TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 167, May 1972

ON SUBGROUPS OF M_{24}. I: STABILIZERS OF SUBSETS

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Abstract. In this paper we study the orbits of the Mathieu group M_{24} on sets of n points, $1 \leqq n \leqq 12$. For $n \geqq 6$, M_{24} is not transitive on these sets, so we may classify the sets into types corresponding to the orbits of M_{24} and then show how to construct a set of each type from smaller sets. We determine the stabilizer of a set of each type and describe its representation on the 24 points. From the conclusions, the class of subgroups which are maximal among the intransitives of M_{24} can be read off. This work forms the first part of a study which yields, in particular, a complete list of the primitive representations of M_{24}.

In this paper we study the orbits of the Mathieu group M_{24} on sets of n points, $1 \leqq n \leqq 12$. We say that two sets are of the same type if they belong to the same orbit. We then classify the sets of given size and show how to construct a set of each type from smaller sets. We determine the stabilizer of a set of each type and describe its representation on the 24 points.

This work forms the first part of a study which yields, in particular, a complete list of the primitive representations of M_{24}. The complete list includes one more in addition to those listed by J. A. Todd [8]. The two studies were done independently, employing completely different methods.

From the conclusions of this paper, the classes of subgroups which are maximal among the intransitive subgroups of M_{24} can be read off immediately (Theorem I). This paper also yields the essential information for the study of transitive subgroups of M_{24} to follow in a second paper, *On subgroups of* M_{24}. II, in which all the maximal subgroups of M_{24} are completely enumerated.

1. **Notation and preliminaries.** Let Ω be the 24 points of M_{24}. An unordered set of n distinct points in Ω will be denoted by $\boldsymbol{n}$, $1 \leqq n \leqq 12$. M_{24} induces a permutation group on these $\binom{24}{n}$ unordered sets of n distinct points. If, in this representation of M_{24}, two sets of n distinct points $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are of the same type (i.e., in the same orbit), we write $\boldsymbol{n}_1 \approx \boldsymbol{n}_2$. Since M_{24} is not n^*-transitive, if $n \geqq 6$, we know that for $n \geqq 6$ there will be more than one type of $\boldsymbol{n}$'s. If there are j different types of sets of n distinct points, they are denoted by $\boldsymbol{n}^{\cdot}, \boldsymbol{n}^{\cdot\cdot}, \ldots, \boldsymbol{n}^{j}$. $G_{(n^i)}$ denotes the setwise stabilizer of $\boldsymbol{n}^i$ in M_{24} and $G_{[n^i]}$ denotes the pointwise stabilizer of $\boldsymbol{n}^i$ in M_{24}. We will adopt one more notation, $\Delta_s(n^i)$. This denotes an orbit of length s of $G_{(n^i)}$ on $\Omega - \boldsymbol{n}^i$.

Presented to the Society, August 30, 1968 under the title *The complete list of maximal subgroups of Mathieu group of degree* 24; received by the editors August 3, 1970.

AMS 1969 *subject classifications*. Primary 2020, 2029.

Key words and phrases. Mathieu groups, 5-fold transitive permutation groups, stabilizers of subsets.

Except for the conventions introduced above, the notation and terminology follow in general Wielandt [9], with occasional slight deviations which are self-evident from the context.

An element which has α 1-cycles, β 2-cycles, γ 3-cycles, ... in its cycle decomposition is denoted by $1^{\alpha}2^{\beta}3^{\gamma}\cdots$, while decomposition of a number is denoted, e.g., as $1^{\alpha}\cdot 2^{\beta}\cdot 3^{\gamma}\cdots$, with a dot between numbers.

The complete lists of transitive permutation groups of degrees 1 through 12 are available in papers by F. N. Cole and G. A. Miller, [2], [3], and [7] respectively. Frequent use is made of these, and each reference to them is noted individually.

The list of conjugacy classes of elements of M_{24} is given for reference at the end of this section. The twenty-six conjugacy classes are listed by Frobenius [4]. A detailed derivation of the conjugacy classes of all the Mathieu groups is available in the author's thesis [1, p. 14, et seq.].

The character table of M_{24} is used and is available in the paper by Frobenius [4]. A detailed derivation and description of all the characters of all the Mathieu groups is also available in the author's thesis [1, p. 76, et seq.].

The following is a list of theorems and lemmas quoted often throughout.

LEMMA 1.1 (JORDAN, 1871) [9, p. 34]. *If G is primitive on Ω and $G_{[\Delta]}$ is primitive on $\Omega-\Delta=\Gamma$, and, in addition, $1<|\Gamma|=m<n=|\Omega|$, then G is $(n-m+1)$-fold primitive.*

LEMMA 1.2 (WITT, 1937) [10, p. 259]. *Let G be k-fold transitive on Ω, and let $\Gamma\subseteq\Omega$, $|\Gamma|=k$. Let the subgroup $U\leqq G_{[\Gamma]}$ be conjugate in $G_{[\Gamma]}$ to every group V which lies in $G_{[\Gamma]}$ and which is conjugate to U in G. Then $N_G(U)$ is k-fold transitive on the set of points left fixed by U.*

LEMMA 1.3 (M. HALL) [5, p. 80]. *The only nontrivial quadruply transitive groups on less than 35 points are M_{11}, M_{12}, M_{23} and M_{24}.*

LEMMA 1.4 (D. LIVINGSTONE, A. D. WAGNER) [6, p. 394]. *Let G be a permutation group on Ω, and let Δ and Γ form a partition of Ω: $\Delta\cup\Gamma=\Omega$, $\Delta\cap\Gamma=\varnothing$, and let the number of points in Δ be n. Then the permutation group derived from G by its action on the unordered sets of n points is permutation isomorphic to the restriction to G of the permutation representation of S_Ω on the cosets of $(S_\Omega)_{(n)}$. S_Ω is the symmetric group on Ω.*

LEMMA 1.5. *We have*
for $1\leqq n\leqq 5$, one type, $\boldsymbol{n}$,
for $n=6, 7$, two types, $\boldsymbol{n}^{\cdot}$ and $\boldsymbol{n}^{\cdot\cdot}$,
for $n=8, 9, 10, 11$, three types, $\boldsymbol{n}^{\cdot}$, $\boldsymbol{n}^{\cdot\cdot}$ and $\boldsymbol{n}^{\cdot\cdot\cdot}$,
for $n=12$, five types, $\boldsymbol{n}^{\cdot}$, $\boldsymbol{n}^{\cdot\cdot}$, $\boldsymbol{n}^{\cdot\cdot\cdot}$, $\boldsymbol{n}^{\mathrm{iv}}$, and $\boldsymbol{n}^{\mathrm{v}}$.

Proof. By Lemma 1.4, $((1_{(S_n\times S_{24-n})})^{S_{24}})M_{24}$ is the permutation character of the representation of M_{24} on the $\binom{24}{n}$ unordered sets of n distinct points. Thus to find the number of orbits of M_{24} in this representation, we must find the multiplicity of the principal character of M_{24} in the above character.

Denote $(1_{(S_n \times S_{24-n})})^{S_{24}}$ by $\phi^{(n)}$ and a class of elements of type $1^\alpha 2^\beta 3^\gamma \cdots$ in S_{24} by ρ. Then $\phi_\rho^{(n)}$ is given by

$$\phi_\rho^{(n)} = \sum \frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\beta!}{\beta_1!\beta_2!}\frac{\gamma!}{\gamma_1!\gamma_2!}\cdots$$

where the summation runs through the solutions of the following equations:

$$*: \qquad n = \alpha_1+2\beta_1+3\gamma_1+\cdots, \qquad 24-n = \alpha_2+2\beta_2+3\gamma_2+\cdots,$$

$$**: \qquad \alpha = \alpha_1+\alpha_2, \qquad \beta = \beta_1+\beta_2, \qquad \gamma = \gamma_1+\gamma_2, \ldots.$$

Now $\phi_\rho^{(n)}$ is computed only for those ρ which are in M_{24}, and the inner product $(\phi^{(n)}, 1_{M_{24}})$ gives the assertion.

TABLE OF CONJUGACY CLASSES OF M_{24}

Element in Cycle Type	No. Elements in a Class	Element in Cycle Type	No. Elements in a Class
1^{24}	1	$14\,7\,2\,1_-$	$2^9\cdot 3^3\cdot 5\cdot 11\cdot 23$
$2^8 1^8$	$3^2\cdot 5\cdot 11\cdot 23$	$23\,1_+$	$2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11$
$3^6 1^6$	$2^7\cdot 7\cdot 11\cdot 23$	$23\,1_-$	$2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11$
$5^4 1^4$	$2^8\cdot 3^2\cdot 7\cdot 11\cdot 23$	12^2	$2^8\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$
$4^4 2^2 1^4$	$2^3\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$	6^4	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$
$7^3 1^3{}_+$	$2^9\cdot 3^2\cdot 5\cdot 11\cdot 23$	4^6	$2^5\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$
$7^3 1^3{}_-$	$2^9\cdot 3^2\cdot 5\cdot 11\cdot 23$	3^8	$2^7\cdot 3\cdot 5\cdot 11\cdot 23$
$8^2 4\,2\,1^2$	$2^6\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$	2^{12}	$2\cdot 3^2\cdot 7\cdot 11\cdot 23$
$6^2 3^2 2^2 1^2$	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$	$10^2 2^2$	$2^8\cdot 3^3\cdot 7\cdot 11\cdot 23$
$11^2 1^2$	$2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 23$	$21\,3_+$	$2^{10}\cdot 3^2\cdot 5\cdot 11\cdot 23$
$15\,5\,3\,1_+$	$2^{10}\cdot 3^2\cdot 7\cdot 11\cdot 23$	$21\,3_-$	$2^{10}\cdot 3^2\cdot 5\cdot 11\cdot 23$
$15\,5\,3\,1_-$	$2^{10}\cdot 3^2\cdot 7\cdot 11\cdot 23$	$4^4 2^4$	$2^3\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$
$14\,7\,2\,1_+$	$2^9\cdot 3^3\cdot 5\cdot 11\cdot 23$	$12\,6\,4\,2$	$2^8\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$

$\pm$ denotes that the classes are inverse to one another. $|M_{24}| = 2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$.

2. $G_{(n)}$, $1 \leqq n \leqq 5$. $G_{(1)} = M_{23}$ has order $2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$ and is obviously a maximal subgroup of M_{24}. Since M_{24} is 5-fold transitive on Ω, for $2 \leqq n \leqq 5$, we have one type of $\boldsymbol{n}$. $G_{(n)}$ is $N_{M_{24}}(M_{24-n})$ where $M_{24-n} = M_{24[n]}$, so by Lemmas 1.2 and 1.3, we have $|G_{(n)}| = |S_n|\cdot|M_{24-n}|$, $2 \leqq n \leqq 5$. Thus $|G_{(2)}| = |S_2|\cdot|M_{22}| = 2^8\cdot 3^2\cdot 5\cdot 7\cdot 11$, $|G_{(3)}| = |S_3|\cdot|M_{21}| = 2^7\cdot 3^3\cdot 5\cdot 7$, $|G_{(4)}| = |S_4|\cdot|M_{20}| = 2^9\cdot 3^2\cdot 5$ and $|G_{(5)}| = |S_5|\cdot|M_{19}| = 2^7\cdot 3^2\cdot 5$. Obviously $G_{(n)}$ is $(5-n)$-fold transitive on $\Omega - \boldsymbol{n}$, $1 \leqq n \leqq 4$.

Next we will see that $G_{(2)}$ and $G_{(3)}$ are maximal subgroups of M_{24}.

PROPOSITION 2.1. *$G_{(2)}$ is a maximal subgroup of M_{24}.*

Proof. Let L be a possible overgroup of $G_{(2)}$ in M_{24}. Then L is transitive on Ω. Consider $L_{[1]}$. Then $L_{[1]} > G_{[2]}$ or $L_{[1]} = G_{[2]}$. If $L_{[1]} > G_{[2]}$, then L is 2-transitive. Since $G_{[2]}$ is 3-transitive on 22 points, L is 5-transitive on Ω. By Lemma 1.3, $L = M_{24}$. If $L_{[1]} = G_{[2]}$, then $|L| = 2^3 \cdot 3 \cdot |M_{22}|$ and $[M_{24}:L] = 23$. No nonlinear character of M_{24} has degree less than 23. Therefore there cannot be a subgroup of index 23 in M_{24}.

PROPOSITION 2.2. *$G_{(3)}$ is a maximal subgroup of M_{24}.*

Proof. Let L be a possible overgroup of $G_{(3)}$ in M_{24}. Then L is transitive on Ω. $G_{[3]}$ has orbits 1, 1, 1, 21. Let Δ be the orbit of length 21. $G_{[3]}$ is primitive on Δ, and $24 < 2 \cdot |\Delta|$, so L is primitive on Ω by a lemma [9, p. 16]. By Lemma 1.1, L is $24 - 21 + 1 = 4$-transitive. By Lemma 1.3, $L = M_{24}$.

For $G_{[5]}$, we have the following for future use. The subgroup of M_{24} which leaves 5 points unchanged, M_{19}, has order $2^4 \cdot 3$, and is intransitive on the remaining 19 points. By a definition of M_{24} [10, pp. 260–261], this M_{19} is a subgroup of $\mathrm{PSL}_3(4)$ leaving 2 points unchanged. Thus M_{19} and $\mathrm{syl}_2(M_{19})$ can be represented as

$$\left\{\left[\begin{array}{cc|c} \lambda^{-1} & \cdot & \cdot \\ x & \lambda & \cdot \\ \hline y & \cdot & 1 \end{array}\right]\right\} \quad \text{and} \quad \left\{\left[\begin{array}{cc|c} 1 & \cdot & \cdot \\ x & 1 & \cdot \\ \hline y & \cdot & 1 \end{array}\right]\right\}$$

respectively, with $\lambda, x, y \in \mathrm{GF}[2^2]$. Apparently $\mathrm{syl}_2(M_{19})$ is normal in M_{19} and is elementary abelian of type (2, 2, 2, 2). It is regular on 16 points and fixes the remaining 3 points. Also, we note that M_{19} has two orbits of lengths 16 and 3. Let $\mathfrak{A}_{16}$ denote $\mathrm{syl}_2(M_{19})$ and $\mathfrak{H}$ denote $N_{M_{24}}(\mathfrak{A}_{16})$.

We have the following characterization of $\mathfrak{H}$ and $\mathrm{syl}_2(M_{24})$.

PROPOSITION 2.3. *$\mathfrak{H}$ is A_8 on the 8 points left fixed by $\mathfrak{A}_{16}$ and is the holomorph of $\mathfrak{A}_{16}$ on the 16 remaining points. Let $\mathbf{8}^{\boldsymbol{\cdot}}$ denote the 8 points left fixed by $\mathfrak{A}_{16}$, then $G_{(8^{\cdot})} = \mathfrak{H}$. Also, $\mathrm{syl}_2(M_{24}) = \mathrm{syl}_2(\mathrm{GL}_5(2))$.*

Proof. $\mathfrak{A}_{16}$ satisfies the conditions of Lemma 1.2, so $\mathfrak{H}$ is 5-fold on the 8 points left fixed by $\mathfrak{A}_{16}$. Therefore $\mathfrak{H}$ is S_8 or A_8 by Lemma 1.3. Thus, $\mathfrak{H}/\mathfrak{A}_{16} \cong A_8$.

Now $\mathfrak{H}$ is represented faithfully on the remaining 16 points, thus $\mathfrak{H}$ is contained in the affine group $A_4(F_2)$, but $|\mathfrak{H}| = 2^4 \cdot |A_8| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7$, so $\mathfrak{H} = A_4(F_2)$. Thus we have $A_4(F_2)/\mathfrak{A}_{16} \cong \mathrm{GL}_4(2) \cong A_8$ and $\mathrm{syl}_2(M_{24}) = \mathrm{syl}_2(\mathrm{GL}_5(2))$. Since $\mathfrak{A}_{16} \lhd M_{19}$, $G_{(8^{\cdot})} \subset \mathfrak{H}$ and $G_{(8^{\cdot})} = \mathfrak{H}$.

For $G_{(4)}$ and $G_{(5)}$, we have the following for future use.

PROPOSITION 2.4. *$G_{(4)}$ is an imprimitive group on the remaining 20 points with systems of imprimitivity of length 4.*

Proof. Let $\boldsymbol{x}$ be one of the 21 points in the 2-dimensional projective geometry of $\mathrm{PSL}_3(2^2)$. Then $G_{[4]} = (\mathrm{PSL}_3(2^2))_{[\boldsymbol{x}]}$. Let $\Gamma_i \cup \boldsymbol{x}$, $i = 1, \ldots, 5$, be the $4 + 1 = 5$ lines through $\boldsymbol{x}$ of the geometry, then $G_{[4]}$ is an imprimitive group with systems of imprimitivity, Γ_i, $i=1,\ldots,5$. We claim these systems are unique for $G_{[4]}$: Let y be one of 20 points of $G_{[4]}$ and B be a block containing y. Since $(G_{[4]})_{[y]}=M_{19}$ has 2 orbits, Δ_3 and Δ_{16} of lengths 3 and 16 respectively, $B=y\cup\Delta_3$. Therefore any block must have length 4. Let $B_1,\ldots,B_5$ be conjugates of $B_1=B$. Then $B_j\cap\Gamma_i$ must be a block unless the intersection is null, so $B_j=\Gamma_i$ or $B_j\cap\Gamma_i=\varnothing$. Therefore, in a suitable ordering, $\Gamma_i=B_i$ for $i=1,2,3,4,5$. Now, if $g\in G_{(4)}$, then Γ_i^g is a block of $g^{-1}G_{[4]}g=G_{[4]}$. So by the uniqueness of the blocks of $G_{[4]}$, $\Gamma_i^g=\Gamma_j$, $i,j=1,2,\ldots,5$. Thus, $G_{(4)}$ is an imprimitive group with systems of imprimitivity of length 4 on the 20 points.

Proposition 2.5. *$G_{(5)}$ has two orbits $\Delta_3(5)$ and $\Delta_{16}(5)$ of lengths 3 and 16. Furthermore, $G_{(5)}\subset G_{(8^{\cdot})}$ and $G_{(5)}$ is S_3 on $\Delta_3(5)$.*

Proof. We have seen that $G_{[5]}$ has two orbits of lengths 3 and 16. These two orbits are $G_{(5)}$-orbits as well. Call them $\Delta_3(5)$ and $\Delta_{16}(5)$, respectively. Since $\mathbf{5}\cup\Delta_3(5)=\mathbf{8}^{\cdot}$, $G_{(5)}\subset G_{(8^{\cdot})}$ and $G_{(5)}{}^{\Delta_3(5)}=S_3$.

