the identity map, and part B of this lemma will be a consequence of Lemma 4.2.

Let δ_t denote the distance from $\lambda_t(M\times\{0\})$ to $\Sigma\cap(E^{n+1}-N)$. It follows from [18] (see also [16, Lemma 5]) in case $n>4$ and from [14, Lemma 4] or [9, Theorem 8.2] in case $n=3$ that there exists a $d_t>0$ such that any locally flat n-manifold in E_t homeomorphically within d_t of Σ_t is δ_t-isotopic to Σ_t in E_t. According to Lemma 4.1 there exist real numbers s and u, with $s'<s<t<u<u'$, such that λ_s and λ_u are homeomorphically within d_t of λ_t. If p_t denotes the map projecting $E^n\times E^1$ onto $E^n\times\{t\}$, then $p_t\lambda_s(M\times\{0\})$ and $p_t\lambda_u(M\times\{0\})$ are each δ_t-isotopic to $\lambda_t(M\times\{0\})$ in E_t. By lifting this isotopy through the levels E_r $(s\leqq r\leqq u)$, we construct an embedding h of $M\times[-\frac{1}{2},\frac{1}{2}]$ into $U\cap N$ such that condition (3), as well as the following conditions, holds:

(4) $h(M\times\{-\frac{1}{2}\})=\lambda_s(M\times\{0\})$,

(5) $h(M\times\{\frac{1}{2}\})=\lambda_u(M\times\{0\})$,

(6) for each $w\in[-\frac{1}{2},\frac{1}{2}]$, there exists a distinct $z\in[s,u]$ such that $h(M\times\{w\})\subset E_z$,

(7) diam $h(\{p\}\times[-\frac{1}{2},\frac{1}{2}])<\varepsilon/18$ for each p in M.

Since both λ_s and λ_u are homeomorphically close to λ_t, we may assume s and u were chosen so that

(8) diam $\lambda_s(\{p\}\times[-1,0])<\varepsilon/18$ for each p in M,

(9) diam $\lambda_u(\{p\}\times[-1,0])<\varepsilon/18$ for each p in M.

Now $h|M\times[-\frac{1}{2},\frac{1}{2}]$ can be extended to a homeomorphism h of $M\times[-1,1]$ onto

$$A_t = \lambda_s(M\times[-1,0])\cup h(M\times[-\tfrac{1}{2},\tfrac{1}{2}])\cup\lambda_u(M\times[-1,0]).$$

To verify that $X_t \subset N$, observe that N has connected boundary, as does X_t (where Bd X_t is taken in E^{n+1}, not in Cl U). Since Bd $X_t \subset N$ by construction and since both X_t and N are bounded, X_t must be a subset of N. The only unverified requirement on the construction, condition (2), follows easily from (7)–(9).

ADDENDUM. Suppose $\varepsilon > 0$, X_t and A_t satisfy the conclusions of Lemma 4.3B, and Z is a compact subset of X_t. Then there exists an ε-map of Z into A_t such that $f(Z) \cap \text{Bd } A_t \subset Z$ and $f|Z \cap A_t = \text{identity}$.

Proof. Follow the ε-retraction of X_t onto A_t by a small homeomorphism g of A_t into $(Z \cap \text{Bd } A_t) \cup \text{Int } A_t$ such that $g|Z \cap A_t = \text{identity}$.

5. **Submanifolds of E^{n+1} whose levels are twice bicollared.** The basic result in this section is the following application of Lemma 4.3.

THEOREM 5.1. *Let Σ denote an n-manifold embedded in E^{n+1} $(n \neq 4)$ as a closed subset, U a component of $E^{n+1} - \Sigma$, and (a, b) an interval such that for each $t \in (a, b)$* (i) *Σ_t is a closed, PL $(n-1)$-manifold that is collared from U_t and* (ii) *Cl $U - \Sigma_t$ is 1-LC at each point of Σ_t. Then U is 1-LC at each point of $\Sigma(a, b)$.*

Proof. Suppose $f: \Delta^2 \to \text{Cl } U$ is a map such that $f(\partial\Delta^2) \subset U$ and $f(\Delta^2) \cap \Sigma \subset \Sigma(a, b)$. Let ε be a positive number less than both $\rho(f(\Delta^2), \Sigma_a \cup \Sigma_b)$ and $\rho(f(\partial\Delta^2), \Sigma)$, and let $\mathscr{D}$ denote the countable subset of (a, b) described in Lemma 4.1. Then by Theorem 3.2 there exists a map $f_0: \Delta^2 \to \text{Cl } U$ such that

(1) $f_0|\partial\Delta^2 = f|\partial\Delta^2$,

(2) $\rho(f, f_0) < \varepsilon/3$,

(3) $f_0(\Delta^2) \cap \Sigma_d = \varnothing$ for each d in D.

