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# TRANSVERSALS TO THE FLOW INDUCED BY A DIFFERENTIAL EQUATION ON COMPACT ORIENTABLE 2-DIMENSIONAL MANIFOLDS(1)

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**Abstract.** Every treatment of the theory of differential equations on a torus uses the fact that given a differential equation on a torus of class $C^k$, there is a non-null-homotopic closed Jordan curve $\Gamma$ of class $C^k$ which is transverse to the trajectories of the differential equation that pass through points of $\Gamma$. Such a curve necessarily cannot separate the torus. Here, we prove that given a differential equation on an $n$-fold torus $T_n$ of class $C^k$, possessing only "simple" singularities of negative index there is a non-null-homotopic closed Jordan curve $\Gamma$ of class $C^k$ which is a transversal. The nonseparating property, however, does not follow immediately. For the particular case $T_2$, we prove the existence of such a transversal that does not separate $T_2$.

1. **Introduction.** We develop some machinery helpful in discussing differential equations, also called vector fields, on compact orientable 2-dimensional manifolds.

In the usual treatment of differential equations on a 2-dimensional torus $T$, due to A. Denjoy (see [1], [2] or [3]), the fact that there exists a non-null-homotopic curve $\Gamma$, transverse to the flow induced by a differential equation of class $C^k$, $k \geqq 1$, on $T$, having no stationary points, is of paramount importance. By non-null-homotopic, we mean that $\Gamma$ cannot be contracted to a point. The importance of the existence of $\Gamma$ lies in the fact that $\Gamma$ is a generator of the first homotopy group of $T$. Once the existence of such a transversal is proven, the further assumption is made that every trajectory of the differential equation intersects the transversal. In order to make an assumption on any compact orientable 2-dimensional manifold analagous to the last one above, we must prove that such a non-null-homotopic transversal $\Gamma$ exists and, in addition, that it does not separate the manifold into disjoint pieces.

Here, given a differential equation of class $C^k$, $k \geqq 1$, on a compact orientable 2-dimensional manifold of class $C^l$, $l \geqq k$, containing only simple singularities of negative index, we prove the existence of a non-null-homotopic transversal to the flow induced by the differential equation. By a simple singularity, we mean a

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singularity with no elliptic or parabolic sectors (see [3] or [5]). In view of the end results achieved by Denjoy on a torus, the restriction to simple singularities of negative index is not so severe; this is the only case in which a trajectory of the differential equation could be dense on the manifold. In the special case that the manifold is of genus 2 and the differential equation is of class $C^2$, we also prove that some transversal $\Gamma$ as above does not separate the manifold.

It is the hope of the author that the proof of the latter result can ultimately be amended to provide a means of proof for any compact orientable 2-dimensional manifold.

2. **Preliminaries.** We consider differential equations on compact orientable manifolds of class $C^k$, $k \geqq 1$ and genus $n$, $n \geqq 2$, denoted $T_n$. The Euler characteristic of such a manifold is $\chi = 2-2n$. Concerning the stationary points of the vector field on $T_n$, we have the following well-known theorem originally due to Poincaré and Hopf (see [4] for a discussion in the case of $C^\infty$ manifolds).

THEOREM 2.1. *The sum $\sum i$ of the indices at the zeros of a $C^k$ vector field, $k \geqq 1$, with isolated zeros on a manifold $M$ of class $C^l$, $l \geqq k$, is equal to the Euler characteristic of the manifold.*

In view of this theorem and the fact that we are dealing with isolated simple singularities of negative index, we see that we can have only a finite number of singular points in our vector field on $T_n$, the sum of whose indices is $2-2n$. Given any combination of singular points on $T_n$, the sum of whose indices is $2-2n$, the fact that a $C^k$ vector field on $T_n$ exists having only these points as singular points can easily be established by use of partitions of unity.

Throughout the paper, $F(p)$ will denote a vector field on $T_n$ and $p' = F(p)$ will be the corresponding differential equation on $T_n$. The flow induced by a vector field will be denoted by $p^t = \phi(t, p)$.

It should be noted that the fact that $T_n$ is of class $C^k$, $k \geqq 1$, implies that $T_n$ is Riemannian [7]. This fact will be assumed whenever needed. It is then clear what we mean by a transversal to the flow and by $F_\perp(p)$, the vector field of the system of trajectories orthogonal to the trajectories of $p' = F(p)$.

3. **Existence of a non-null-homotopic transversal $\Gamma$.** In this section, we prove the following theorem.

THEOREM 3.1. *Let $p^t = \phi(t, p)$ be a $C^k$ flow on $T_n$, $k \geqq 1$, having only simple singularities of negative index. Then there exists a Jordan curve $\Gamma$ of class $C^k$ on $T_n$ which is transverse to the flow. Furthermore, $\Gamma$ is non-null-homotopic.*

We first state and prove a lemma and then proceed with the proof of Theorem 3.1.