3. $G_{(n^{\cdot})}$ **and** $G_{(n^{\cdot\cdot})}$, $n=6$ **and** 7. An appendix is placed at the end of this paper. In discussions of the constructions of $\boldsymbol{n}^j$ from $(\boldsymbol{n}-\mathbf{1})^i$, the figures in the appendix will be of great assistance.

Now in Figure 1 in the appendix, the three sets of six points, $\mathbf{5}\cup X$, $X\in\Delta_3(5)$, are all the same type, and the sixteen sets of six points, $\mathbf{5}\cup Y$, $Y\in\Delta_{16}(5)$, are all the same type. By Lemma 1.5, $\{\mathbf{5}\cup X\}\not\approx\{\mathbf{5}\cup Y\}$. Call $\mathbf{5}\cup X$ $\mathbf{6}^{\cdot}$ and $\mathbf{5}\cup Y$ $\mathbf{6}^{\cdot\cdot}$. Then, we have

Proposition 3.1. *$G_{[6^{\cdot}]}=\mathrm{syl}_2\,(M_{19})$ and $G_{[6^{\cdot\cdot}]}=\mathrm{syl}_3\,(M_{19})$. $G_{(6^{\cdot})}/G_{[6^{\cdot}]}\cong S_6$ and $G_{(6^{\cdot\cdot})}/G_{[6^{\cdot\cdot}]}\cong S_6$. $|G_{(6^{\cdot})}|=2^4\cdot|S_6|$ and $|G_{(6^{\cdot\cdot})}|=3\cdot|S_6|$.*

Proof. $|G_{[5\cup X]}|=|G_{[6^{\cdot}]}|=2^4$ and $|G_{[5\cup Y]}|=|G_{[6^{\cdot\cdot}]}|=3$. Thus, $G_{[6^{\cdot}]}=\mathrm{syl}_2\,(M_{19})$ and $G_{[6^{\cdot\cdot}]}=\mathrm{syl}_3\,(M_{19})$. Next, let $|G_{(6^{\cdot})}|=g_1$, and $|G_{(6^{\cdot\cdot})}|=g_2$. Then $|M_{24}|\cdot(1/g_1+1/g_2)=\binom{24}{6}$. Now let $|G_{(6^{\cdot})}/G_{[6^{\cdot}]}|=u_1$, $|G_{(6^{\cdot\cdot})}/G_{[6^{\cdot\cdot}]}|=u_2$. Then $48\cdot(6!/16u_1+6!/3u_2)=19$. Now, $3(6!/u_1)+16(6!/u_2)=19$ and $u_i\leqq 6!$, $i=1,2$. So $u_1=6!$ and $u_2=6!$. Thus

$$G_{(6^{\cdot})}/G_{[6^{\cdot}]}\cong S_6 \quad\text{and}\quad G_{(6^{\cdot\cdot})}/G_{[6^{\cdot\cdot}]}\cong S_6.$$

Next, we consider the action of $G_{(6^{\cdot})}$ on the remaining 18 points of Ω and find

Proposition 3.2. *$G_{(6^{\cdot})}$ has 2 orbits $\Delta_2(6^{\cdot})$ and $\Delta_{16}(6^{\cdot})=\Delta_{16}(5)$, and $G_{(6^{\cdot})}\subset G_{(8^{\cdot})}$.*

Proof. $G_{[6^{\cdot}]}$ fixes $\Delta_3(5)-X$ pointwise and is regular on $\Delta_{16}(5)$. $G_{(5)}$ is S_3 on $\Delta_3(5)$. So $G_{(6^{\cdot})}$ has an orbit of length 2, $\Delta_2(6^{\cdot})=\Delta_3(5)-X$, and $\Delta_{16}(6^{\cdot})=\Delta_{16}(5)$. Clearly $G_{(6^{\cdot})}\subset G_{(8^{\cdot})}$.

Next we consider the action of $G_{(6^{\cdot\cdot})}$ on $\Omega-\mathbf{6}^{\cdot\cdot}$. $G_{[6^{\cdot\cdot}]}=\mathrm{syl}_3\,(M_{19})$. Let $\tau^3=1$ and $\langle\tau\rangle=\mathrm{syl}_3\,(M_{19})$. Then τ has cycle type $1^6 3^6$ and $\mathbf{6}^{\cdot\cdot}$ is the six points left fixed by τ. We find the following

Proposition 3.3. *$G_{(6^{\cdot\cdot})}$ is S_6 on the six 3-cycles of τ and thus transitive on $\Omega-\mathbf{6}^{\cdot\cdot}$.*

Proof. Obviously $N_{M_{24}}(\langle\tau\rangle)\subset G_{(6^{\cdot\cdot})}$, but by Lemma 1.2, $N_{M_{24}}(\mathrm{syl}_3\,(G_{[5]}))$ is S_6 on the $\mathbf{6}^{\cdot\cdot}$, so $G_{(6^{\cdot\cdot})}=N_{M_{24}}(\mathrm{syl}_3\,(G_{[5]}))$. Thus, it can be represented as a subgroup of S_6 on the six 3-cycles. If the kernel K of this representation is greater than $\mathrm{syl}_3\,(G_{[5]})$, $K/\mathrm{syl}_3\,(G_{[5]})$ is normal in S_6 and $|K|=2^3\cdot 3^3\cdot 5$. This is impossible because elements of order 5 cannot fix all the six 3-cycles setwise. So $K=\mathrm{syl}_3\,(G_{[5]})$ and $G_{(6^{\cdot\cdot})}$ is S_6 on these six 3-cycles of $\mathrm{syl}_3\,(G_{[5]})$.

The following further characterization of $G_{(6^{\cdot\cdot})}$ is necessary for the future discussion of the maximality of $G_{(6^{\cdot\cdot})}$.

PROPOSITION 3.4. *Let the representation of $G_{(6^{\cdot\cdot})}$ on the six 3-cycles of τ be denoted by $\tilde{S}_6$. Then $G_{(6^{\cdot\cdot})}{}^{\mathbf{6}^{\cdot\cdot}}=S_6$ and $\tilde{S}_6$ are in permutation isomorphism.*

Proof. As has been seen, every $\mathbf{5}$ determines a unique $\Delta_3(5)$ and $\Delta_{16}(5)$. If $Y\in\Delta_{16}(5)$ then $\mathbf{5}\cup Y$ is a $\mathbf{6}^{\cdot\cdot}$.

Now $\mathbf{5}\cup\Delta_3(5)$ is the set of points left fixed by a $\mathrm{syl}_2\,(M_{19})$, of which the normalizer in M_{24} is the group $\mathfrak{H}$. In particular $\mathfrak{H}$ is faithfully represented on $\Delta_{16}(5)$ as the holomorph of the elementary abelian group $\mathrm{syl}_2\,(M_{19})$. The subgroup K of $\mathfrak{H}$ fixing the point Y is thus faithfully represented as the transitive group $\mathrm{GL}_4\,(2)\cong A_8$ on the remaining 15 points. A homomorphic image of K acts nontrivially on $\mathbf{5}\cup\Delta_3(5)$. By the simplicity of A_8, this image is precisely A_8 on the 8 points.

Consider now the subgroup L of K which stabilizes $\mathbf{5}$. $L\cong S_5$, whence L is the subgroup of $G_{(6^{\cdot\cdot})}$ fixing Y. L normalizes τ and the corresponding group $\tilde{L}$ on the six 3-cycles is isomorphic to S_5, since elements of order 5 cannot be in the kernel. $\tilde{L}$ fixes $\Delta_3(5)$, the correspondence $S_6\leftrightarrow\tilde{S}_6$ of $G_{(6^{\cdot\cdot})}$ associates permutation isomorphic S_5's, and the theorem follows.

We refer to Figure 2 in the appendix during the discussion of $\mathbf{7}$. Construct two $\mathbf{7}$'s by $\mathbf{6}^{\cdot}\cup\alpha_1$ and $\mathbf{6}^{\cdot}\cup Y$. Then $G_{[\mathbf{6}^{\cdot}\cup\alpha_1]}$ is the elementary abelian group of order 2^4 and $G_{[\mathbf{6}^{\cdot}\cup Y]}=\mathrm{syl}_2\,(M_{19})\cap\mathrm{syl}_3\,(M_{19})=E$. Therefore $\{\mathbf{6}^{\cdot}\cup\alpha_1\}\not\approx\{\mathbf{6}^{\cdot}\cup Y\}$. Call $\{\mathbf{6}^{\cdot}\cup\alpha_1\}\ \mathbf{7}^{\cdot}$ and $\{\mathbf{6}^{\cdot}\cup Y\}\ \mathbf{7}^{\cdot\cdot}$. The study of $G_{[7]}$ and $G_{(7)}$ leads to

PROPOSITION 3.5. *$G_{(7^{\cdot})}/G_{[7^{\cdot}]}=A_7$ and $G_{(7^{\cdot})}\subset H$. And $G_{(7^{\cdot\cdot})}=S_6$ fixing Y and $G_{(7^{\cdot\cdot})}\subset G_{(6^{\cdot})}$. Also, $\mathbf{7}^{\cdot\cdot}=\mathbf{6}^{\cdot\cdot}\cup z$, $z\in\Delta_{18}(6^{\cdot\cdot})$.*

Proof. $G_{(7^{\cdot})}<\mathfrak{H}_{[\alpha_2]}$ and $\mathfrak{H}_{[\alpha_2]}<G_{(7^{\cdot})}$. So $G_{(7^{\cdot})}=\mathfrak{H}_{[\alpha_2]}$ and $G_{(7^{\cdot})}/G_{[7^{\cdot}]}=A_7$ on $\mathbf{7}^{\cdot}$. Then $|G_{(7^{\cdot\cdot})}|=6!$. Now $(G_{(6^{\cdot})})_{[Y]}$ is in $G_{(7^{\cdot\cdot})}$ and has order $6!$, so $G_{(7^{\cdot\cdot})}=(G_{(6^{\cdot})})_{[Y]}$, and $G_{(7^{\cdot\cdot})}\subset G_{(6^{\cdot})}$.

We have $\mathbf{7}^{\cdot\cdot}=\mathbf{6}^{\cdot}\cup Y$, $Y\in\Delta_{16}(6^{\cdot})=\Delta_{16}(5)$. But $\mathbf{6}=\mathbf{5}\cup X$, $X\in\Delta_3(5)$. Therefore $\mathbf{7}^{\cdot\cdot}=\mathbf{5}\cup X\cup Y$ and $\mathbf{7}^{\cdot\cdot}=\mathbf{6}^{\cdot\cdot}\cup X$. Therefore $\mathbf{7}^{\cdot\cdot}\supset\mathbf{6}^{\cdot\cdot}$ and $\mathbf{7}^{\cdot\cdot}=\mathbf{6}^{\cdot\cdot}\cup Z$, $Z\in\Delta_{18}(6^{\cdot\cdot})$. Since $G_{(7^{\cdot\cdot})}$ fixes a point from $\Delta_{16}(6^{\cdot})$, $\mathbf{7}^{\cdot\cdot}$ contains one $\mathbf{6}^{\cdot}$ and six $\mathbf{6}^{\cdot\cdot}$'s, and $\mathbf{7}^{\cdot}$ contains only $\mathbf{6}^{\cdot}$'s.

Next we will consider the actions of $G_{(7^{\cdot})}$ and $G_{(7^{\cdot\cdot})}$ on the remaining 17 points and find

PROPOSITION 3.6. (A) *$G_{(7^{\cdot})}$ has two orbits of lengths 1 and 16, and $\Delta_1(7^{\cdot})=\Delta_2(6^{\cdot})-\alpha_1$ and $\Delta_{16}(7^{\cdot})=\Delta_{16}(6^{\cdot})$ on the remaining 17 points.*

(B) $G_{(7^{\cdot\cdot})}$ *has two orbits of lengths* 2 *and* 15, *and* $\Delta_2(7^{\cdot\cdot})=\Delta_2(6^{\cdot})$ *and* $\Delta_{15}(7^{\cdot\cdot})=\Delta_{16}(6^{\cdot})-Y$.

Proof. (A) is obvious since $G_{[7^{\cdot}]}$ has orbits of lengths 1 and 16.

(B) We have seen $G_{(7^{\cdot\cdot})}$ is $(G_{(6^{\cdot})})_{[Y]}$. Since $(G_{(6^{\cdot})})_{[Y]}$ is S_6 on the six points, there are elements in $(G_{(6^{\cdot})})_{[Y]}$, α, β and γ, such that

$\alpha = 5\ 1$	1	5^3	1^2
$\beta = 3\ 1^3$	1	3^5	1^2
$\gamma = 6^1$	1	$6\ 3^2\ 2\ 1$	2
on $\mathbf{6}^{\cdot}$	Y	$\Delta_{16}(6^{\cdot})-Y$	$\Delta_2(6^{\cdot})$

The elements assert (B).

4. $G_{(8^{\cdot})}$, $G_{(8^{\cdot\cdot})}$ **and** $G_{(8^{\cdot\cdot\cdot})}$. We have

PROPOSITION 4.1.

$\mathbf{8}^{\cdot}=\mathbf{7}^{\cdot}\cup\alpha_2$, $\alpha_2\in\Delta_1(7^{\cdot})$.

$\mathbf{8}^{\cdot\cdot}=\mathbf{7}^{\cdot}\cup X$, $X\in\Delta_{16}(7^{\cdot})$; $\mathbf{8}^{\cdot\cdot}=\mathbf{7}^{\cdot\cdot}\cup\delta$, $\delta\in\Delta_2(7^{\cdot\cdot})$.

$\mathbf{8}^{\cdot\cdot\cdot}=\mathbf{7}^{\cdot\cdot}\cup Z$, $Z\in\Delta_{15}(7^{\cdot\cdot})$.

$G_{(8^{\cdot})}$ *is the holomorph of the* $\mathfrak{A}_{16}$, *as described in* §2.

$G_{(8^{\cdot\cdot})}$ *is* A_7 *fixing* X, $G_{[8^{\cdot\cdot}]}=E$ *and* $|G_{(8^{\cdot\cdot})}|=|A_7|$.

$G_{(8^{\cdot\cdot\cdot})}$ *is the imprimitive group of degree* 8 *of order* 384 *with systems of imprimitivity of length* 2, *the kernel of imprimitivity* $C_2\times C_2\times C_2\times C_2$, *and the image of the imprimitivity*, S_4. $G_{[8^{\cdot\cdot\cdot}]}=E$.

Proof. The assertion for $\mathbf{8}^{\cdot}$ and $G_{(8^{\cdot})}$ is now trivial.

Let $\mathbf{8}=\mathbf{7}^{\cdot}\cup X$, $X\in\Delta_{16}(7^{\cdot})$. Then $G_{[8]}=E$, so $\mathbf{8}\not\approx\mathbf{8}^{\cdot}$. Call this $\mathbf{8}$ an $\mathbf{8}^{\cdot\cdot}$. By Lemma 1.5, there is one more type of $\mathbf{8}$. The $\mathbf{8}^{\cdot\cdot\cdot}$ must come from $\mathbf{7}^{\cdot\cdot}$. Therefore either $\mathbf{7}^{\cdot\cdot}\cup\delta=\mathbf{8}^{\cdot\cdot\cdot}$ with $\delta\in\Delta_2(7^{\cdot\cdot})$ or $\mathbf{7}^{\cdot\cdot}\cup Z=\mathbf{8}^{\cdot\cdot\cdot}$ with $Z\in\Delta_{15}(7^{\cdot\cdot})$. In any case $G_{[8^{\cdot\cdot\cdot}]}=E$.

Now,

$$(*)\qquad 8!/|G_{(8^{\cdot\cdot})}|+8!/|G_{(8^{\cdot\cdot\cdot})}| = 121.$$

Since $G_{[8^{\cdot\cdot}]}=G_{[8^{\cdot\cdot\cdot}]}=E$, the above equation shows the sum of the indices of the two groups in S_8 is 121.

$(G_{(7^{\cdot})})_{[X]}$ has order $7!/2$ and is contained in $G_{(8^{\cdot\cdot})}$. Therefore $|A_7|$ divides $|G_{(8^{\cdot\cdot})}|$, so that the index of $G_{(8^{\cdot\cdot})}$ in S_8 is 2^0, 2^1, 2^2, 2^3 or 2^4. Since $2^4+3\cdot5\cdot7=121$ and $2^0+2^3\cdot3\cdot5=121$, $G_{(8^{\cdot\cdot})}$ can only have index either 2^0 or 2^4 in S_8. If it has index 2^0 then $G_{(8^{\cdot\cdot})}=S_8$, then $(G_{(8^{\cdot\cdot})})_{[X]}$ must be S_7 on $\mathbf{7}^{\cdot}$ which is impossible because $G_{(7^{\cdot})}=A_7$. Therefore $G_{(8^{\cdot\cdot})}$ must be A_7 on $\mathbf{8}^{\cdot\cdot}$ fixing X, and $(G_{(7^{\cdot})})_{[X]}=G_{(8^{\cdot\cdot})}$.

By (*), $[S_8 : G_{(8^{\cdot\cdot\cdot})}]=105$ and $|G_{(8^{\cdot\cdot\cdot})}|=2^7\cdot3$.

Now take $\mathbf{7}^{\cdot\cdot}\cup Z$, $Z\in\Delta_{15}(7^{\cdot\cdot})$. Since $G_{(7^{\cdot\cdot})}=S_6$ and $(G_{(7^{\cdot\cdot})})_{[Z]}\subset G_{(7^{\cdot\cdot}\cup Z)}$, $2^4\cdot3$ divides $|G_{(7^{\cdot\cdot}\cup Z)}|$. Therefore $\{\mathbf{7}^{\cdot\cdot}\cup Z\}\not\approx\mathbf{8}^{\cdot\cdot}$, and $\{\mathbf{7}^{\cdot\cdot}\cup Z\}\approx\mathbf{8}^{\cdot\cdot\cdot}$. Since $(G_{(8^{\cdot\cdot\cdot})})_{[Z]}=(G_{(7^{\cdot\cdot})})_{[Z]}$ and $|G_{(8^{\cdot\cdot\cdot})}|=2^7\cdot3$, $G_{(8^{\cdot\cdot\cdot})}$ is transitive on $\mathbf{8}^{\cdot\cdot\cdot}$. The transitive group of degree 8, order 384, is not a primitive group. There is only one imprimitive group of degree 8, order 384. It has four blocks of length 2. The kernel of imprimitivity of the group is $C_2 \times C_2 \times C_2 \times C_2$ and the image is S_4.