For each $t \in (a, b) - \mathscr{D}$, application of Lemma 4.3 yields an $\varepsilon/3$-retraction r_t of the X_t onto the A_t associated (by Lemma 4.3B) with t and $\varepsilon/3$, where $\Sigma \cap A_t = \Sigma_{s(t)} \cup \Sigma_{u(t)}$. Let π denote the projection of $E^n \times E^1$ onto the second factor. Then $\pi(\Sigma \cap f_0(\Delta^2))$, a subset of $(a, b) - \mathscr{D}$, is covered by the collection $\mathscr{C}$ of open intervals $\mathscr{C} = \{(s(t), u(t)) \mid t \in \pi(\Sigma \cap f_0(\Delta^2))\}$, from which we can extract a finite subcovering $\mathscr{F}$ of $\pi(\Sigma \cap f_0(\Delta^2))$. After eliminating unnecessary elements of $\mathscr{F}$, we can regard $\mathscr{F}$ as the union of two (finite) collections $\mathscr{F}_1$ and $\mathscr{F}_2$ such that for $i = 1, 2$ no two elements of $\mathscr{F}_i$ intersect. Let F_1 and F_2 denote the underlying point sets of $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively.

In case $(s(t), u(t)) \in \mathscr{F}_1$, the addendum to Lemma 4.3 implies the existence of an $\varepsilon/3$ map R_t of $X_t \cap f_0(\Delta^2)$ into A_t such that

(4) $R_t(X_t \cap f_0(\Delta^2)) \cap \text{Bd } A_t \subset X_t \cap f_0(\Delta^2)$,

(5) $R_t|f_0(\Delta^2) \cap A_t = \text{identity}$.

Define a map $f_1: \Delta^2 \to \text{Cl } U$ by the rule

$$\begin{aligned} f_1(x) &= R_t f_0(x), && \text{if } f_0(x) \in X_t \text{ and } (s(t), u(t)) \in \mathscr{F}_1, \\ &= f_0(x), && \text{otherwise.} \end{aligned}$$

It follows from (5) and the fact that the X_t's considered are pairwise disjoint that f_1 is well defined and continuous. Note that

(6) $f_1|\partial\Delta^2 = f_0|\partial\Delta^2$,

(7) $\rho(f_0, f_1) < \varepsilon/3$,

(8) $\pi(\Sigma \cap f_1(\Delta^2)) \subset F_2$.

Similarly, in case $(s(t), u(t)) \in \mathscr{F}_2$, the addendum to Lemma 4.3 implies the existence of an $\varepsilon/3$-map R_t of $X_t \cap f_1(\Delta^2)$ into A_t such that

(9) $R_t(X_t \cap f_1(\Delta^2)) \cap \operatorname{Bd} A_t = \varnothing$,

(10) $R_t|f_1(\Delta^2) \cap A_t = \text{identity}$.

Define $f_2\colon \Delta^2 \to \operatorname{Cl} U$ by the rule

$$\begin{aligned} f_2(x) &= R_t f_1(x), && \text{if } f_1(x) \in X_t \text{ and } (s(t), u(t)) \in \mathscr{F}_2, \\ &= f_1(x), && \text{otherwise.} \end{aligned}$$

As before, f_2 is a continuous function satisfying

(11) $f_2|\partial\Delta^2 = f_1|\partial\Delta^2$,

(12) $\rho(f_1, f_2) < \varepsilon/3$,

(13) $f_2(\Delta^2) \cap \Sigma = \varnothing$.

Since $f_2|\partial\Delta^2 = f|\partial\Delta^2$ and $\rho(f, f_2) < \varepsilon$, it follows immediately that U is 1-LC at each point of $\Sigma(a, b)$.

REMARKS. Although condition (ii) clearly is a necessary hypothesis for Theorem 5.1, one questions whether it might be superfluous. Even without this condition it follows from [4], by means of the trick emerging here in Lemma 4.1, that $\operatorname{Cl} U - \Sigma_t$ is 1-LC at each point of Σ_t for all but at most countably many points t in (a, b), but this fact, obviously, is no help. By attacking the problem differently in the next section, we shall prove, under the hypotheses of Theorem 5.1 without condition (ii), that U is 1-LC at many points of Σ (see Corollary 6.2).

Without much extra effort one can prove the following slightly stronger version of Theorem 5.1. Statements of the other results in this section can be altered in a similar manner.