Now consider $\mathbf{7}^{\cdot\cdot} \cup \delta$, $\delta \in \Delta_2(7^{\cdot\cdot})$. Since $G_{(8^{\cdot})}$ is transitive on $\mathbf{8}^{\cdot}$, $\{\mathbf{7}^{\cdot\cdot} \cup \delta\} \not\approx \mathbf{8}^{\cdot}$. Since $|(G_{(7^{\cdot\cdot})})_{[\delta]}| = 2^3 \cdot 3^2 \cdot 5$ and 5 does not divide $|G_{(8^{\cdot\cdot\cdot})}|$, $\{\mathbf{7}^{\cdot\cdot} \cup \delta\} \not\approx \mathbf{8}^{\cdot\cdot\cdot}$, so $\{\mathbf{7}^{\cdot\cdot} \cup \delta\} \approx \mathbf{8}^{\cdot\cdot}$. The above can also be seen in the following way. $\mathbf{8}^{\cdot\cdot} = \mathbf{7}^{\cdot} \cup X$, $X \in \Delta_{16}(7^{\cdot}) = \Delta_{16}(6^{\cdot})$. But $\mathbf{7}^{\cdot} = \mathbf{6}^{\cdot} \cup \alpha_1$, $\alpha_1 \in \Delta_2(6^{\cdot})$. So $\mathbf{8}^{\cdot\cdot} = \mathbf{6}^{\cdot} \cup \alpha_1 \cup X = \mathbf{7}^{\cdot\cdot} \cup \alpha_1$. Since $\alpha_1 \in \Delta_2(6^{\cdot}) = \Delta_2(7^{\cdot\cdot})$, $\mathbf{8}^{\cdot\cdot} = \mathbf{7}^{\cdot\cdot} \cup \delta$, $\delta \in \Delta_2(7^{\cdot\cdot})$.

Proposition 4.2. *$G_{(8^{\cdot\cdot})}$ has two orbits $\Delta_1(8^{\cdot\cdot}) = \Delta_1(7^{\cdot})$ and $\Delta_{15}(8^{\cdot\cdot})$ on the remaining 16 points. And $G_{(8^{\cdot\cdot})} \subset G_{(7^{\cdot})}$.*

Proof. By Proposition 4.1, $G_{(8^{\cdot\cdot})} = (G_{(7^{\cdot})})_{[X]}$, so $G_{(8^{\cdot\cdot})} \subset G_{(7^{\cdot})}$ and $\Delta_1(7^{\cdot})$ remains as an orbit of $G_{(8^{\cdot\cdot})}$, i.e. $\Delta_1(7^{\cdot}) = \Delta_1(8^{\cdot\cdot})$.

Since $G_{(7^{\cdot})}$ is A_7 on $\mathbf{7}^{\cdot}$, there is an element of order 7, α, in $G_{(8^{\cdot\cdot})}$ such that

$$\alpha = \underbrace{(\,.\,.\,.\,.\,.\,.\,.\,)}_{\mathbf{7}^{\cdot}}(\alpha_2)(X)(.)(\,.\,.\,.\,.\,.\,.\,.\,)(\,.\,.\,.\,.\,.\,.\,.\,)$$

and also an element of order 5, β, in $G_{(8^{\cdot\cdot})}$ such that

$$\beta = \underbrace{(.)(.)(\,.\,.\,.\,.\,.\,)}_{\mathbf{7}^{\cdot}}(\alpha_2)(X)(\,.\,.\,.\,.\,.\,)(\,.\,.\,.\,.\,.\,)(\,.\,.\,.\,.\,.\,).$$

Therefore $G_{(8^{\cdot\cdot})}$ has an orbit of length 15, $\Delta_{15}(8^{\cdot\cdot})$.

As for the action of $G_{(8^{\cdot\cdot\cdot})}$ on $\Omega - \mathbf{8}^{\cdot\cdot\cdot}$, we have

Proposition 4.3. *$G_{(8^{\cdot\cdot\cdot})}$ has two orbits of lengths 8 and 8. One of them is an $\mathbf{8}^{\cdot\cdot\cdot}$ and the other is an $\mathbf{8}^{\cdot}$. We will denote them by $\Delta_{8^{\cdot\cdot\cdot}}(8^{\cdot\cdot\cdot})$ and $\Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$ to distinguish them. Also, $G_{(8^{\cdot\cdot\cdot})} \subset G_{(8^{\cdot})}$.*

Proof. We have seen that $G_{(8^{\cdot})}$, the holomorph of $\mathfrak{A}_{16}$, can be represented as a group of matrices of the form $\left[\begin{smallmatrix} A & \alpha \\ \cdot & 1 \end{smallmatrix}\right]$ where $A \in \mathrm{GL}_4(2)$ and $\alpha \in V_4(F_2)$. Identify $\Omega - \mathbf{8}^{\cdot}$ with the vectors $\left[\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right]$, $\alpha \in V_4(F_2)$. Then the action of $\mathfrak{H} = G_{(8^{\cdot})}$ on $\Omega - \mathbf{8}^{\cdot}$ is matrix multiplication.

Let Γ be a set of vectors of the form

$$\begin{bmatrix} xy \\ x \\ y \\ z \\ 1 \end{bmatrix}$$

and Δ be a set of vectors of the form

$$\begin{bmatrix} xy+1 \\ x \\ y \\ z \\ 1 \end{bmatrix}.$$

Then $|\Gamma|=|\Delta|=8$ and $\Omega-8^{\cdot}=\Gamma\cup\Delta$.

Let K be the subgroup of $\mathfrak{H}$ consisting of all

$$\begin{bmatrix} 1 & (a+e)c+fa & (b+e)d+fb & 0 & ef \\ 0 & a & b & 0 & e \\ 0 & c & d & 0 & f \\ g & h & k & 1 & l \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\in GL_2(2)$. Then $|K|=2^6\cdot|GL_2(2)|=384=|G_{(8^{\cdots})}|$.

It is easy to verify that Γ is invariant under K and that K is faithful on Γ. Since $G_{(8^{\cdot})}$ induces A_8 on $\mathbf{8}^{\cdot}$ and $G_{(8^{\cdot\cdot})}$ induces A_7 on $\mathbf{8}^{\cdot\cdot}$, and 384 does not divide $|A_8|$ or $|A_7|$, Γ must be an $\mathbf{8}^{\cdots}$, and $K=G_{(\Gamma)}$ is transitive on Γ. Similarly Δ is an $\mathbf{8}^{\cdots}$ and $K=G_{(\Delta)}$.

Finally let L be the subgroup of K consisting of the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ g & h & k & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then L is elementary abelian of order 8 and its involutions are of type $1^8 2^4$ on $\Omega-\mathbf{8}^{\cdot}$, and so of type 2^4 on $\mathbf{8}^{\cdot}$. Hence L is regular on $\mathbf{8}^{\cdot}$. Thus the orbits of $K=G_{(8^{\cdots})}$ are Γ, Δ and the original $\mathbf{8}^{\cdot}$.

5. $G_{(9^{\cdot})}$, $G_{(9^{\cdot\cdot})}$ **and** $G_{(9^{\cdots})}$. We have

PROPOSITION 5.1. (A) $\mathbf{9}^{\cdot}=\mathbf{8}^{\cdot}\cup\alpha$, $\alpha\in\Delta_{16}(8^{\cdot})$.

(B) $G_{(9^{\cdot})}=(G_{(8^{\cdot})})_{[\alpha]}$ *is* A_8 *on* $\mathbf{8}^{\cdot}$ *and* S_1 *on* α.

(C) *Also,* $\mathbf{9}^{\cdot}=\mathbf{8}^{\cdot\cdot}\cup\alpha_2$, $\alpha_2\in\Delta_1(8^{\cdot\cdot})$, *and* $\mathbf{9}^{\cdot}$ *contains one* $\mathbf{8}^{\cdot}$ *and eight* $\mathbf{8}^{\cdot\cdot}$*'s.*

(D) $\mathbf{9}^{\cdot\cdot}=\mathbf{8}^{\cdot\cdot}\cup\beta$, $\beta\in\Delta_{15}(8^{\cdot\cdot})$.

(E) $G_{(9^{\cdot\cdot})}=(G_{(8^{\cdot\cdot})})_{[x,\beta]}\times C_2$, *where* x *is the point of* $\mathbf{8}^{\cdot\cdot}$ *left fixed by* $G_{(8^{\cdot\cdot})}$, *has two orbits of lengths* 7 *and* 2 *on* $\mathbf{9}^{\cdot\cdot}$ *and* $|G_{(9^{\cdot\cdot})}|=2^3\cdot3\cdot7\cdot2=336$.

(F) $\Delta_{15}(7^{\cdot\cdot})=\Delta_{15}(8^{\cdot\cdot})$.

(G) $\mathbf{9}^{\cdots}=\mathbf{8}^{\cdots}\cup\delta$, $\delta\in\Delta_{8^{\cdot}}(8^{\cdots})$.

(H) $\mathbf{9}^{\cdot\cdot}=\mathbf{8}^{\cdots}\cup\gamma$, $\gamma\in\Delta_{8^{\cdots}}(8^{\cdots})$, *and* $\mathbf{9}^{\cdot\cdot}$ *contains two* $\mathbf{8}^{\cdot\cdot}$*'s and seven* $\mathbf{8}^{\cdots}$*'s.*

(I) $G_{(9^{\cdots})}$ *is transitive on* $\mathbf{9}^{\cdots}$ *and has order* $3^2\cdot2^4\cdot3=432$, *and* $G_{(9^{\cdots})}$ *is the holomorph of an elementary abelian group of order* 3^2.

Proof. Let

(A) $\mathbf{9}^{\cdot}=\mathbf{8}^{\cdot}\cup\alpha$, $\alpha\in\Delta_{16}(8^{\cdot})$. $(G_{(8^{\cdot})})_{[\alpha]}=A_8$, so $G_{(9^{\cdot})}$ is not transitive on $\mathbf{9}^{\cdot}$, otherwise, $|G_{(9^{\cdot})}|=3^2\cdot|A_8|=3^2\cdot2^6\cdot3^2\cdot5\cdot7$. Therefore,

(B) $G_{(9^{\cdot})}=(G_{(8^{\cdot})})_{[\alpha]}$.

Since $G_{(8^{\cdot})}$ is 3-transitive on $\Delta_{16}(8^{\cdot})$, $G_{(9^{\cdot})}$ is 2-transitive on $\Delta_{15}(9^{\cdot})=\Delta_{16}(8^{\cdot})-\alpha$.

Now $\mathbf{9}^{\cdot}=\mathbf{8}^{\cdot} \cup \alpha$, $\alpha \in \Delta_{16}(8^{\cdot})$. But $\mathbf{8}^{\cdot}=\mathbf{7}^{\cdot} \cup \alpha_2$, $\alpha_2 \in \Delta_1(7^{\cdot})=\Delta_1(8^{\cdot\cdot})$, $\alpha \in \Delta_{16}(7^{\cdot})$. Therefore $\mathbf{9}^{\cdot}=\mathbf{7}^{\cdot} \cup \alpha \cup \alpha_2=\mathbf{8}^{\cdot\cdot} \cup \alpha_2$. Therefore

(C) $\mathbf{9}^{\cdot}=\mathbf{8}^{\cdot\cdot} \cup \alpha_2$, $\alpha_2 \in \Delta_1(8^{\cdot\cdot})$.

If we take out α which is the point left fixed by $G_{(9^{\cdot})}$ from $\mathbf{9}^{\cdot}$ we obtain an $\mathbf{8}^{\cdot}$. If we take out a point $\neq \alpha$ from $\mathbf{9}^{\cdot}$ we obtain an $\mathbf{8}^{\cdot\cdot}$. Therefore, $\mathbf{9}^{\cdot}$ contains one $\mathbf{8}^{\cdot}$ and eight $\mathbf{8}^{\cdot\cdot}$'s.

Next, let $\mathbf{9}=\mathbf{8}^{\cdot\cdot} \cup \beta$, $\beta \in \Delta_{15}(8^{\cdot\cdot})$. $(G_{(8^{\cdot\cdot})})_{[\beta]}=(G_{(9)})_{[\beta]}$ and $|(G_{(8^{\cdot\cdot})})_{[\beta]}|=|A_7|/15=2^3 \cdot 3 \cdot 7$. If $G_{(9)}$ is transitive on $\mathbf{9}$, $|G_{(9)}|=3^2 \cdot 2^3 \cdot 3 \cdot 7=1512$. By Cole [2, p. 258], a transitive group of order 1512, degree 9, is unique and contains an element of order 9. M_{24} does not contain such an element, so $G_{(9)}$ is intransitive. Now, $(G_{(8^{\cdot\cdot})})_{[\beta]}=(G_{(8^{\cdot\cdot})})_{[x,\beta]}$ where x is the point left fixed by $G_{(8^{\cdot\cdot})}$. $(G_{(8^{\cdot\cdot})})_{[x,\beta]} \subset G_{(9)}$ and, since 7 divides $|(G_{(8^{\cdot\cdot})})_{[x,\beta]}|$, $(G_{(8^{\cdot\cdot})})_{[x,\beta]}$ has three orbits $\mathbf{7}^{\cdot}$, $\{x\}$, and $\{\beta\}$. Since $\mathfrak{A}_{16}$ fixes all the 8 points of $\mathbf{8}^{\cdot}=\mathbf{7}^{\cdot} \cup \alpha_2$, where $\alpha_2=\Delta_1(7^{\cdot})$ and is regular on the remaining 16 points, there is an element g such that g fixes $\mathbf{8}^{\cdot}$ pointwise and interchanges x and β. Therefore $g \in G_{(9)}$. Thus $G_{(9)}$ must have two orbits $\mathbf{7}^{\cdot}$ and $\{x, \beta\}$. Therefore $\mathbf{9} \not\approx \mathbf{9}^{\cdot}$. Call this $\mathbf{9}$ a $\mathbf{9}^{\cdot\cdot}$. Now we have

(D) $\mathbf{9}^{\cdot\cdot}=\mathbf{8}^{\cdot\cdot} \cup \beta$, $\beta \in \Delta_{15}(8^{\cdot\cdot})$ and $G_{(9^{\cdot\cdot})}{}^{\mathbf{7}^{\cdot}}=(G_{(8^{\cdot\cdot})})_{[x,\beta]}$ and $G_{(9^{\cdot\cdot})}{}^{\{x,\beta\}}=C_2$. Therefore

(E) $G_{(9^{\cdot\cdot})}=(G_{(8^{\cdot\cdot})})_{[x,\beta]} \times C_2$ and has order $2 \cdot 2^3 \cdot 3 \cdot 7=336$.

Now we show that $\Delta_{15}(8^{\cdot\cdot})=\Delta_{15}(7^{\cdot\cdot})$. $\Delta_2(7^{\cdot\cdot})=\Delta_2(6^{\cdot})$ by Proposition 3.6(B) and $\Delta_1(7^{\cdot})=\{\Delta_2(6^{\cdot})$ minus a point$\}$ by Proposition 3.6(A), so $\Delta_1(7^{\cdot})=\{\Delta_2(7^{\cdot\cdot})$ minus a point$\}$, but $\Delta_1(8^{\cdot\cdot})=\Delta_1(7^{\cdot})$ by Proposition 4.2, so $\Delta_1(8^{\cdot\cdot})=\{\Delta_2(7^{\cdot\cdot})$ minus a point$\}$. Let $\Delta_2(7^{\cdot\cdot})=\{\xi, \eta\}$, and let $\mathbf{8}^{\cdot\cdot}=\mathbf{7}^{\cdot\cdot} \cup \xi$, then by the above $\{\eta\}=\Delta_1(8^{\cdot\cdot})$. Therefore we have

(F) $\Delta_{15}(7^{\cdot\cdot})=\Delta_{15}(8^{\cdot\cdot})$.

Now $\mathbf{9}^{\cdot\cdot}=\mathbf{8}^{\cdot\cdot} \cup \beta$, $\beta \in \Delta_{15}(8^{\cdot\cdot})$. But $\mathbf{8}^{\cdot\cdot}=\mathbf{7}^{\cdot\cdot} \cup \alpha_1$, $\alpha_1 \in \Delta_2(7^{\cdot\cdot})$. Therefore $\mathbf{9}^{\cdot\cdot}=\mathbf{7}^{\cdot\cdot} \cup \beta \cup \alpha_1$, $\beta \in \Delta_{15}(7^{\cdot\cdot})$ by the above. So $\mathbf{9}^{\cdot\cdot}=\mathbf{8}^{\cdot\cdot\cdot} \cup \alpha_1$, $\alpha_1 \in \Delta_{8^{\cdot\cdot\cdot}}(8^{\cdot\cdot\cdot})$ or $\alpha_1 \in \Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$.

Next, let another $\mathbf{9}=\mathbf{8}^{\cdot\cdot\cdot} \cup \delta$, $\delta \in \Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$. $G_{(8^{\cdot\cdot\cdot})}$ has two orbits $\Delta_{8^{\cdot\cdot\cdot}}(8^{\cdot\cdot\cdot})$ and $\Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$ and there is an element α in $G_{(8^{\cdot\cdot\cdot})}$ such that

$$\begin{array}{cccc} \alpha = 8 & 8 & (\delta)(.)(\ldots.)(..). \\ \mathbf{8}_L^{\cdot\cdot\cdot} & \mathbf{8}_R^{\cdot\cdot\cdot} & \mathbf{8}^{\cdot} \end{array}$$

Therefore $(G_{(8^{\cdot\cdot\cdot})})_{[\delta]}$ is transitive on $\mathbf{8}_L^{\cdot\cdot\cdot}$.Therefore this $\mathbf{9} \not\approx \mathbf{9}^{\cdot\cdot}$, and

(G) $\mathbf{9}^{\cdot\cdot\cdot}=\mathbf{8}^{\cdot\cdot\cdot} \cup \delta$, $\delta \in \Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$ and

(H) $\mathbf{9}^{\cdot\cdot}=\mathbf{8}^{\cdot\cdot\cdot} \cup \gamma$, $\gamma \in \Delta_{8^{\cdot\cdot\cdot}}(8^{\cdot\cdot\cdot})$.

Now if we take a point ξ from the orbit of length 2 of $G_{(9^{\cdot\cdot})}$, we get $\mathbf{8}^{\cdot\cdot}$ and $\xi \in \Delta_{15}(8^{\cdot\cdot})$ and if we take away a point η from the orbit of length 7 of $G_{(9^{\cdot\cdot})}$ we get $\mathbf{8}^{\cdot\cdot\cdot}$ and $\eta \in \Delta_{8^{\cdot\cdot\cdot}}(8^{\cdot\cdot\cdot})$.

Now we have $9!/|G_{(9^{\cdot\cdot})}|=840$ and $|(G_{(8^{\cdot\cdot\cdot})})_{[\delta]}|=2^4 \cdot 3$. If $G_{(9^{\cdot\cdot\cdot})}$ is intransitive, then $(G_{(8^{\cdot\cdot\cdot})})_{[\delta]}=G_{(9^{\cdot\cdot\cdot})}$ and $|G_{(9^{\cdot\cdot\cdot})}|=2^4 \cdot 3$. Then $9!/|G_{(9^{\cdot\cdot\cdot})}|=2^3 \cdot 3^3 \cdot 5 \cdot 7=7560>840$. Therefore

(I) $G_{(9^{\cdots})}$ is transitive on $\mathbf{9}^{\cdots}$ and has order $3^2 \cdot 2^4 \cdot 3 = 432$, so is the holomorph of $\mathfrak{A}_{3^2}$, an elementary abelian group of order 3^2, since there is only one transitive group of degree 9 with this order.

The fact that $G_{(9^{\cdot})}$ is 2-transitive on $\Delta_{15}(9^{\cdot})$ is already mentioned. For the actions of $G_{(9^{\cdot\cdot})}$ on the remaining 15 points, we have

PROPOSITION 5.2. *$G_{(9^{\cdot\cdot})}$ has two orbits of lengths* 1 *and* 14, $\Delta_1(9^{\cdot\cdot})$ *and* $\Delta_{14}(9^{\cdot\cdot})$.

Proof. $G_{(9^{\cdot\cdot})}$ is a direct product of $(G_{(8^{\cdot\cdot})})_{[x,\beta]}$ on $\mathbf{7}^{\cdot}$ and C_2 on $\{x, \beta\}$ where $\mathbf{9}^{\cdot\cdot} = \mathbf{8}^{\cdot\cdot} \cup \beta$, $\beta \in \Delta_{15}(8^{\cdot\cdot})$. $(G_{(8^{\cdot\cdot})})_{[x,\beta]}$ has order 168. The group of degree 7 and order 168 is unique, by Miller [7, p. 395], and has, in particular, an element of type 7. Then $G_{(9^{\cdot\cdot})}$ has an element of type $7^1 2^1$. Action of this element on $\Omega - \mathbf{9}^{\cdot\cdot}$ is $14^1 1^1$. Since 15 does not divide $|G_{(9^{\cdot\cdot})}|$, $G_{(9^{\cdot\cdot})}$ has two orbits of lengths 1 and 14.

PROPOSITION 5.3. (A) *$G_{(9^{\cdots})}$ has two orbits,* $\Delta_3(9^{\cdots})$ *and* $\Delta_{12}(9^{\cdots})$ *of lengths* 3 *and* 12.

(B) $G_{(9^{\cdots})}^{\Delta_3(9^{\cdots})} \cong S_4/V$, *$V$ is a Klein group.*

Proof. (A) $G_{(9^{\cdots})}$ is represented as

$$\left\{ \left[\begin{array}{cc|c} \alpha & \beta & x \\ \gamma & \delta & y \\ \hline \cdot & \cdot & 1 \end{array} \right] \right\}, \qquad \alpha, \beta, \gamma, \delta, x, y \in \mathrm{GF}\,[3] \text{ and } \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}_2\,(3),$$

and is doubly transitive. The conjugacy of $G_{(9^{\cdot\cdot})}$ is determined as follows.

On $\mathbf{9}^{\cdots}$	Class Length	On $\Omega - \mathbf{9}^{\cdots}$	Points Fixed on **15**
1^9	1	1^{15}	15
$1^3 2^3$	36	$1^5 2^5$	180
$1^3 3^2$	24	$1^3 3^4$	72
18	54	1248	54
18	54	1248	54
14^2	54	$1^3 2^2 4^2$	162
126	72	$123^2 6$	72
12^4	9	$1^7 2^4$	63
36	72	$1^2 2^2 36$	144

In addition to the above, there are two classes of elements of type 3^3 of class lengths 8 and 48 respectively. These elements can each act either as $1^6 3^3$ or 3^5 on **15**. If the elements in the class of length 8 act as $1^6 3^3$ on **15** then 48 points are fixed, and if they act as 3^5, then no point is fixed. Again, if the elements in the class of length 48 act as $1^6 3^3$, then 288 points are fixed, and if they act as 3^5, then no point is fixed.

Thus the minimum possible number of points in $\Omega - \mathbf{9}^{\cdots}$ left fixed is 816 and the maximum possible number of points in $\Omega-\mathbf{9}^{\cdots}$ left fixed is 1152. Therefore, $G_{(9\cdots)}$ has 2 orbits and $2\cdot 432=864$ points must be left fixed. And 8 elements of type 3^3 must act as $1^6 3^3$ fixing $864-816=48$ points and 48 elements of type 3^3 must act as 3^5 fixing no points.

The existence of cycle type 1248 on 15 points shows orbits are either of lengths 9, 6 or 12, 3. We will show that 9, 6 is impossible.

Suppose $G_{(9\cdots)}$ has two orbits Δ_9 and Δ_6 of lengths 9, 6 respectively. Since $G_{(9\cdot)}$ and $G_{(9\cdot\cdot)}$ are intransitive, Δ_9 is a $\mathbf{9}^{\cdots}$. If Δ_6 is a $\mathbf{6}^{\cdot}$, $|\mathrm{syl}_3\,(G_{(9\cdots)})^{\Delta_6}|=3^3$, but $|\mathrm{syl}_3\,(S_6)|=3^2$, so Δ_6 is a $\mathbf{6}^{\cdot\cdot}$. Therefore we have $\Delta_9=\mathbf{9}^{\cdots}$ and $\Delta_6=\mathbf{6}^{\cdot\cdot}$. The intransitive group of order 432, degree 18 on $\mathbf{9}^{\cdots}\cup\mathbf{9}^{\cdots}$ must be constructed by an automorphism of $G_{(9\cdots)}$. We have seen there is only one class of length 9 of elements of order 2 of type 12^4. Therefore, under any automorphism elements of type 12^4 must be combined with elements of type 12^4 to produce elements of $12^4+12^4+1^6$. The elements of type 1^6 must be on $\mathbf{6}^{\cdot\cdot}$, which is impossible. Therefore $G_{(9\cdots)}$ has two orbits of lengths 12, 3 on the remaining 15 points.

(B) First we note $G_{(9\cdots)}$ is S_3 on $\Delta_3(9^{\cdots})$. This is seen by the existence of the following types of elements:

$\mathbf{9}^{\cdots}$	$\Delta_{12}(9^{\cdots})$	$\Delta_3(9^{\cdots})$
18	84	12
3^3	3^4	3

Next denote by T the subgroup of the translations in $G_{(9\cdots)}$.

$$T=\left\{\left[\begin{array}{cc|c} 1 & \cdot & x \\ \cdot & 1 & y \\ \hline \cdot & \cdot & 1 \end{array}\right]\right\}.$$

Now we consider the representation of $G_{(9\cdots)}$ on $\Delta_3(9^{\cdots})$. Let K be the kernel of the representation. Obviously $T\cap K\neq E$. Now $T\cap K\lhd G_{(9\cdots)}$ and since $G_{(9\cdots)}$ is primitive, $T\cap K$ is transitive on $\mathbf{9}^{\cdots}$ and 3^2 divides $|T\cap K|$. Thus $T\subset K$. Therefore we have the representation of $G_{(9\cdots)}/T$ ($\cong\mathrm{GL}_2(3)$) on $\Delta_3(9^{\cdots})$. Since $\boldsymbol{Z}(\mathrm{GL}_2(3))$ maps to $\boldsymbol{Z}(S_3)=E$, we have the representation of $\mathrm{GL}_2(3)/\boldsymbol{Z}(\mathrm{GL}_2(3))\cong S_4$ on $\Delta_3(9^{\cdots})$. Now S_4 has unique normal subgroup of order 4, V, the Klein group, and $S_4/V\cong S_3$. Therefore $G_{(9\cdots)}{}^{\Delta_3(9^{\cdots})}\cong S_4/V$.

6. $G_{(10\cdot)}$, $G_{(10\cdot\cdot)}$, and $G_{(10\cdots)}$. First we have

Proposition 6.1. *Let* $\mathbf{10}^{\cdot}=\mathbf{9}^{\cdot}\cup\varepsilon$, $\varepsilon\in\Delta_{15}(9^{\cdot})$. *Then* $G_{(10\cdot)}=G_{1344}\times C_2$ *where* G_{1344} *is the holomorph of an elementary abelian group of order* 2^3 *and has order* 1344, *and* $G_{(10\cdot)}$ *is transitive on the remaining* 14 *points.*

Proof. Recall $G_{(9\cdot)}$ is A_8 on $\mathbf{8}^{\cdot}$, and S_1 on α. Let $\mathbf{10}^{\cdot}=\mathbf{9}^{\cdot}\cup\varepsilon$. Then

$$(G_{(10\cdot)})_{[\varepsilon]}=(G_{(9\cdot)})_{[\varepsilon]}\quad\text{and}\quad |(G_{(9\cdot)})_{[\varepsilon]}|=2^6\cdot 3\cdot 7=1344.$$

Denote this group by G_{1344}. Since $|\mathrm{syl}_2(G_{1344})| = |\mathrm{syl}_2(A_8)|$ and G_{1344} contains an element of order 7, G_{1344} is transitive on $\mathbf{8}^{\cdot}$. If $G_{(10^{\cdot})}$ is transitive on $\mathbf{10}^{\cdot}$, $|G_{(10^{\cdot})}| = 2^7\cdot 3\cdot 5\cdot 7$, so that 7 divides $|G_{(10^{\cdot})}|$ and $G_{(10^{\cdot})}$ must be primitive. But $(G_{(10^{\cdot})})_{[\varepsilon]} = (G_{(9^{\cdot})})_{[\varepsilon]} = (G_{(9^{\cdot})})_{[\alpha,\varepsilon]}$, so $G_{(10^{\cdot})}$ cannot be primitive. Therefore $G_{(10^{\cdot})}$ must be intransitive on $\mathbf{10}^{\cdot}$, with suborbits $\mathbf{8}^{\cdot}$, $\{\alpha\}$, $\{\varepsilon\}$. Now we recall, $\mathfrak{A}_{2^4}$ has an element which is identity on $\mathbf{8}^{\cdot}$ and $(\alpha\varepsilon)$ on $\{\alpha, \varepsilon\}$. So

$$G_{(10^{\cdot})} = G_{1344}\times C_2 \quad\text{and}\quad |G_{(10^{\cdot})}| = 2\cdot 1344.$$

G_{1344} is of order 1344 and degree 8, and so is the holomorph of an elementary abelian group of order 2^3. G_{1344} has an element of order 7, type 71, therefore $G_{(10^{\cdot})}$ contains an element of type 71 2, and action of this element on the remaining 14 points is uniquely determined as 71 2 14. Therefore, $G_{(10^{\cdot})}$ is transitive on the remaining 14 points.

As for $\mathbf{10}^{\cdot\cdot}$ and $G_{(10^{\cdot\cdot})}$, we have

PROPOSITION 6.2. $\mathbf{10}^{\cdot\cdot} = \mathbf{9}^{\cdot\cdot\cdot} \cup \mu$, $\mu \in \Delta_3(9^{\cdot\cdot\cdot})$ *and* $G_{(10^{\cdot\cdot})}$ *is* 3-*transitive and is* $P\Gamma L_2(3^2)$ *of order* 1440.

Proof. Consider $\mathbf{10} = \mathbf{9}^{\cdot\cdot\cdot} \cup \mu$, $\mu \in \Delta_3(9^{\cdot\cdot\cdot})$. $(G_{(10)})_{[\mu]} = (G_{(9^{\cdot\cdot\cdot})})_{[\mu]}$ and $|(G_{(10)})_{[\mu]}| = 2^4\cdot 3^2 = 144$. We show that $(G_{(9^{\cdot\cdot\cdot})})_{[\mu]} = (G_{(10)})_{[\mu]}$ is 2-transitive on $\mathbf{9}^{\cdot\cdot\cdot}$.

The character table of the holomorph of the elementary abelian group of order 3^2, $G_{(9^{\cdot\cdot\cdot})}$, shows clearly there is only one subgroup of $G_{(9^{\cdot\cdot\cdot})}$ of index 3, order 144. Call this subgroup of $G_{(9^{\cdot\cdot\cdot})}$ as G_{144}. The compound character of G_{144} in $G_{(9^{\cdot\cdot\cdot})}$ is easily computed and from it the following conjugacy of G_{144} is obtained. There are 12 elements of type $1^3 2^2$, 36 elements of type 18, 54 elements of type 14^2, 9 elements of type 12^4, 8 elements of type 3^3, 24 elements of type 36 and lastly 1 element of type 1^9.

The existence of elements of type 18 and 3^3 assures that the group G_{144} is doubly transitive.

Now we show that $G_{(10)}$ is transitive on $\mathbf{10}$, thus 3-transitive on $\mathbf{10}$. Recall $\mathbf{9}^{\cdot\cdot\cdot} = \mathbf{8}^{\cdot\cdot\cdot} \cup \delta$, $\delta \in \Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$. $G_{(8^{\cdot\cdot\cdot})}$ has an element σ of type

$$\begin{array}{ccc} 8 & (w)(x)(yz)(\ldots.) & 8 \\ \mathbf{8}^{\cdot\cdot\cdot} & \mathbf{8}^{\cdot} & \mathbf{8}^{\cdot\cdot\cdot} \end{array}$$

$\mathbf{9}^{\cdot\cdot\cdot}$ can be regarded as $\mathbf{8}^{\cdot\cdot\cdot} \cup w$, and $\Delta_3(9^{\cdot\cdot\cdot}) = \{x, y, z\}$. Thus $\mathbf{10}$ can be regarded as $\mathbf{8}^{\cdot\cdot\cdot} \cup w \cup x$. Now $|\mathrm{syl}_2(G_{(8^{\cdot\cdot\cdot})}{}^{\mathbf{8}^{\cdot}})| = 2^6$, so $\mathrm{syl}_2(G_{(8^{\cdot\cdot\cdot})}{}^{\mathbf{8}^{\cdot}}) = \mathrm{syl}_2(A_8)$. Therefore, there exists an element β in $G_{(8^{\cdot\cdot\cdot})}$, which interchanges w and x. Thus $\beta \in G_{(10)}$ and $G_{(10)}$ is transitive on $\mathbf{10}$.

Since $|(G_{(10)})_{[\mu]}| = 144$, $|G_{(10)}| = 1440$. Since 5 does not divide $|G_{(10^{\cdot})}|$, $\mathbf{10}^{\cdot} \not\approx \mathbf{10}$. Let $\mathbf{10} = \mathbf{10}^{\cdot\cdot}$.

The 3-transitive group of order 1440, degree 10 is unique and is $P\Gamma L_2(3^2)$.

PROPOSITION 6.3. (A) $\mathbf{10}^{\cdot\cdot\cdot} = \mathbf{9}^{\cdot\cdot\cdot} \cup \rho$, $\rho \in \Delta_{12}(9^{\cdot\cdot\cdot})$ *and* $G_{(10^{\cdot\cdot\cdot})}$ *has order* 144 *and is intransitive on* $\mathbf{10}^{\cdot\cdot\cdot}$ *with* 2 *orbits* Δ_6 *and* Δ_4 *of lengths* 6 *and* 4. $G_{(10^{\cdot\cdot\cdot})}{}^{\Delta_6}$ *is an imprimitive group of order* 36, *and* $G_{(10^{\cdot\cdot\cdot})}{}^{\Delta_4} = S_4$.

(B) $G_{(10\cdots)}$ *has two orbits on the remaining* 14 *points*, $\Delta_2(10^{\cdots})$ *and* $\Delta_{12}(10^{\cdots})$ *of lengths* 2 *and* 12.

Proof. (A) First we know that $|G_{(10\cdots)}|=144$.

Now construct a set of 10 points by $\mathbf{10}=\mathbf{9}^{\cdots}\cup\rho$, $\rho\in\Delta_{12}(9^{\cdots})$. We determine $(G_{(9\cdots)})_{[\rho]}$. $G_{(9\cdots)}$ has been seen to be $GA_2(3)$ of order $3\cdot 2^4\cdot 3^2$. Let T denote the translations of $GA_2(3)$. Now, if $\tau\neq E$ and $\tau\in T$, then τ fixes, by Proposition 5.3, $\Delta_3(9^{\cdots})$ pointwise, and fixes no point of $\mathbf{9}^{\cdots}$. Therefore, every element of T must fix exactly 3 points of $\Delta_{12}(9^{\cdots})$. We may assume at least one element of T fixes $\rho\in\Delta_{12}(9^{\cdots})$, but T does not fix ρ since any subgroup of order 3^2 of M_{24} cannot fix 4 points. Therefore, $|(G_{(9\cdots)})_{[\rho]}\cap T|=3$. We may also assume $(G_{(9\cdots)})_{[\rho]}$ contains

$$\alpha=\left[\begin{array}{cc|c} 1 & \cdot & 1\\ \cdot & 1 & \cdot\\ \hline \cdot & \cdot & 1\end{array}\right].$$

Then we see $(G_{(9\cdots)})_{[\rho]}$ is in the normalizer of $\langle\alpha\rangle$. Now the normalizer can be represented as

$$\left\{\left[\begin{array}{cc|c} a & b & x\\ \cdot & c & y\\ \hline \cdot & \cdot & 1\end{array}\right]\right\}.$$

It has order $12\cdot 9=108$. Since $|(G_{(9\cdots)})_{[\rho]}\cap T|=3$, we have

$$(G_{(9\cdots)})_{[\rho]}=\left\{\left[\begin{array}{cc|c} a & b & x\\ \cdot & c & \cdot\\ \hline \cdot & \cdot & 1\end{array}\right]\right\}\quad\text{of order } 12\cdot 3=36.$$

Denote $(G_{(9\cdots)})_{[\rho]}$ by G_{36}. The computation shows G_{36} is intransitive on $\mathbf{9}^{\cdots}$ with two systems of intransitivity $\overline{\Delta}_6$ and $\overline{\Delta}_3$ of lengths 6 and 3. The transitive constituent of G_{36} on $\overline{\Delta}_6$ is faithful and $G_{36}{}^{\overline{\Delta}_3}$ is S_3. The $G_{36}{}^{\Delta_6}$ is an imprimitive group with 2 blocks B_1 and B_2 of lengths 3, 3. The kernel is even part of $S_{B_1}\times S_{B_2}$ and the image is C_2. Consequently, there is a normal subgroup of order 6; denote it by G_6 in G_{36} such that

$$G_{36}{}^{\overline{\Delta}_6}/G_6\cong S_3\cong G_{36}{}^{\overline{\Delta}_3}$$

and the above isomorphism describes the intransitive group, $(G_{(9\cdots)})_{[\rho]}$ on $\mathbf{9}^{\cdots}$.