THEOREM 5.2. *Let Σ denote an n-manifold embedded in E^{n+1} $(n \neq 4)$ as a closed subset, U a component of $E^{n+1} - \Sigma$, and (a, b) an interval such that* (i) *for each $t \in (a, b)$, $\operatorname{Cl} U - \Sigma_t$ is 1-LC at each point of Σ_t and* (ii) *for all but countably many points t in (a, b), Σ_t is a closed, PL $(n-1)$-manifold that is collared from U_t. Then U is 1-LC at each point of $\Sigma(a, b)$.*

Proof. Simply incorporate those countably many t's of (a, b) that fail to satisfy condition (ii) into the set $\mathscr{D}$ and reapply the proof of Theorem 5.1.

THEOREM 5.3. *Let Σ denote an n-manifold embedded in E^{n+1} $(n \neq 4)$ as a closed subset, U a component of $E^{n+1} - \Sigma$, and (a, b) an interval such that for each $t \in (a, b)$* (i) *Σ_t is a closed, PL $(n-1)$-manifold that is collared from U_t and* (ii) *$\Sigma - \Sigma_t$ is 1-LC at each point of Σ_t. Then U is 1-LC at each point of $\Sigma(a, b)$.*

Proof. Let p be a point of Σ_t and $\varepsilon>0$. There exists a $\delta>0$ such that any loop in $N_\delta(p)\cap(\Sigma-\Sigma_t)$ is contractible in $N_\varepsilon(p)\cap(\Sigma-\Sigma_t)$. Since U is locally 1-connected at p in the homology sense [17, Theorem II, 5.35], it follows from [6, Proposition 3.3] that there exists an $\alpha>0$ such that any loop in $N_\alpha(p)\cap U$ is contractible in $N_\delta(p)\cap(E^{n+1}-\Sigma_t)$. By cutting off such a contraction in $\Sigma-\Sigma_t$ one can show that each loop in $N_\alpha(p)\cap U$ is contractible in $N_\varepsilon(p)\cap(\mathrm{Cl}\ U-\Sigma_t)$. Using this property one easily can prove that $\mathrm{Cl}\ U-\Sigma_t$ is 1-LC at p. Hence, Theorem 5.1 gives the desired result.

THEOREM 5.4. *Let Σ denote a closed n-manifold in E^{n+1} $(n\neq 4)$, U a component of $E^{n+1}-\Sigma$, and $[a, b]$ the interval such that $\Sigma=\Sigma([a, b])$. Suppose that for each t in (a, b), Σ_t is a closed PL $(n-1)$-manifold that is collared from U_t and that for each t in $[a, b]$, $\mathrm{Cl}\ U-\Sigma_t$ is 1-ULC. Then U is 1-ULC.*

Proof. Obviously U is 1-LC at points of $\Sigma(a, b)$. Furthermore, by hypothesis, any small loop in U near a point of Σ_a is contractible in a small subset of $\mathrm{Cl}\ U-\Sigma_a$. According to Theorem 3.2 such a contraction can be modified slightly so that the range of the resulting map is a small subset of U. Thus, U is 1-LC at points of Σ_a. The same argument applies to points of Σ_b. Consequently, U is 1-ULC.

Using Theorem 5.4 one can extend results of [11] and [12] to higher dimensions in the following ways.

COROLLARY 5.5. *Suppose Σ is a closed n-manifold in E^{n+1} $(n\neq 4)$ such that* (i) $\Sigma=\Sigma([-1, 1])$, (ii) $\Sigma-(\Sigma_{-1}\cup\Sigma_1)$ *is 1-ULC, and* (iii) *for each $t\in(-1, 1)$, Σ_t is a closed, PL $(n-1)$-manifold bicollared in E_t and $\Sigma-\Sigma_t$ is 1-ULC. Then $E^{n+1}-\Sigma$ is 1-ULC.*

Proof. For either component U of $E^{n+1}-\Sigma$ and each t in $[-1, 1]$, the proof of Theorem 5.3 indicates that $\mathrm{Cl}\ U-\Sigma_t$ is 1-ULC. Although the hypotheses of Proposition 3.3 of [6] do not apply when $t=\pm 1$, the argument there can be used to establish the property employed in proving Theorem 5.3, namely, for each $\delta>0$ there exists an $\alpha>0$ such that each α-loop in U is contractible in a δ-subset of $E^{n+1}-\Sigma_t$ $(t\neq\pm 1)$.