Recall that we started with a set of 10 points $\mathbf{10}=\mathbf{9}^{\cdots}\cup\rho$, $\rho\in\Delta_{12}(9^{\cdots})$. Now we construct $G_{(10)}$. $|(G_{(10)})_{[\rho]}|=36$, therefore $|G_{(10)}|=36\cdot l$ where l is the length of the orbit of $G_{(10)}$ which contains ρ. Consider

$$\begin{aligned}36\cdot l &= |G_{(10)}| = 2688,\\ &= |G_{(10)}| = 1440,\\ &= |G_{(10)}| = 144.\end{aligned}$$

36 does not divide 2688. $36\cdot 40 = 1440$ but cannot have $l=40$. Therefore we have

$$l = 4 \quad \text{and} \quad \mathbf{10} = \mathbf{9}^{\cdots} \cup \rho = \mathbf{10}^{\cdots}, \qquad \rho \in \Delta_{12}(9^{\cdots}).$$

Since $(G_{(9\cdots)})_{[\rho]}$ has orbits $\overline{\Delta}_6$, $\overline{\Delta}_3$ and ρ, $G_{(10\cdots)}$ has two orbits of lengths 6 and 4. Call them Δ_6 and Δ_4, respectively. Then $\Delta_6 = \overline{\Delta}_6$ and $\overline{\Delta}_3 \cup \rho = \Delta_4$. Apparently $G_{(10\cdots)}{}^{\Delta_4} = S_4$, and $G_{(10\cdots)}{}^{\Delta_6} = (G_{(10\cdots)})_{[\rho]}{}^{\overline{\Delta}_6} = G_{36}$. Therefore, $G_{(10\cdots)}$ can be described by the following isomorphism of quotient groups of transitive constituents

$$G_{36}/G_6 \cong S_3 \cong S_4/V,$$

where V is the Klein group.

(B) The existence of elements of type 6 4 12 2 and the fact that 14 does not divide 144 shows that $G_{(10\cdots)}$ has two orbits $\Delta_{12}(10^{\cdots})$ and $\Delta_2(10^{\cdot\cdot})$.

Now we will discuss the action of $G_{(10\cdot\cdot)}$ on $\Omega - \mathbf{10}^{\cdot\cdot}$.

Proposition 6.4. *$G_{(10\cdot\cdot)}$ has 2 orbits of lengths 2 and 12 on the remaining 14 points, $\Delta_2(10^{\cdot\cdot})$ and $\Delta_{12}(10^{\cdot\cdot})$. $G_{(10\cdot\cdot)}{}^{\Delta_{12}(10^{\cdot\cdot})}$ is an imprimitive group with image C_2.*

Proof. We have seen that $G_{(10\cdot\cdot)}$ with order 1440 and 3-transitive on $\mathbf{10}^{\cdot\cdot}$ is $\mathrm{P\Gamma L}_2(3^2) = \mathrm{Aut}(S_6)$. The compound character corresponding to $\mathrm{P\Gamma L}_2(3^2)$ in S_{10}, if computed and split into irreducible characters of S_{10}, is found to have the following decomposition (using Littlewood notation).

$$\begin{aligned} 1_{(\mathrm{P\Gamma L}_2(3^2))}S_{10} = {} & [10]+[64]+[62^2]+[52^21]+[4^22] \\ & +[3^31]+[42^3]+[431^3]+[2^5]+[421^4] \end{aligned}$$

and from the above characters the number of elements of each cycle type in $\mathrm{P\Gamma L}_2(3^2)$ is computed as follows:

type:	1^{10}	1^42^3	1^28	1^24^2	1^22^4	13^3	136	10
number:	1	30	180	270	45	80	240	144

type:	82	2^5	4^22	5^2
number:	180	36	90	144

Now for the elements of each type, the actions on the remaining 14 points are uniquely determined, and it is seen that $\mathrm{P\Gamma L}_2(3^2)$ fixes $2880 = 1440\cdot 2$ points on the remaining 14 points, so $G_{(10\cdot\cdot)}$ has 2 orbits on the 14 points. Therefore, $G_{(10\cdot\cdot)}$ has either two orbits of lengths 10, 4 or two of lengths 12, 2.

Now suppose $G_{(10\cdot\cdot)}$ has two orbits of lengths 10 and 4. Denote them by Δ_{10} and Δ_4, respectively.

Since 5 does not divide either $|G_{(10\cdot)}|$ or $|G_{(10\cdots)}|$, Δ_{10} must be a $\mathbf{10}^{\cdot\cdot}$. We note in $\mathrm{P\Gamma L}_2(3^2)$ that all elements of order 3 and 6 have cycle types 3^31 and 631 respectively. Therefore

10	Δ_{10}	Δ_4	
3^31	3^31	1^4	and
631	631	2^2	so 3 does not divide $\lvert G_{(10\cdot\cdot)}{}^{\Delta_4}\rvert$.

We further note that an element of order 8 must act as a cycle of length 4 on Δ_4 as seen by

10	Δ_{10}	Δ_4
81^2	82	4

Furthermore, $|G_{(10^{\cdot\cdot})}{}^{\Delta_4}| \neq 8$. For if this is not true then $(G_{(10^{\cdot\cdot})})_{[\Delta_4]}$ has order $2^2 \cdot 3^2 \cdot 5 = 180$ and, since $G_{(10^{\cdot\cdot})}$ is primitive and $(G_{(10^{\cdot\cdot})})_{[\Delta_4]}$ is a normal subgroup of $G_{(10^{\cdot\cdot})}$, $(G_{(10)})_{[\Delta_4]}$ must be transitive on $\mathbf{10}^{\cdot\cdot}$, but there is no transitive group of degree 10 and order 180. Therefore, $|G_{(10^{\cdot\cdot})}{}^{\Delta_4}| = 4$ and $G_{(10^{\cdot\cdot})}{}^{\Delta_4} \cong C_4$. Hence $(G_{(10^{\cdot\cdot})})_{[\Delta_4]} \cong A_6$ and $\mathrm{P\Gamma L}_2(3^2)/A_6 \cong C_4$. It is known, however, that $G_{(10^{\cdot\cdot})} = \mathrm{P\Gamma L}_2(3^2) = \mathrm{Aut}(A_6)$ contains 3 distinct primitive groups of order 720, degree 10, say $\mathrm{PGL}_2(3^2)$, $\mathrm{P\Gamma L}_2^+(3^2)$, and $\mathrm{P\Gamma L}_2^-(3^2)$, and one primitive group of order 360, A_6, and one of $\mathrm{P\Gamma L}_2^{\pm}(3^2)$ is S_6. Therefore, $\mathrm{P\Gamma L}_2(3^2)/A_6 \cong V_4$. This is a contradiction. Therefore $G_{(10^{\cdot\cdot})}$ must have two orbits of lengths 12 and 2. Denote them by $\Delta_{12}(10^{\cdot\cdot})$ and $\Delta_2(10^{\cdot\cdot})$.

For future reference, we look into the representation of $G_{(10^{\cdot\cdot})} = \mathrm{P\Gamma L}_2(3^2)$ on $\Delta_{12}(10^{\cdot\cdot})$.

All the primitive groups of degree 12 are known by Miller [7, p. 20] and it is seen that $\mathrm{P\Gamma L}_2(3^2)$ is represented as an imprimitive group of degree 12 on $\Delta_{12}(10^{\cdot\cdot})$ with two blocks of lengths 6, 6, denoted by B_1 and B_2. The kernel of imprimitivity is obtained by connecting two S_6's on B_i, $i=1, 2$, by the outer automorphism of S_6.

We add the following geometry of **10**'s.

Proposition 6.5. (A) $\mathbf{10}^{\cdot}$ *is also obtainable by* $\mathbf{9}^{\cdot\cdot} \cup \xi$, $\xi \in \Delta_1(9^{\cdot\cdot})$, *and* $\mathbf{10}^{\cdot}$ *contains two* $\mathbf{9}^{\cdot}$*'s and eight* $\mathbf{9}^{\cdot\cdot}$*'s.*

(B) $\mathbf{10}^{\cdot\cdot\cdot}$ *is also obtainable by* $\mathbf{9}^{\cdot\cdot} \cup \eta$, $\eta \in \Delta_{14}(9^{\cdot\cdot})$, *and* $\mathbf{10}^{\cdot\cdot\cdot}$ *contains four* $\mathbf{9}^{\cdot\cdot\cdot}$*'s and six* $\mathbf{9}^{\cdot\cdot}$*'s.*

Proof. (A) $\mathbf{10}^{\cdot} = \mathbf{9}^{\cdot} \cup \varepsilon$, $\varepsilon \in \Delta_{15}(9^{\cdot})$, but $\mathbf{9}^{\cdot} = \mathbf{8}^{\cdot\cdot} \cup \alpha_2$, $\alpha_2 \in \Delta_1(8^{\cdot\cdot})$, and $\varepsilon \in \Delta_{15}(8^{\cdot\cdot})$. Therefore, $\mathbf{10}^{\cdot} = \mathbf{8}^{\cdot\cdot} \cup \varepsilon \cup \alpha_2 = \mathbf{9}^{\cdot\cdot} \cup \alpha_2$, and $\mathbf{10}^{\cdot} \supset \mathbf{9}^{\cdot\cdot}$. Now consider $G_{(9^{\cdot\cdot} \cup \xi)}$, $\xi \in \Delta_1(9^{\cdot\cdot})$. Its order is $2^4 \cdot 3 \cdot 7$. Since 7 does not divide $|G_{(10^{\cdot\cdot\cdot})}|$ and $G_{(10^{\cdot\cdot})}$ is transitive, we have $\mathbf{10}^{\cdot} = 9^{\cdot\cdot} \cup \xi$, $\xi \in \Delta_1(9^{\cdot\cdot})$, and, by construction, $\mathbf{10}^{\cdot}$ minus ε is a $\mathbf{9}^{\cdot}$, and the assertion follows.

(B) $\mathbf{10}^{\cdot\cdot\cdot} = \mathbf{9}^{\cdot\cdot\cdot} \cup \rho$, $\rho \in \Delta_{12}(9^{\cdot\cdot\cdot})$. But $\mathbf{9}^{\cdot\cdot\cdot} = \mathbf{8}^{\cdot\cdot\cdot} \cup \delta$, $\delta \in \Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$. So $\mathbf{10}^{\cdot\cdot\cdot} = \mathbf{8}^{\cdot\cdot\cdot} \cup \delta \cup \rho$.

Now $\delta \in \Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$ and we have seen that $\Delta_3(9^{\cdot\cdot\cdot})$ is in $\Delta_{8^{\cdot}}(8^{\cdot\cdot\cdot})$ in the proof of Proposition 6.2. Therefore ρ could be taken as in $\Delta_{8^{\cdot\cdot\cdot}}(8^{\cdot\cdot\cdot})$. Then $\mathbf{8}^{\cdot\cdot\cdot} \cup \rho = \mathbf{9}^{\cdot\cdot}$ and we have $\mathbf{10}^{\cdot\cdot\cdot} = \mathbf{9}^{\cdot\cdot} \cup \delta$. Therefore $\mathbf{10}^{\cdot\cdot\cdot} \supset \mathbf{9}^{\cdot\cdot}$ and $\mathbf{10}^{\cdot\cdot\cdot} = \mathbf{9}^{\cdot\cdot} \cup \eta$, $\eta \in \Delta_{14}(9^{\cdot\cdot})$. By construction, $\mathbf{10}^{\cdot\cdot\cdot}$ minus the point ρ from $\Delta_{12}(9^{\cdot\cdot\cdot})$ is $\mathbf{9}^{\cdot\cdot\cdot}$ and ρ is in the orbit of length 4, so there are four $\mathbf{9}^{\cdot\cdot\cdot}$'s and six $\mathbf{9}^{\cdot\cdot}$'s.

7. $G_{(11^{\cdot})}$, $G_{(11^{\cdot\cdot})}$ **and** $G_{(11^{\cdot\cdot\cdot})}$. We first study $\mathbf{11}^{\cdot}$ and $G_{(11^{\cdot})}$ and obtain

Proposition 7.1. (A) *Let* $\mathbf{11}^{\cdot} = \mathbf{10}^{\cdot\cdot} \cup \tau$, $\tau \in \Delta_2(10^{\cdot\cdot})$, *then* $G_{(11^{\cdot})} = M_{11}$.

(B) $G_{(11^{\cdot})}$ *has two orbits of lengths* 12, 1, *denoted by* $\Delta_{12}(11^{\cdot})$ *and* $\Delta_1(11^{\cdot})$ *respec-*

tively. Furthermore, $\Delta_{12}(11^{\cdot})=\Delta_{12}(10^{\cdot\cdot})=\Delta_{12}(9^{\cdot\cdot\cdot})$. $\Delta_1(11^{\cdot})=\Delta_2(10^{\cdot\cdot})-\tau=\Delta_3(9^{\cdot\cdot\cdot})-\{\mu,\tau\}$.

Proof. (A) Let $\mathbf{11}^{\cdot}=\mathbf{10}^{\cdot\cdot}\cup\tau$, $\tau\in\Delta_2(10^{\cdot\cdot})$. Recall $G_{(10^{\cdot\cdot})}$ of order 1440 is $P\Gamma L_2(3^2)$. $(G_{(10^{\cdot\cdot})})_{[\tau]}$ has order 720 and is one of the three normal, primitive subgroups of $P\Gamma L_2(3^2)$. Two of these primitive groups of degree 10 are triply transitive and one is doubly transitive. Therefore, $(G_{(10^{\cdot\cdot})})_{[\tau]}$ is at least doubly transitive.

By Proposition 5.3(B), $G_{(9^{\cdot\cdot\cdot})}$ acts as S_3 on $\Delta_3(9^{\cdot\cdot\cdot})$, so there is an element, σ, in $G_{(9^{\cdot\cdot\cdot})}$ such that

$$\sigma^{\Delta_3(9^{\cdot\cdot\cdot})} = (\mu\tau)(y), \quad \text{where } \Delta_3(9^{\cdot\cdot\cdot}) = \{\mu,\tau,y\}.$$

Therefore, $G_{(11^{\cdot})}$ is 3-transitive on $\mathbf{11}^{\cdot}=\mathbf{10}^{\cdot\cdot}\cup\tau=\mathbf{9}^{\cdot\cdot}\cup\mu\cup\tau$. Now $|(G_{(11^{\cdot})})_{[\tau]}|=720$, so

$$|G_{(11^{\cdot})}| = 720\times 11 = 7920 \quad\text{and}\quad G_{(11^{\cdot})} = M_{11}.$$

(B) Now we will consider the action of $M_{11}=G_{(11^{\cdot})}$ on the remaining 13 points. By Proposition 6.4, $G_{(10^{\cdot\cdot})}$ is represented on $\Delta_{12}(10^{\cdot\cdot})$ as an imprimitive group with its kernel of imprimitivity constructed by two S_6's on two blocks connected by the outer automorphism of S_6. Also, in the proof of Proposition 6.4, actions of the elements of $G_{(10^{\cdot\cdot})}$ on $\Omega-\mathbf{10}^{\cdot\cdot}$ is completely determined as far as the cycle types on $\Omega-\mathbf{10}^{\cdot\cdot}$ of the elements are concerned.

Now, we note $P\Gamma L_2(3^2)$ has the following elements:

$\mathbf{10}^{\cdot\cdot}$	$\Delta_{12}(10^{\cdot\cdot})$	$\Delta_2(10^{\cdot\cdot})$
82	84	1^2
5^2	5^21^2	1^2

Therefore $(G_{(10^{\cdot\cdot})})_{[\tau]}=(G_{(11^{\cdot})})_{[\tau]}$ is still transitive on $\Delta_{12}(10^{\cdot\cdot})$, so $(G_{(11^{\cdot})})_{[\tau]}$ must be represented on $\Delta_{12}(10^{\cdot\cdot})$ as an imprimitive group with its kernel of imprimitivity constructed by two A_6's on the two blocks connected by the outer automorphism of A_6. Anyway, $(G_{(11^{\cdot})})_{[\tau]}$ has two orbits of lengths 1 and 12. Therefore, since $13\nmid|G_{(11^{\cdot})}|$, $G_{(11^{\cdot})}$ must have two orbits of lengths 1 and 12. Call them $\Delta_1(11^{\cdot})$ and $\Delta_{12}(11^{\cdot})$, respectively.

Now we have $\Delta_{12}(11^{\cdot})=\Delta_{12}(10^{\cdot\cdot})$ and $\Delta_1(11^{\cdot})=\Delta_2(10^{\cdot\cdot})-\tau$, where $\mathbf{11}^{\cdot}=\mathbf{10}^{\cdot\cdot}\cup\tau$, $\tau\in\Delta_2(10^{\cdot\cdot})$.

Next we claim $\Delta_{12}(10^{\cdot\cdot})=\Delta_{12}(9^{\cdot\cdot})$ and $\Delta_2(10^{\cdot\cdot})=\Delta_3(9^{\cdot\cdot\cdot})-\mu$, where $\mathbf{9}^{\cdot\cdot\cdot}\cup\mu=\mathbf{10}^{\cdot\cdot}$. $(G_{(10^{\cdot\cdot})})_{[\mu]}=(G_{(9^{\cdot\cdot\cdot})})_{[\mu]}$ has order 144 and its conjugacy is given in the proof of Proposition 6.2, and is called G_{144}. The action of G_{144} on the remaining 14 points is uniquely determined. It is seen that $(G_{(9^{\cdot\cdot\cdot})})_{[\mu]}$ fixes 288 points out of the 14 points, so $(G_{(9^{\cdot\cdot\cdot})})_{[\mu]}$ has two orbits on the 14 points. Therefore $(G_{(9^{\cdot\cdot\cdot})})_{[\mu]}$ has two orbits $\Delta_3(9^{\cdot\cdot\cdot})-\mu$ and $\Delta_{12}(9^{\cdot\cdot\cdot})$. Since $G_{(10^{\cdot\cdot})}$ has two orbits of lengths 2, 12, and $(G_{(9^{\cdot\cdot\cdot})})_{[\mu]}\subset G_{(10^{\cdot\cdot})}$, $\Delta_2(10^{\cdot\cdot})=\Delta_3(9^{\cdot\cdot\cdot})-\mu$ and $\Delta_{12}(10^{\cdot\cdot})=\Delta_{12}(9^{\cdot\cdot\cdot})$.

Since $G_{(11^{\cdot})}=M_{11}$ has an orbit of length 12, $\Delta_{12}(11^{\cdot})$, M_{11} is seen represented transitively on 12 points. This is the representation of M_{11} on the cosets of $PSL_2(11)$

in M_{11}. ($[M_{11} : \mathrm{PSL}_2(11)]=12$.) Then, since the representation is faithful, $M_{11}{}^{\mathbf{11}^{\cdot}} \cong M_{11}{}^{\Delta_{12}(11^{\cdot})}$. Now we note, in M_{11}, we have elements of type 821 and $4^2 1^3$. Action of these elements on the remaining 13 points can be uniquely determined as follows:

$\mathbf{11}^{\cdot}$	$\Delta_{12}(11^{\cdot})$	$\Delta_1(11^{\cdot})$
821	84	1
$4^2 1^3$	$4^2 2^2$	1

Therefore the isomorphism connecting two constituents $M_{11}{}^{\mathbf{11}^{\cdot}}$ and $M_{11}{}^{\Delta_{12}(11^{\cdot})}$ must be an outer automorphism of M_{12} such that $821 \leftrightarrow 84$ and $4^2 1^3 \leftrightarrow 4^2 2^2$.

The existence of such an outer automorphism is noted in Theorem 9 in E. Witt's paper [10, p. 262].

Now we proceed to construct the second set of 11 points and obtain

PROPOSITION 7.2. (A) $\mathbf{11}^{\cdot\cdot} = \mathbf{10}^{\cdot\cdot} \cup \nu$, $\nu \in \Delta_{12}(10^{\cdot\cdot})$. *$G_{(11^{\cdot\cdot})}$ is S_5 on the* 10 *cosets of subgroup $S_2 \times S_3$, thus has order* 120. *This group will be denoted by $S_5|_{S_2 \times S_3}$ from here on. $G_{(11^{\cdot\cdot})}$ is intransitive on $\mathbf{11}^{\cdot\cdot}$ having two orbits $\mathbf{10}^{\cdot\cdot}$ and $\{\nu\}$.*

(B) *$G_{(11^{\cdot\cdot})}$ has* 3 *orbits on $\Omega - \mathbf{11}^{\cdot\cdot}$ of lengths* 2, 5, 6, *denoted as $\Delta_2(11^{\cdot\cdot})$, $\Delta_5(11^{\cdot\cdot})$ and $\Delta_6(11^{\cdot\cdot})$ respectively, and $\Delta_2(10^{\cdot\cdot}) = \Delta_2(11^{\cdot\cdot})$.*

Proof. (A) Construct an **11** by $\mathbf{10}^{\cdot\cdot} \cup \nu$, $\nu \in \Delta_{12}(10^{\cdot\cdot})$. Then $|(G_{(10^{\cdot\cdot})})_{[\nu]}| = 120$. Call this group of order 120, G_{120}. First we will show G_{120} is transitive on $\mathbf{10}^{\cdot\cdot}$.

Let α be an element of order 3 in G_{120}. Since α must fix ν and two more points in $\Delta_2(10^{\cdot\cdot})$ and α can fix only 6 points out of 24 points, $\alpha^{\mathbf{10}^{\cdot\cdot}}$ is of type $3^3 1$. Let β be an element of order 5. β can fix only 4 points out of 24 points, so $\beta^{\mathbf{10}^{\cdot\cdot}}$ is of type 5^2. α and β assure the transitivity of G_{120} over $\mathbf{10}^{\cdot\cdot}$.

Next we will see G_{120} is primitive on $\mathbf{10}^{\cdot\cdot}$. Suppose G_{120} is imprimitive with 2 blocks of length 5. Then the kernel of imprimitivity has order $2^2 \cdot 3 \cdot 5 = 60$. Then the kernel must contain an element of order 3 of type α above, which is impossible.

Suppose G_{120} is imprimitive with 5 blocks of length 2, denoted by Δ_i, $i = 1, 2, 3, 4, 5$. Then an element of order 3 cannot be in the kernel of imprimitivity, so elements of order 3 must act as $(\Delta_i)(\Delta_j)(\Delta_k \Delta_l \Delta_m)$ fixing 4 points, but this is impossible since any element of order 3 in G_{120} must fix 3 points outside $\mathbf{10}^{\cdot\cdot}$ and can fix only 6 points out of the total 24 points.

Since a primitive group of degree 10, order 120 is known to be isomorphic to S_5, G_{120} must be the representation of S_5 on the cosets of $S_3 \times S_2$ of order 12, so $G_{120} \cong S_5|_{S_3 \times S_2}$. Next consider $G_{(11)}$. $(G_{(11)})_{[\nu]} = (G_{(10^{\cdot\cdot})})_{[\nu]}$ and we have seen $(G_{(11)})_{[\nu]}$ is transitive of order 120 on $\mathbf{11} - \nu$. If $G_{(11)}$ is transitive, then $|G_{(11)}| = 120 \times 11 = 1320$. But there is no group of degree 11, order 1320 (Cole, [2, p. 49]), so $G_{(11)} = (G_{(11)})_{[\nu]} = (G_{(10^{\cdot\cdot})})_{[\nu]}$. Since transitive extension is impossible, $\mathbf{11} \not\cong \mathbf{11}^{\cdot}$. Call $\mathbf{11} = \mathbf{10}^{\cdot\cdot} \cup \nu$, $\nu \in \Delta_{12}(10^{\cdot\cdot})$, an $\mathbf{11}^{\cdot\cdot}$.

(B) Next we will consider action of $G_{(11^{\cdot\cdot})}$ on the remaining 13 points. We have seen $G_{(11^{\cdot\cdot})} = S_5|_{S_3 \times S_2}$, i.e. the representation of S_5 on the 10 cosets of $S_3 \times S_2$ of order 12. A computation shows $G_{(11^{\cdot\cdot})}$ contains an element of type 631. Therefore on the rest, it must act as

$$\begin{array}{cccc} \mathbf{10}^{\cdot\cdot} & \{\nu\} & \Delta_{12}(10^{\cdot\cdot})-\nu & \Delta_2(10^{\cdot\cdot}) \\ 631 & 1 & 632 & 2 \end{array}$$

Therefore $G_{(11^{\cdot\cdot})}$ must retain $\Delta_2(10^{\cdot\cdot})$ as an orbit of length 2.

Now 5 divides 120, and an element of order 5 has only one way of action as follows:

$$\begin{array}{cccc} \mathbf{10}^{\cdot\cdot} & \{\nu\} & \Delta_{12}(10^{\cdot\cdot})-\nu & \Delta_2(10^{\cdot\cdot}) \\ 5^2 & 1 & 5^21 & 1^2 \end{array}$$

Since 11 does not divide 120, G_{120} must have two orbits of lengths 5 and 6 on $\Delta_{12}(10^{\cdot\cdot})-\nu$. Thus $G_{(11^{\cdot\cdot})}$ has 3 orbits on the remaining 13 points, of lengths 6, 5 and 2. Call them $\Delta_6(11^{\cdot\cdot})$, $\Delta_5(11^{\cdot\cdot})$ and $\Delta_2(11^{\cdot\cdot})$.

Next we turn to the construction of the third type of set of 11 points, $\mathbf{11}^{\cdots}$ and obtain

PROPOSITION 7.3. (A) $G_{(11^{\cdots})}$ *has order* 576, *and is intransitive on* $\mathbf{11}^{\cdots}$, *having two orbits* Ω_3 *and* Ω_8 *of lengths* 3 *and* 8, *respectively, and* $\mathbf{11}^{\cdots}=\mathbf{10}^{\cdots}\cup\varphi$, $\varphi\in\Delta_2(10^{\cdots})$, *and also*

(B) $\mathbf{11}^{\cdots}=\mathbf{10}^{\cdot}\cup\sigma$, $\sigma\in\Delta_{14}(10^{\cdot})$, *thus* $\mathbf{11}^{\cdots}$ *contains three* $\mathbf{10}^{\cdot\cdot}$*'s and eight* $\mathbf{10}^{\cdots}$*'s.*

(C) $G_{(11^{\cdots})}$ *has two orbits* $\Delta_1(11^{\cdots})$ *and* $\Delta_{12}(11^{\cdots})$ *and* $\Delta_1(11^{\cdots})=\Delta_2(10^{\cdots})-\varphi$ *and* $\Delta_{12}(11^{\cdots})=\Delta_{12}(10^{\cdots})$.

Proof. (A) The stabilizer of the third type of set of 11 points must have order 576.

Consider a set of 11 points, $\mathbf{11}=\mathbf{10}^{\cdots}\cup\varphi$, $\varphi\in\Delta_2(10^{\cdots})$. Since $G_{(10^{\cdots})}$ has order 144, $|(G_{(10^{\cdots})})_{[\varphi]}|=72$. Call this group of order 72 as G_{72}. Since $(G_{(11)})_{[\varphi]}=(G_{(10^{\cdots})})_{[\varphi]}$, $|G_{(11)}|=72\cdot l$, l is the length of the orbit to which φ belongs. Therefore

$$\begin{aligned} |G_{(11)}| = 72\cdot l &= 7920 = |G_{(11^{\cdot})}|, \\ &= 120 = |G_{(11^{\cdot\cdot})}|, \\ &= 576 = |G_{(11^{\cdots})}|. \end{aligned}$$

If $|G_{(11)}|=7920$, $l=110$, which is impossible. $|G_{(11)}|=120$ is also impossible since $72\nmid 120$. So $|G_{(11)}|=576$ and $l=8$, and $\mathbf{11}=\{\mathbf{10}^{\cdots}\cup\varphi\}\approx\mathbf{11}^{\cdots}$.

We shall determine the orbits of $G_{(11^{\cdots})}$ on $\mathbf{11}^{\cdots}$. By Proposition 6.3, $G_{(10^{\cdots})}$ has two orbits Δ_6 and Δ_4 on $\mathbf{10}^{\cdots}$, and $G_{(10^{\cdots})}{}^{\Delta_4}$ is S_4. Therefore $G_{72}{}^{\Delta_4}$ is S_4 or A_4 and G_{72} is still transitive on Δ_4. The orbit of length 8, denoted by Ω_8, which $G_{(11^{\cdots})}$ is supposed to have must contain this Δ_4 along with φ. So $\Omega_8\supset\Delta_4\cup\varphi$. We need 3 more points to complete Ω_8 of length 8. Now consider the action of G_{72} on Δ_6 of $G_{(10^{\cdots})}$. By Proposition 6.3, $G_{(10^{\cdots})}{}^{\Delta_6}$ is an imprimitive group with 2 blocks B_1, B_2 of lengths 3, 3 and its kernel of primitivity consists of even permutations of $S_{B_1}\times S_{B_2}$. Therefore $G_{72}{}^{\Delta_6}$ contains elements of type 3^2 on $B_1\cup B_2$. Now $G_{(11^{\cdots})}$

has an orbit of length 8, so $G_{72}{}^{\Delta_6}$ cannot be transitive on $B_1 \cup B_2$, so $G_{72}{}^{\Delta_6}$ must have orbits B_1 and B_2 of lengths 3, 3. Therefore $G_{(11^{\cdots})}$ must have two orbits of lengths 8, 3; $B_i \cup \Delta_4 \cup \varphi$, and B_j on $\mathbf{11}^{\cdots}$, $\{i, j\}=\{1, 2\}$. Call them Ω_8 and Ω_3.

(B) Let us consider another set of 11 points $\mathbf{11}=\mathbf{10}^{\cdot} \cup \sigma$, $\sigma \in \Delta_{14}(10^{\cdot})$. Let two orbits of length 8 and length 2 of $G_{(10^{\cdot})}$ on $\mathbf{10}^{\cdot}$ be Δ_8 and Δ_2. By Proposition 6.1, $G_{(10^{\cdot})}{}^{\Delta_8} \cong$ the holomorph of $\mathfrak{A}_{2^3}$ of order 1344 and $G_{(10^{\cdot})}{}^{\Delta_2} \cong C_2$ of order 2 and $G_{(10^{\cdot})}=G_{1344} \times C_2$. Now $(G_{(11)})_{[\sigma]}=(G_{(10^{\cdot})})_{[\sigma]}$ has order $2^6 \cdot 3$. Therefore $2^6 \cdot 3$ divides $|G_{(11)}|$. Since $2^6 \cdot 3$ does not divide either $|G_{(11^{\cdot})}|$ or $|G_{(11^{\cdot\cdot})}|$, $G_{(11)} \approx G_{(11^{\cdots})}$ and $\mathbf{11} \approx \mathbf{11}^{\cdots}$. Now $G_{(11^{\cdots})}$ has two orbits of lengths 8 and 3 on $\mathbf{11}^{\cdots}$. Since $|G_{(11)}|=2^6 \cdot 3 \cdot 3$ the orbit of length 3 must contain the new point σ. Thus $\mathbf{11}^{\cdots}$ contains three $\mathbf{10}^{\cdot}$'s and eight $\mathbf{10}^{\cdots}$'s.

(C) We note $G_{(10^{\cdot})}=G_{1344} \times C_2$ contains an element of type

$$\begin{matrix} \Delta_8 & \Delta_2 & \Delta_{14}(10^{\cdot}) \\ 4^2 & 1^2 & 1^2 2^2 4^2 \end{matrix}$$

Then $(G_{(11^{\cdots})})_{[\sigma]}$ contains an element of type

$$4^2\,1^2 \quad \text{on } \mathbf{10}^{\cdot}.$$

Therefore Ω_8 of $G_{(11^{\cdots})}$ coincides with Δ_8 of $G_{(10^{\cdot})}$ and Ω_3 of $G_{(11^{\cdots})}$ is $\{\Delta_2$ of $G_{(10^{\cdot})}\} \cup \sigma$.

Now by construction, Δ_8 of $G_{(10^{\cdot})}$ is an $\mathbf{8}^{\cdot}$, therefore $G_{(11^{\cdots})}$ is in the holomorph of $\mathfrak{A}_{2^4}$ which is 3-fold transitive on the 16 points, $\{\Delta_2$ of $G_{(10^{\cdot})} \cup \Delta_{14}(10^{\cdot})\}$. Furthermore the holomorph $\mathfrak{H}$ has order $16 \cdot 15 \cdot 14 \cdot 96$, so $\mathfrak{H}_{[\Omega_3]}$ has order 96 and $\mathfrak{H}_{(\Omega_3)}$ has order $96 \times 6 = 576$. Therefore $G_{(11^{\cdots})}=\mathfrak{H}_{(\Omega_3)}$. Now the subgroup of $\mathfrak{H}$ fixing a certain 3 points may be represented as follows:

$$\left\{\left[\begin{array}{cccc|c} 1 & \cdot & \alpha & \varepsilon & \cdot \\ \cdot & 1 & \beta & \zeta & \cdot \\ \cdot & \cdot & \gamma & \eta & \cdot \\ \cdot & \cdot & \delta & \theta & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right]\right\} \quad \text{of order 96; } \alpha, \beta, \ldots, \theta \in \mathrm{GF}\,[2].$$

The computation shows that the above subgroup fixes not only the 3 points but also fixes one more point and is transitive on the remaining 12 points of $V_4(F_2)$. Therefore $\mathfrak{H}_{(\Omega_3)}=G_{(11^{\cdots})}$ has 2 orbits of lengths 1 and 12; call them $\Delta_1(11^{\cdots})$ and $\Delta_{12}(11^{\cdots})$ respectively.

Now for future reference we characterize $\Delta_1(11^{\cdots})$. In Figure 6, Appendix, $\mathbf{11}^{\cdots}=\mathbf{10}^{\cdots} \cup \varphi$, $\varphi \in \Delta_2(10^{\cdots})$. We have just seen $G_{(11^{\cdots})}$ has two orbits $\Delta_1(11^{\cdots})$ and $\Delta_{12}(11^{\cdots})$.

$G_{(10^{\cdots})}=G_{144}$ contains an element of type

$$\begin{matrix} 3^2 1^4 & 1^2 & 3^4 \\ \mathbf{10}^{\cdots} & \Delta_2(10^{\cdots}) & \Delta_{12}(10^{\cdots}) \end{matrix}$$

Thus $\Delta_1(11^{\cdots})=\Delta_2(10^{\cdots})-\varphi$ and $\Delta_{12}(11^{\cdots})=\Delta_{12}(10^{\cdots})$.

To complete the scheme of **11**, we add the following

PROPOSITION 7.4. $\mathbf{11}^{\cdot\cdot}$ *can also be constructed by* $\mathbf{10}^{\cdot\cdot\cdot} \cup \psi$, $\psi \in \Delta_{12}(10^{\cdot\cdot\cdot})$.