COROLLARY 5.6. *Suppose Σ is an n-sphere in E^{n+1} $(n\neq 4)$ such that both Σ_1 and Σ_{-1} are points and, for each t in $(-1, 1)$, Σ_t is an $(n-1)$-sphere bicollared in E_t and $\Sigma-\Sigma_t$ is 1-ULC. Then $E^{n+1}-\Sigma$ is 1-ULC.*

From Theorem 9 of [16] we obtain the following flatness conditions.

THEOREM 5.7. *Suppose Σ is a closed PL n-manifold in E^{n+1} $(n\geqq 5)$ satisfying the hypothesis of Corollary 5.5. Then Σ is locally flat if and only if Σ can be homeomorphically approximated by locally flat manifolds.*

THEOREM 5.8. *Let Σ denote the boundary of an $(n+1)$-cell B in E^{n+1} $(n\geqq 5)$ and U the complement of B. Suppose* (i) $\Sigma=\Sigma([a, b])$, (ii) *for each $t\in(a, b)$, Σ_t is a PL $(n-1)$-manifold that is collared from U_t, and* (iii) *for each $t\in[a, b]$, $\mathrm{Cl}\ U-\Sigma_t$ is 1-ULC. Then Σ is locally flat.*

6. **Submanifolds of E^{n+1} whose levels satisfy 1-ULC conditions.** In this section we give sufficient conditions, in the spirit of Cannon's work in E^3 [7], for a complementary domain of a closed n-manifold in E^{n+1} to be 1-ULC. In place of the hypothesis typical of the results found in §5 that the Σ_t's be collared PL manifolds stands the weakened hypothesis that the U_t's be 1-ULC, together with strong restrictions on the embeddings of the Σ_t's in Σ. As one advantage of this approach, the case $n=4$ need not be excluded.

THEOREM 6.1. *Suppose that Σ is a closed n-manifold in E^{n+1}, Z a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that* (i) $\Sigma_t = \mathrm{Bd}\, Z_t$ *and* (ii) Z_t *is both* 0-*ULC and* 1-*ULC for each t in (a, b). Then (a, b) contains a dense G_δ-subset G such that Z is* 1-*LC at each point of $\Sigma(G)$.*

Proof. Equivalently we shall show that the set F of levels t in (a, b) at which Σ_t contains a point q such that Z fails to be 1-LC at q is a 0-dimensional F_σ-set.

For each positive integer n let X_n denote the set of points x in Σ such that for no neighborhood V of x is every loop of $V \cap Z$ contractible in a $(1/n)$-subset of Z. Then X_n is a compact subset of Σ, and therefore $\pi(X_n)$ is a closed subset of E^1, where π denotes the map projecting $E^n \times E^1$ onto the second factor. Define F_n as $(a, b) \cap \pi(X_n)$. Obviously $F = \bigcup F_n$. Hence, we only need show F_n to be 0-dimensional.

Suppose to the contrary that some F_n contains a subinterval $[a', b']$ of (a, b). By the Baire Category Theorem $[a', b']$ then contains a subinterval $[c, d]$ such that corresponding to some dense subset Y of $[c, d]$ there exists a positive number δ with the property that each δ-loop in Z_t is null homotopic in a $(1/3n)$-subset of Z_t $(t \in Y)$. To reach the required contradiction we shall apply Hosay's argument [12] to prove that each point q of $\Sigma(c, d)$ has a neighborhood V such that every loop in $V \cap Z$ is null-homotopic in a $(1/n)$-subset of Z.

Given any such point q let U be a round open ball in E^{n+1} containing q of diameter less than $\min\{\delta, 1/3n\}$. Assume further that U misses E_c and E_d. Let V be a neighborhood of q such that $V \subset U$ and $V \cap \Sigma$ lies in an n-cell in $U \cap (\Sigma - (\Sigma_c \cup \Sigma_d))$. We must show that any map f of the boundary of a 2-cell D into $V \cap Z$ has an extension g sending D into a $(1/n)$-subset of Z.

Using the notation developed in [12] we trace the argument given on pp. 371–373 there with certain modifications. First, replace part A by the following observation: the hypotheses that $\Sigma_t = \mathrm{Bd}\, Z_t$ and Z_t is 0-ULC imply that there exists an arc in $U \cap Z_t$ connecting each pair of points of $h(A_j^t) \cap f(\partial D)$. Let K_j^t denote the union of the (finitely many) arcs obtained in this way. Second, in part B observe that the special levels (the t's) can be chosen from Y, since Y is dense in $[c, d]$. The last two paragraphs of B may be ignored, noting instead that any map of a simple closed curve into

$$(K^{t_i} \cup K^{t_{i+1}} \cup h(\partial D)) \cap \bigcup \{E_t | t_i \leqq t \leqq t_{i+1}\}$$

is homotopic, in the intersection of Z and the $(1/3n)$-neighborhood of U, to a constant map. Making use of this fact at the end of part D, we can construct the required extension g sending D into Z.