Proof. We have seen $\mathbf{11}^{\cdot\cdot} = \mathbf{10}^{\cdot\cdot} \cup \nu$, $\nu \in \Delta_{12}(10^{\cdot\cdot})$. But $\mathbf{10}^{\cdot\cdot} = \mathbf{9}^{\cdot\cdot\cdot} \cup \mu$, $\mu \in \Delta_3(9^{\cdot\cdot\cdot})$. Therefore $\mathbf{11}^{\cdot\cdot} = \mathbf{9}^{\cdot\cdot\cdot} \cup \mu \cup \nu$ where $\nu \in \Delta_{12}(10^{\cdot\cdot}) = \Delta_{12}(9^{\cdot\cdot\cdot})$ by Proposition 7.1(B). So $\mathbf{11}^{\cdot\cdot} = \mathbf{10}^{\cdot\cdot\cdot} \cup \mu$ and μ cannot be in $\Delta_2(10^{\cdot\cdot\cdot})$. Therefore $\mathbf{11}^{\cdot\cdot} = \mathbf{10}^{\cdot\cdot\cdot} \cup \psi$, $\psi \in \Delta_{12}(10^{\cdot\cdot\cdot})$.

8. $G_{(12^{\cdot})}$, $G_{(12^{\cdot\cdot})}$, $G_{(12^{\cdot\cdot\cdot})}$, $G_{(12^{\mathrm{iv}})}$ **and** $G_{(12^{\mathrm{v}})}$. By Lemma 1.5, there are 5 types of sets of 12 distinct points. The construction of these sets will be discussed in two propositions.

PROPOSITION 8.1. (A) *Let* $\mathbf{12}^{\cdot} = \mathbf{11}^{\cdot} \cup \theta$, $\theta \in \Delta_1(11^{\cdot})$, *then* $G_{(12^{\cdot})}$ *is* M_{12} *on* $\mathbf{12}^{\cdot}$.

(B) $\mathbf{12}^{\cdot\cdot} = \mathbf{11}^{\cdot\cdot\cdot} \cup \pi$, $\pi \in \Delta_1(11^{\cdot\cdot\cdot})$. $G_{(12^{\cdot\cdot})}$ *is transitive on* $\mathbf{12}^{\cdot\cdot}$ *and has order* $2^8 \cdot 3^3 = 6912$.

(C) $\mathbf{12}^{\cdot\cdot\cdot} = \mathbf{11}^{\cdot\cdot} \cup k = \mathbf{11}^{\cdot} \cup i$, $k \in \Delta_2(11^{\cdot\cdot})$ *and* $i \in \Delta_{12}(11^{\cdot})$. $G_{(12^{\cdot\cdot\cdot})}$ *is* $\mathrm{PSL}_2(11)$ *on* $\mathbf{11}^{\cdot}$ *and* S_1 *on* i, *or* $G_{(12^{\cdot\cdot\cdot})} = (G_{(11^{\cdot})})_{[i]}$.

Proof. (A) Let $\mathbf{12}^{\cdot} = \mathbf{11}^{\cdot} \cup \theta$, $\theta \in \Delta_1(11^{\cdot})$. Then since by Proposition 7.1, $\Delta_1(11^{\cdot}) = \Delta_2(10^{\cdot\cdot}) - \tau = \Delta_3(9^{\cdot\cdot\cdot}) - \{\mu, \tau\}$, $\mathbf{12}^{\cdot} = \mathbf{9}^{\cdot\cdot\cdot} \cup \Delta_3(9^{\cdot\cdot\cdot})$. Now by Proposition 5.3(B), $G_{(9^{\cdot\cdot\cdot})}$ is S_3 on $\Delta_3(9^{\cdot\cdot\cdot})$, therefore $G_{(12^{\cdot})}$ is transitive on $\mathbf{12}^{\cdot}$ and $(G_{(12^{\cdot})})_{[\theta]} = G_{(11^{\cdot})} = M_{11}$. Therefore $G_{(12^{\cdot})}$ is 5-fold on $\mathbf{12}^{\cdot}$ and is M_{12}. And $|G_{(12^{\cdot})}| = |M_{11}| \cdot 12 = 2^6 \cdot 3^3 \cdot 5 \cdot 11$.

(B) Consider a $\mathbf{12} = \mathbf{11}^{\cdot\cdot\cdot} \cup \pi$, $\pi \in \Delta_1(11^{\cdot\cdot\cdot})$. In $G_{(12)}$, since $\mathbf{12} = \mathbf{10}^{\cdot\cdot\cdot} \cup \Delta_2(10^{\cdot\cdot\cdot})$, the orbit containing π has at least length 9. If the orbit containing π has indeed length 9, then $|G_{(12)}| = |G_{(11^{\cdot\cdot\cdot})}| \cdot 3^2 = 2^6 \cdot 3^2 \cdot 3^2 = 2^6 \cdot 3^4$. This is impossible since $|\mathrm{syl}_3\,(M_{24})| = 3^3$. Therefore $G_{(12)}$ must be transitive on **12** and $|G_{(12)}| = 2^6 \cdot 3^2 \cdot 2^2 \cdot 3 = 2^8 \cdot 3^3$, and $\mathbf{12} \not\approx \mathbf{12}^{\cdot}$. Call this **12** a $\mathbf{12}^{\cdot\cdot}$.

(C) Let $\mathbf{12} = \mathbf{11}^{\cdot\cdot} \cup k$, $k \in \Delta_2(11^{\cdot\cdot}) = \Delta_2(10^{\cdot\cdot})$. Then $\mathbf{12} = \mathbf{10}^{\cdot\cdot} \cup \nu \cup k$, $\nu \in \Delta_{12}(10^{\cdot\cdot}) = \Delta_{12}(11^{\cdot})$ and this decomposition gives $\mathbf{12} = \mathbf{11}^{\cdot} \cup \nu$.

Since $G_{(12^{\cdot})}$ and $G_{(12^{\cdot\cdot})}$ are transitive, $\mathbf{12} \not\approx \mathbf{12}^{\cdot}$ and $\mathbf{12} \not\approx \mathbf{12}^{\cdot\cdot}$. Call **12** a $\mathbf{12}^{\cdot\cdot\cdot}$. Then $\mathbf{12}^{\cdot\cdot\cdot} = \mathbf{11}^{\cdot\cdot} \cup k = \mathbf{11}^{\cdot} \cup i$, $k \in \Delta_2(11^{\cdot\cdot})$, $i \in \Delta_{12}(11^{\cdot})$. Now, we find $G_{(12^{\cdot\cdot\cdot})}$. We know $G_{(11^{\cdot})} = M_{11}$ is represented on $\Delta_{12}(11^{\cdot})$ as a transitive group of degree 12. This representation is $M_{11}|_{\mathrm{PSL}_2(11)}$ and is triply transitive on the $\Delta_{12}(11^{\cdot})$. Therefore $(G_{(11^{\cdot})})_{[i]}$, $i \in \Delta_{12}(11^{\cdot})$, is $\mathrm{PSL}_2\,(11)$ of order 660. $\mathrm{PSL}_2\,(11)$ on $\mathbf{11}^{\cdot}$ is doubly transitive as $\mathrm{PSL}_2\,(11)|_{A_5}$. Since $G_{(12^{\cdot\cdot\cdot})}$ is intransitive, $G_{(12^{\cdot\cdot\cdot})}$ is $\mathrm{PSL}_2\,(11)$ (in S_{11}) on $\mathbf{11}^{\cdot}$ and fixes i.

Now we have two more distinct sets of 12 distinct points to be built. The additional points for these two new sets $\mathbf{12}^{\mathrm{iv}}$ and $\mathbf{12}^{\mathrm{v}}$ must come from $\Delta_5(11^{\cdot\cdot})$, $\Delta_6(11^{\cdot\cdot})$ and $\Delta_{12}(11^{\cdot\cdot\cdot})$. The construction for these new sets follows:

PROPOSITION 8.2. (A) $\mathbf{12}^{\mathrm{iv}} = \mathbf{11}^{\cdot\cdot\cdot} \cup \omega$, $\omega \in \Delta_{12}(11^{\cdot\cdot\cdot})$, *also*.

(B) $\mathbf{12}^{\mathrm{iv}} = \mathbf{11}^{\cdot\cdot} \cup \lambda$, $\lambda \in \Delta_5(11^{\cdot\cdot})$.

(C) $\mathbf{12}^{\mathrm{v}} = \mathbf{11}^{\cdot\cdot} \cup \mu$, $\mu \in \Delta_6(11^{\cdot\cdot})$.

(D) $G_{(12^{\mathrm{v}})}$ *is transitive on* $\mathbf{12}^{\mathrm{v}}$.

(E) $G_{(12^{\mathrm{v}})}$ *has order* 240 *and is an imprimitive group with six blocks of length* 2.

The kernel has order 2 *and its image is* $\bar{S}_5$ (S_5 *in* S_6).

(F) $G_{(12^{\mathrm{iv}})}$ *has order* 192 *and is intransitive on* $\mathbf{12}^{\mathrm{iv}}$ *with two orbits of lengths* 8 *and* 4.

Proof. Before proceeding we note two results.

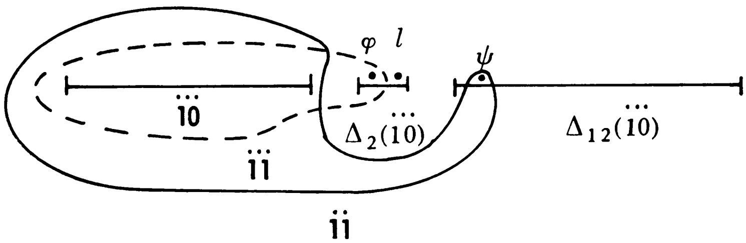
(a) We have

$$|M_{24}|/|G_{(12^{\mathrm{iv}})}|+|M_{24}|/|G_{(12^{\mathrm{v}})}| = 2295216.$$

Next we need

(b) $\Delta_2(10^{\cdots}) \cap \Delta_2(11^{\cdot\cdot})=\varnothing$.

Suppose $\Delta_2(10^{\cdots}) \cap \Delta_2(11^{\cdot\cdot}) \ni \varphi$. Look at the diagram.



Then by Proposition 7.4 we have $\mathbf{11}^{\cdot\cdot}=\mathbf{10}^{\cdots} \cup \psi$, $\psi \in \Delta_{12}(10^{\cdots})$. By assumption $\varphi \in \Delta_2(11^{\cdot\cdot})$, so $\mathbf{11}^{\cdot\cdot} \cup \varphi=\mathbf{12}^{\cdot\cdot}$. On the other hand, $\mathbf{10}^{\cdots} \cup \psi \cup \varphi=\mathbf{11}^{\cdots} \cup \psi$, $\psi \in \Delta_{12}(11^{\cdots})$. Thus $\mathbf{12}^{\cdots}$ contains not only $\mathbf{11}^{\cdot}$ and $\mathbf{11}^{\cdot\cdot}$ by Proposition 8.1 but also contains $\mathbf{11}^{\cdots}$. This is impossible because, as an intransitive group with two orbits, $\mathbf{12}^{\cdots}$ can contain at most two **11**'s.

Now in the above diagram let $\mathbf{12}=\mathbf{11}^{\cdots} \cup \psi$, $\psi \in \Delta_{12}(11^{\cdots})$ since $\Delta_1(11^{\cdots})=\{l\}$. But $\mathbf{11}^{\cdots}=\mathbf{10}^{\cdots} \cup \varphi$, $\varphi \in \Delta_2(10^{\cdots})$, so $\mathbf{12}=\mathbf{10}^{\cdots} \cup \varphi \cup \psi$. But $\mathbf{10}^{\cdots} \cup \psi=\mathbf{11}^{\cdot\cdot}$, so $\mathbf{12}=\mathbf{11}^{\cdot\cdot} \cup \varphi$. By (b) above, $\varphi \notin \Delta_2(11^{\cdot\cdot})$, so $\varphi \in \Delta_5(11^{\cdot\cdot})$ or $\varphi \in \Delta_6(11^{\cdot\cdot})$. Therefore **12** contains $\mathbf{11}^{\cdot\cdot}$ and $\mathbf{11}^{\cdots}$. Obviously $\mathbf{12} \not\cong \mathbf{12}^{\cdot}$, $\mathbf{12}^{\cdot\cdot}$ or $\mathbf{12}^{\cdots}$. Call this **12** a $\mathbf{12}^{\mathrm{iv}}$. Therefore,

(A) $\mathbf{12}^{\mathrm{iv}}=\mathbf{11}^{\cdots} \cup \omega$, $\omega \in \Delta_{12}(11^{\cdots})$, and also $\mathbf{12}^{\mathrm{iv}}=\mathbf{11}^{\cdot\cdot} \cup \varphi$, $\varphi \in \Delta_5(11^{\cdot\cdot})$ or $\varphi \in \Delta_6(11^{\cdot\cdot})$, and $G_{(12^{\mathrm{iv}})}$ is intransitive on $\mathbf{12}^{\mathrm{iv}}$.

We will decide where φ belongs. Look at Figure 7. If $\varphi=\mu$, i.e., $\varphi \in \Delta_6(11^{\cdot\cdot})$, then $\mathbf{11}^{\cdots} \cup \omega=\mathbf{11}^{\cdot\cdot} \cup \mu$. In $G_{(12^{\mathrm{iv}})}$, the stabilizer of ω has order $48=576/12$. $(G_{(11^{\cdot\cdot})})_{[\mu]}$ has order 20 and contains an element of type

$$\begin{array}{cc} \mathbf{11}^{\cdot\cdot} & \{\mu\} \\ 5^2 1 & 1 \end{array}$$

Furthermore, 5 does not divide the order of stabilizer of ω, so $|G_{(12^{\mathrm{iv}})}| \geqq 5\cdot 48=240$. Then the orbit containing μ must have length at least 12, which is impossible. Therefore $\varphi=\lambda$, i.e.

(B) $\mathbf{12}^{\mathrm{iv}}=\mathbf{11}^{\cdot\cdot} \cup \lambda$, $\lambda \in \Delta_5(11^{\cdot\cdot})$.

The last type of set of 12 points is seen then

(C) $\mathbf{12}^{\mathrm{v}}=\mathbf{11}^{\cdot\cdot} \cup \mu$, $\mu \in \Delta_6(11^{\cdot\cdot})$.

Next we will show

(D) $G_{(12^{\mathrm{v}})}$ is transitive on $\mathbf{12}^{\mathrm{v}}$.

First we investigate $(G_{(12^{\mathrm{v}})})_{[\mu]}$ and show that $(G_{(12^{\mathrm{v}})})_{[\mu]}=(G_{(11^{\cdot\cdot})})_{[\mu]}$ is transitive on $\mathbf{12}^{\mathrm{v}}-\{\nu,\mu\}$ where ν is the point left fixed by $G_{(11^{\cdot\cdot})}$ and $\mathbf{12}^{\mathrm{v}}=\mathbf{11}^{\cdot\cdot}\cup\mu$. We have seen in Proposition 7.2(A) that $G_{(11^{\cdot\cdot})}$ of order 120 is $S_5|_{S_2\times S_3}$ and transitive on $\mathbf{10}^{\cdot\cdot}$ and fixes ν, where $\mathbf{11}^{\cdot\cdot}=\mathbf{10}^{\cdot\cdot}\cup\nu$, $\nu\in\Delta_{12}(10^{\cdot\cdot})$. First we determine cycle types of elements of $S_5|_{S_2\times S_3}$. The generators of $S_5|_{S_2\times S_3}$ are known [3, p. 44] as follows:

$$S_5|_{S_2\times S_3}=\langle(abc)(def)(ghi),\ (ad)(bf)(ce)(gh),\ (iafbd)(jehgc),\ (ad)(be)(cf)\rangle.$$

Since $S_5|_{S_2\times S_3}\cong S_5$, $G_{(11^{\cdot\cdot})}$ has two classes of elements of order 2, and by the above generators it is easy to see that they have types 1^22^4 and 1^42^3. $G_{(11^{\cdot\cdot})}$ has one class of elements of order 5 and they have type 5^2. Now, $(ad)(be)(cf)(g)(h)(i)(j)\cdot(iafbd)(jehgc)=(ai)(bhgc)(dfje)$. Therefore $G_{(11^{\cdot\cdot})}\cong S_5$ has one class of elements of order 4 and of type 24^2. Now $(G_{(12^{\mathrm{v}})})_{[\mu]}$ has order 20 in $G_{(11^{\cdot\cdot})}$. Therefore, $(G_{(12^{\mathrm{v}})})_{[\mu]}$ is isomorphic to a subgroup of index 6 in S_5. Let us call this subgroup of index 6 in S_5 a G_{20}. Since 3 does not divide $|G_{20}|$, the compound character corresponding to G_{20} is uniquely determined as follows:

$$(1_{G_{20}})^{S_5}=\psi_1+\psi_{5^2}.$$

Indeed there are two irreducible characters of degree 5, $\psi_{5^1}=[32]$ and $\psi_{5^2}=[2^27]$ but $\psi_{5^1}=[32]$ would bring in elements of order 6 in G_{20}, which is impossible. From this compound character, we can find the number of elements in each class of G_{20} $(\cong(G_{(11^{\cdot\cdot})})_{[\mu]})$. They have 1 element of 1^{10}, 10 elements of 24^2, 4 elements of 5^2 and 5 elements of type either 1^22^4 or 1^42^3. Now element 1^{10} fixes 10 points, elements of type 24^2 fix no point, elements of type 5^2 fix no point. Therefore elements of type 1^22^4 are in $(G_{(11^{\cdot\cdot})})_{[\mu]}$ and fix 10 more points. Therefore $(G_{(11^{\cdot\cdot})})_{[\mu]}$ is transitive on $\mathbf{10}^{\cdot\cdot}$ and fixes ν and μ.

Therefore, the orbit of $G_{(12^{\mathrm{v}})}$ which contains the new point μ, call it l_μ, has length either $|l_\mu|=2$ or 11 or 12. If $|l_\mu|=2$, then $|G_{(12^{\mathrm{v}})}|=40$ and $[M_{24}:G_{(12^{\mathrm{v}})}]=12241152>2295216$, which is impossible by (a). If $|l_\mu|=11$, then $|G_{(12^{\mathrm{v}})}|=220$. In this case 17 must divide $|G_{(12^{\mathrm{iv}})}|$, which is impossible. Therefore, $|l_\mu|=12$ and $G_{(12^{\mathrm{v}})}$ is transitive on $\mathbf{12}^{\mathrm{v}}$, and $|G_{(12^{\mathrm{v}})}|=20\cdot 12=240$.