COROLLARY 6.2. *Suppose that Σ is a closed n-manifold in E^{n+1}, U a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that for each $t \in (a, b)$, Σ_t is a closed $(n-1)$-manifold that is collared from U_t. Then (a, b) contains a dense G_δ-subset G such that U is 1-LC at each point of $\Sigma(G)$.*

THEOREM 6.3. *Suppose that Σ is a closed n-manifold in E^{n+1}, Z a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that* (i) $\mathrm{Cl}\,(Z)-\Sigma_t$ *is 1-ULC and* (ii) Z_t *is 1-ULC for each t in (a, b). Then (a, b) contains a dense G_δ-subset G such that Z is 1-LC at each point of $\Sigma(G)$.*

Proof. We begin by repeating the first three paragraphs of the proof of Theorem 6.1. Then, with minor modifications similar to those given in the preceding proposition, Cannon's argument [7, Theorem 1] can be applied to complete the proof.

It would be interesting to know whether the hypotheses of either Theorem 6.1 or 6.3 actually imply that Z is 1-LC at each point of $\Sigma(a, b)$. One should note that some restrictions on the set Z are necessary, for it is quite simple to describe a connected open subset Z of E^{n+1} $(n \geqq 2)$ such that each Z_t is 1-ULC but Z fails to be 1-LC at certain points of $\mathrm{Bd}\, Z$. In particular, a bounded open subset Z of E^{n+1} $(n \geqq 2)$ need not be 1-ULC even if each Z_t is 1-ULC.

THEOREM 6.4. *Suppose Σ is a closed n-manifold in E^{n+1}, U a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that* (1) *for each $t \in (a, b)$, U_t is 1-ULC and* (2) *for each compact 0-dimensional subset C of (a, b), $\mathrm{Cl}\, U-\Sigma(C)$ is 1-ULC. Then U is 1-LC at each point of $\Sigma(a, b)$.*

Proof. Let f be a map of the disk Δ^2 into $\mathrm{Cl}\, U$ such that $f(\partial\Delta^2) \subset U$ and $f(\Delta^2) \cap \Sigma \subset \Sigma(a, b)$. If G denotes the dense G_δ-subset of (a, b) promised by Theorem 6.3, then from hypothesis (2) above and Theorem 3.2 we find that f can be adjusted slightly, not changing the map on $\partial\Delta^2$, such that $f(\Delta^2) \cap \Sigma \subset \Sigma(G)$ and $f^{-1}(\Sigma \cap f(\Delta^2))$ is 0-dimensional. As a result, U is 1-LC at each point of $\Sigma \cap f(\Delta^2)$, and it is then a simple matter to alter f further so that $f(\Delta^2)$ misses Σ entirely. Hence, U is 1-LC at each point of $\Sigma(a, b)$.

COROLLARY 6.5. *Suppose Σ is a closed n-manifold in E^{n+1} such that each component U of $E^{n+1}-\Sigma$ satisfies* (1) *for each $t \in E^1$, U_t is 1-ULC and* (2) *for each compact, 0-dimensional subset C of E^1, $\mathrm{Cl}\, U-\Sigma(C)$ is 1-ULC. Then $E^{n+1}-\Sigma$ is 1-ULC.*

Corollary 6.5 can be interpreted as a generalization of [7, Corollary 3].

REMARK. Variations on Theorem 6.4 and Corollary 6.5 can be obtained by exchanging condition (2) in each for the following condition:

(2*) for each compact 0-dimensional subset C of (a, b), or E^1, as the context requires, $\Sigma-\Sigma(C)$ is 1-ULC.

The proof given for Theorem 2 of [7] establishes that to each $\delta>0$ there corresponds an $\alpha>0$ such that each α-loop in U is contractible in an ε-subset of $E^{n+1}-\Sigma(C)$. Using this we can prove, as in Theorem 5.3, that $\mathrm{Cl}\, U-\Sigma(C)$ is 1-ULC.

Finally, Corollary 6.5 and [16, Theorem 9] can be combined, as in §5, to produce a local flatness criterion.

THEOREM 6.6. *Suppose Σ is a closed PL n-manifold in E^{n+1} $(n\geqq 4)$ satisfying the hypotheses of Corollary* 6.5. *Then Σ is locally flat if and only if Σ can be homeomorphically approximated by locally flat manifolds.*

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