Now we show

(E) $G_{(12^{\mathrm{v}})}$ is an imprimitive group with kernel of order 2, image $\bar{S}_5$ (S_5 in S_6).

All the primitive groups of degree 12 are known and none of them has order 240, so $G_{(12^{\mathrm{v}})}$ is an imprimitive group of degree 12. There are two imprimitive groups of degree 12, order 240 [7, p. 118]. Let us call the first of them $G_{240^{\mathrm{I}}}$. $G_{240^{\mathrm{I}}}$ is described as an imprimitive group with kernel system $\bar{S}_5|\bar{S}_5$, image C_2, and $G_{240^{\mathrm{II}}}$ is described as an imprimitive group with kernel of order 2, image $\bar{S}_5$, and furthermore, $G_{240^{\mathrm{II}}}$ cannot be represented as an imprimitive group with two blocks of lengths 6, 6. By (D), we have seen $(G_{(12^{\mathrm{v}})})_{[\mu]}$ is transitive on $\mathbf{10}^{\cdot\cdot}$ and fixes ν and μ. Consider a subgroup of $G_{240^{\mathrm{I}}}$ which fixes a point. Then the subgroup must be in the kernel and cannot have an orbit of length 10. Therefore $G_{(12^{\mathrm{v}})}$ is $G_{240^{\mathrm{II}}}$.

(F) $|G_{(12^{iv})}|=192$, by (a), and l_λ (the orbit of $G_{(12^{iv})}$ which contains λ) must have length 8 and l_ω has length 4. Thus $\mathbf{12}^{iv}$ contains eight $\mathbf{11}^{\cdot\cdot}$'s and four $\mathbf{11}^{\cdot\cdot\cdot}$'s.

As for the action of $G_{(12^i)}$ on $(\mathbf{12}^i)^c$, the complement of $\mathbf{12}^i$ in Ω, we have

PROPOSITION 8.3. (A) $\mathbf{12}^i \approx (\mathbf{12}^i)^c$ *and*

(B) $G_{(12^i)}{}^{\mathbf{12}^i} \approx G_{(12^i)}{}^{(\mathbf{12}^i)^c}$.

Proof. Since no nonidentity element can fix 9 points, $|G_{(12^i)}|=|G_{((12^i)^c)}|$. Furthermore, no two orders of $G_{(12^i)}$ are equal, therefore, $\mathbf{12}^i \approx (\mathbf{12}^i)^c$, and $G_{(12^i)}{}^{\mathbf{12}^i} \approx G_{(12^i)}{}^{(\mathbf{12}^i)^c}$.

It is interesting to note that, for $\mathbf{12}^{\cdot}$, we have $M_{12}{}^{\mathbf{12}^{\cdot}} \approx M_{12}{}^{(\mathbf{12}^{\cdot})^c}$. These two M_{12}'s are seen to be connected by the outer automorphism of M_{12} such that

$$\begin{array}{ccc} \mathbf{12}^{\cdot} & & (\mathbf{12}^{\cdot})^c \\ 4^2 1^4 & \leftrightarrow & 4^2 2^2 \\ 821^2 & \leftrightarrow & 84 \end{array}$$

and the effect of this outer automorphism on M_{11} is observed in Proposition 7.1(B). For $\mathbf{12}^{\cdot\cdot\cdot}$, $(\mathbf{12}^{\cdot\cdot\cdot})^c=\theta \cup \{\Delta_{12}(11^{\cdot})-i\}$. $G_{(12^{\cdot\cdot\cdot})^c}$ is $\mathrm{PSL}_2(11)$ on $\{\Delta_{12}(11^{\cdot})-i\}$ and fixes θ.

Further, we note that since $\mathbf{12}^i \approx (\mathbf{12}^i)^c$ and $G_{(12^i)}{}^{\mathbf{12}^i} \approx G_{(12^i)}{}^{(\mathbf{12}^i)^c}$, there exists an element ρ in M_{24} such that $(\mathbf{12}^i)^\rho=(\mathbf{12}^i)^c$. Thus we have an imprimitive group with kernel $G_{(12^i)}{}^{\mathbf{12}^i} | G_{(12^i)}{}^{(\mathbf{12}^i)^c}$ and C_2 as its image except for $\mathbf{12}^{\cdot\cdot\cdot}$ and $\mathbf{12}^{iv}$. These imprimitive groups are $N_{M_{24}}(G_{(12^i)})$, $i=1, 2, 5$, and will be studied in the following paper, *On subgroups of* M_{24}. II.

9. **The maximal subgroups among the intransitives of** M_{24}. Any intransitive subgroup of M_{24} is clearly contained in a $G_{(n^i)}$ we have studied. Any maximal subgroup among the intransitives is to be found among the 26 $G_{(n^i)}$'s and, if not contained in a transitive subgroup of M_{24}, is a maximal intransitive subgroup of M_{24}. We will now show that there are nine such maximals among the intransitives, four of which will be shown, in Part II, to remain as maximal intransitive subgroups of M_{24}.

We have the following inclusion relations.

PROPOSITION 9.1. (a) $G_{(8^{\cdot})}$ *contains*: $G_{(5)}$, $G_{(8^{\cdot\cdot\cdot})}$, $G_{(9^{\cdot})}$, $G_{(12^{iv})}$, $G_{(6^{\cdot})}$ *which contains* $G_{(7^{\cdot\cdot})}$, $G_{(7^{\cdot})}$ *which contains* $G_{(8^{\cdot\cdot})}$, *and* $G_{(10^{\cdot})}$ *which contains* $G_{(9^{\cdot\cdot})}$.

(b) $G_{(12^{\cdot})}$ *contains*: $G_{(9^{\cdot\cdot\cdot})}$, $G_{(10^{\cdot\cdot})}$ *which contains* $G_{(11^{\cdot\cdot})}$, *and* $G_{(11^{\cdot})}$ *which contains* $G_{(12^{\cdot\cdot\cdot})}$.

(c) $G_{(12^{\cdot\cdot})}$ *contains*: $G_{(10^{\cdot\cdot\cdot})}$ *and* $G_{(11^{\cdot\cdot\cdot})}$.

Proof. (a) $G_{(5)}$, $G_{(8^{\cdot\cdot\cdot})}$, $G_{(9^{\cdot})}$ and $G_{(12^{iv})}$ are contained in $G_{(8^{\cdot})}$ by Propositions 2.5, 4.3, 5.1, and 8.2, respectively. By Proposition 3.2, $G_{(6^{\cdot})}$ is in $G_{(8^{\cdot})}$ and contains $G_{(7^{\cdot\cdot})}$ by 3.5. By 3.5, $G_{(7^{\cdot})}$ is in $G_{(8^{\cdot})}$ and contains $G_{(8^{\cdot\cdot})}$ by 4.2. By 6.1, $G_{(10^{\cdot})}$ is in $G_{(8^{\cdot})}$ and contains $G_{(9^{\cdot\cdot})}$ by 6.5.

(b) By 5.3, $G_{(9^{\cdot\cdot\cdot})}$ has $\Delta_{12}(9^{\cdot\cdot\cdot})$, which is equal to $\Delta_{12}(11^{\cdot})$ by 7.1; by construction of $\mathbf{12}^{\cdot}$, $G_{(12^{\cdot})} \supset G_{(9^{\cdots})}$. By 6.4, $G_{(10^{\cdot\cdot})}$ has $\Delta_{12}(10^{\cdot\cdot})$ which is equal to $\Delta_{12}(11^{\cdot})$ by 7.1; by 8.1, again, $G_{(12^{\cdot})} \supset G_{(10^{\cdot\cdot})}$; by 7.2, $G_{(10^{\cdot\cdot})} \supset G_{(11^{\cdot\cdot})}$. By 7.1, $G_{(12^{\cdot})} \supset G_{(11^{\cdot})}$ and $G_{(11^{\cdot})} \supset G_{(12^{\cdots})}$ by 8.1.

(c) By 6.3, $G_{(10^{\cdots})}$ has $\Delta_{12}(10^{\cdots})$ which is equal to $\Delta_{12}(11^{\cdots})$ by 7.3; by 8.1, $G_{(12^{\cdot\cdot})} \supset G_{(10^{\cdots})}$. By 8.1, $G_{(12^{\cdot\cdot})} \supset G_{(11^{\cdots})}$.

Thus 17 out of 26 $G_{(n^i)}$'s are contained in some $G_{(n^i)}$. The remaining nine $G_{(n^i)}$'s will be shown in the following to be the maximal subgroups among the intransitives of M_{24}.

THEOREM I. *Let G be an intransitive subgroup of M_{24}, then G is contained in one of the following nine maximal subgroups among the intransitives of M_{24}: $G_{(1)}$, $G_{(2)}$, $G_{(3)}$, $G_{(4)}$, $G_{(6^{\cdot\cdot})}$, $G_{(8^{\cdot})}$, $G_{(12^{\cdot})}$, $G_{(12^{\cdot\cdot})}$ and $G_{(12^{\mathrm{v}})}$.*

Proof. By Proposition 9.1, the maximals among the intransitives are to be found among the nine remaining $G_{(n^i)}$'s, namely, $G_{(1)}$, $G_{(2)}$, $G_{(3)}$, $G_{(4)}$, $G_{(6^{\cdot\cdot})}$, $G_{(8^{\cdot})}$, $G_{(12^{\cdot})}$, $G_{(12^{\cdot\cdot})}$ and $G_{(12^{\mathrm{v}})}$. $G_{(1)}$, $G_{(2)}$ and $G_{(3)}$ are intransitive maximal subgroups by Propositions 2.0, 2.1, and 2.2, respectively. The rest is proved by simple comparison of orders. Thus we have, by Propositions 2.0, 3.1, 2.3, 8.1 and 8.2, $|G_{(4)}| = 2^9\cdot 3^2\cdot 5$, $|G_{(6^{\cdot\cdot})}| = 2^4\cdot 3^3\cdot 5$, $|G_{(8^{\cdot})}| = 2^{10}\cdot 3^2\cdot 5\cdot 7$, $|G_{(12^{\cdot})}| = 2^6\cdot 3^3\cdot 5\cdot 11$, $|G_{(12^{\cdot\cdot})}| = 2^8\cdot 3^3$ and $|G_{(12^{\mathrm{v}})}| = 2^4\cdot 3\cdot 5$, respectively. $G_{(8^{\cdot})}$, $G_{(12^{\cdot})}$, and $G_{(12^{\cdot\cdot})}$ are seen to be maximals among the intransitives by simple comparison of orders among them.

$G_{(4)}$ can be in $G_{(8^{\cdot})}$ by comparison of the orders, but $G_{(4)}$ has an orbit of length 20 by Proposition 2.4, therefore $G_{(4)}$ is a maximal among the intransitives.

$G_{(6^{\cdot\cdot})}$ can be in $G_{(12^{\cdot})}$. But $G_{(6^{\cdot\cdot})}$ has an orbit of length 18 by Proposition 3.3, and $G_{(6^{\cdot\cdot})}$ is a maximal among the intransitives.

$G_{(12^{\mathrm{v}})}$ cannot be contained in $G_{(4)}$ or $G_{(6^{\cdot\cdot})}$ or $G_{(8^{\cdot})}$. $\mathbf{12}^{\mathrm{v}} \not\approx \mathbf{12}^{\cdot}$, so $G_{(12^{\mathrm{v}})} \not\subset G_{(12^{\cdot})}$ and $G_{(12^{\mathrm{v}})}$ is also a maximal among the intransitives.

Appendix.

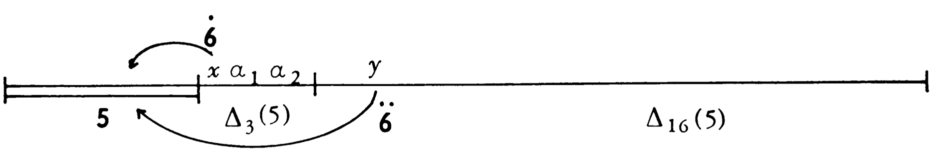


Figure 1

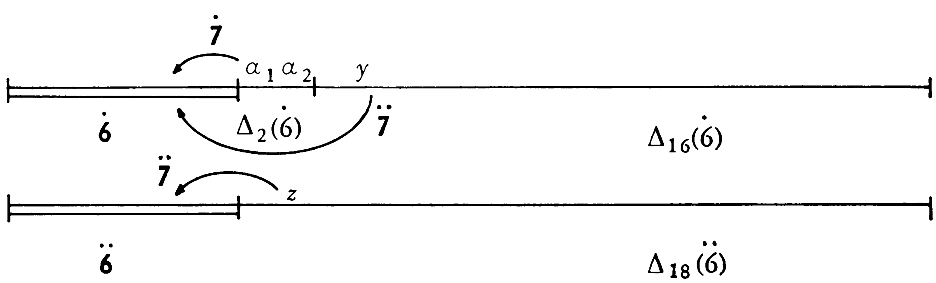


Figure 2

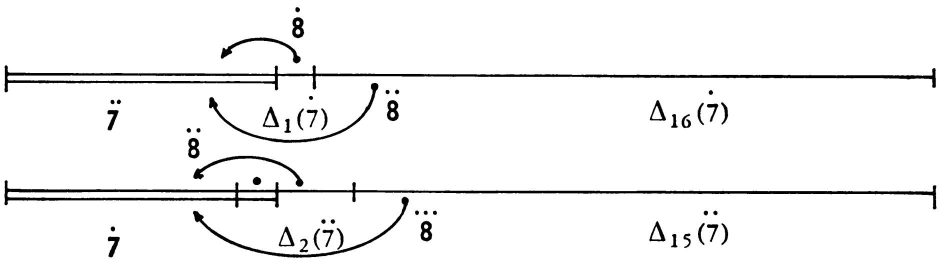


Figure 3

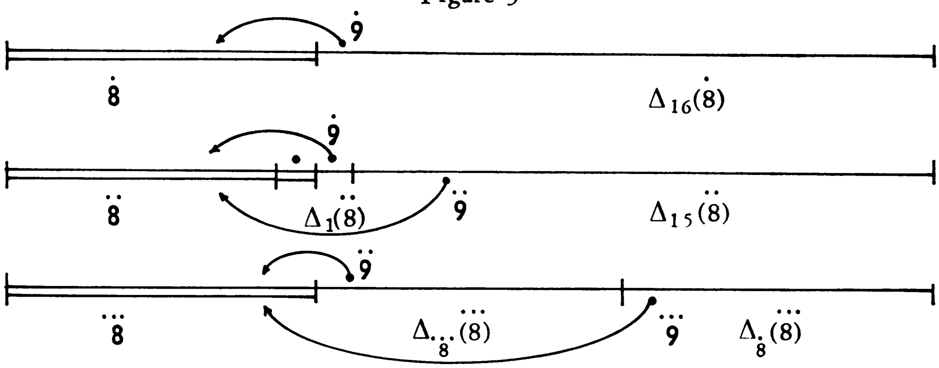


Figure 4

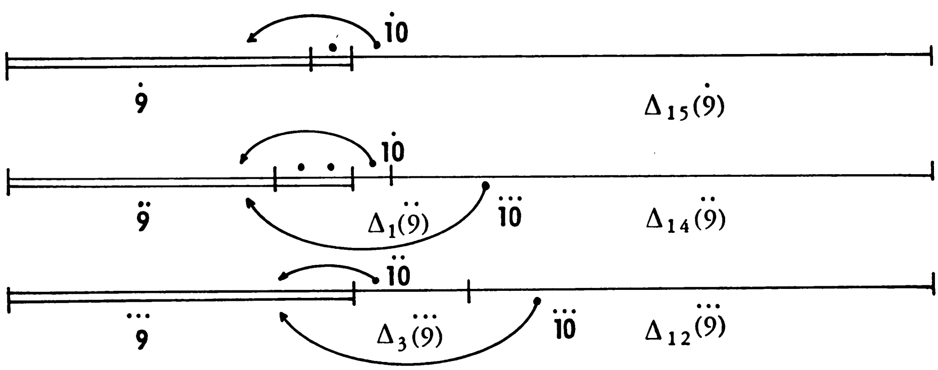


Figure 5

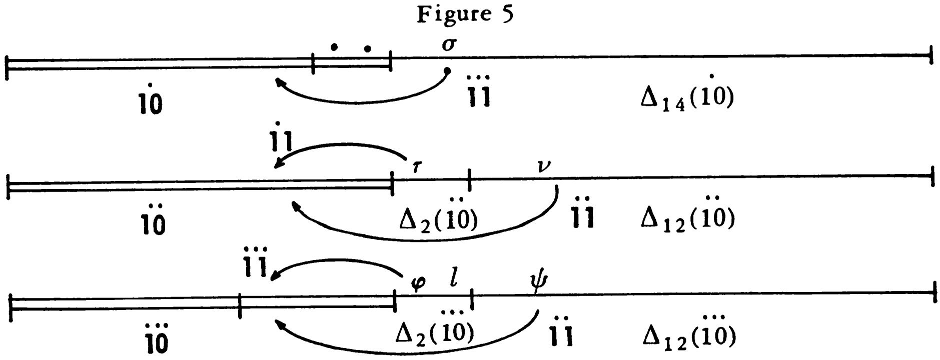


Figure 6

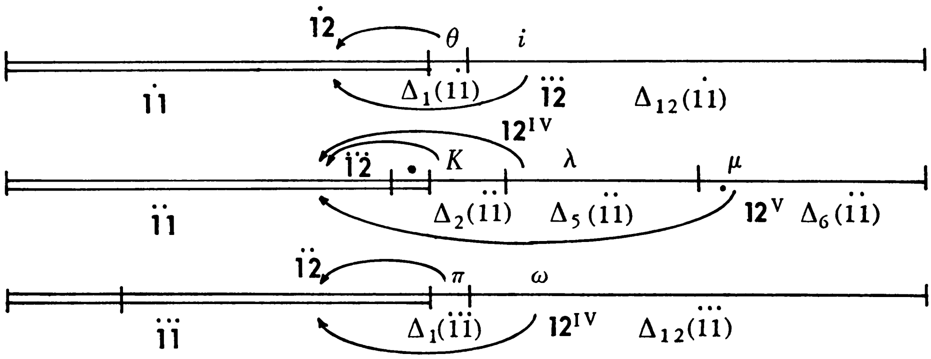


Figure 7

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