DEFINITION 3.1. Let $p' = F(p)$ be a differential equation on $T_n$. Let $p_0$ be an isolated singular point of $F(p)$. A solution $p(t)$ of $p' = F(p)$ is called a positive (or negative) base solution at $p_0$, if $p(t)$ is a separatrix of the differential equation at

$p_0$ and $p_0 \in \Omega(p)$ (or $p_0 \in A(p)$), where $\Omega(p)$ is the $\omega$-limit set and $A(p)$ is the $\alpha$-limit set of the trajectory through $p$.

LEMMA 3.1. *Given a $C^k$ vector field on $T_n$, $k \geqq 1$, which has only simple singular points of negative index, there is a point $q$ such that the solution of $p' = F(p)$ through $q$ is not both a positive and negative base solution to the singular points.*

**Proof.** About each critical point, consider a small Jordan curve, so small that the neighborhood of the critical points determined by the curves are 2-cells such that the only sectors inside the neighborhoods are hyperbolic. There are only a finite number of arcs of trajectories that are base to each singular point. Suppose every solution $p = p(t)$ on $T_n$ is positively base to a singular point and negatively base to a singular point. The Jordan curve $C$ may, in fact, be chosen so small that once a base solution enters the neighborhood bounded by $C$, it stays in that neighborhood and the neighborhoods are disjoint.

Consider a solution $p_i = p_i(t)$. $p_i(t)$ approaches a singular point both as $t \to \infty$ and $t \to -\infty$. Thus there exists a $t_i^+$ such that if $t > t_i^+$, then $p_i(t)$ belongs to a neighborhood bounded by one of the $C$'s and there exists a $t_i^-$ such that if $t < t_i^-$, $p_i(t)$ belongs to a neighborhood bounded by one of the $C$'s. Let $T_1 = \max_i t_i^+$ and $T_2 = \min_i t_i^-$ (under the assumptions, there are only a finite number of trajectories). The only portions of a trajectory that can be outside the regions bounded by the $C$'s is $p(t)$ for $T_2 \leqq t \leqq T_1$. The measure of these portions of the trajectories is zero. Hence, there is a point $q$ through which the trajectories do not pass; a contradiction. Q.E.D.

**Proof of Theorem 3.1.** Consider the differential equation for the orthogonal trajectories

$$p' = F_\perp(p) \tag{3.1}$$

to the trajectories of

$$p' = F(p). \tag{3.2}$$

We note that a simple singular point of negative index of $F$ is also one of $F_\perp$. Let $p_0(t)$ be a solution of (3.1) through a point $p_0$ such that $p_0(t) \not\equiv$ constant and is not positively base to a singular point (Lemma 3.1). If $p = p_0(t)$ is a closed curve $\Gamma$, it will be clear by the end of the proof that $\Gamma$ is the transversal curve we want. If $p = p_0(t)$ is not a closed curve, the semitrajectory $p_0(t)$, $t \geqq 0$, has at least one $\omega$-limit point, say $p_1$. The point $p_1$ is contained in an arbitrarily small curvilinear rectangle $R{:}ABCD$ on $T_n$ in which the arcs $BC$ and $AD$ are solution arcs of (3.1) and $AB$ and $CD$ are solution arcs of (3.2). The point $p_0(t)$ is in $R$ for some large time $t = t_0$ and leaves $R$ at some point $q_1$ on $CD$ (or $AB$) at a first time $t_1 > t_0$. It then enters $R$ at some point $q_2$ on $AB$ (or $CD$) for some first time $t_2 > t_1$. It is clear that if $R$ is small enough, there exists an arc $q_1q_2$ in $R$ which together with the arc $p_0(t)$, $t_1 \leqq t \leqq t_2$, constitutes a transversal curve $\Gamma$ of class $C^k$.

We show that $\Gamma$ is non-null-homotopic. Suppose that $\Gamma$ is null-homotopic; then

$\Gamma$ has an image in the plane as a Jordan curve bounding an open set. Since $\Gamma$ is a transversal, $\Gamma$ consists entirely of ingress or egress points of trajectories and the index of $\Gamma$ is $+1$ [5, p. 133]. But $\Gamma$ has either no singularities or only singularities of negative index in its interior. Hence, the index of $\Gamma$ is $\leqq 0$; a contradiction. Q.E.D.

4. There exists a $\Gamma$ that does not separate $T_2$ into disjoint parts. The following theorem is proved.

THEOREM 4.1. *Let $p^t = \phi(t, p)$ be a $C^k$ flow on $T_2$, $k \geqq 2$. Suppose that $p^t$ has two simple stationary points of index $-1$ or one simple stationary point of index $-2$. Then there exists a transversal $\Gamma$ to the flow that does not separate $T_2$.*

The property of "separating" in the above theorem will be illustrated in the following sequence of diagrams which also show that Theorem 4.1 is the best possible in the sense that, under the given hypothesis, there may also be transversals $\Gamma$ that do separate $T_2$. In Diagram 4.1a, $p$ is a simple saddle point, $\gamma_1$ and $\gamma_2$ are trajectories which are both positively and negatively base to $p$, $\Gamma$ is the transversal and arrows through $\Gamma$ indicate the direction of flow across $\Gamma$. The diagram can be completed by putting a symmetric picture, with arrows reversed, to the right of

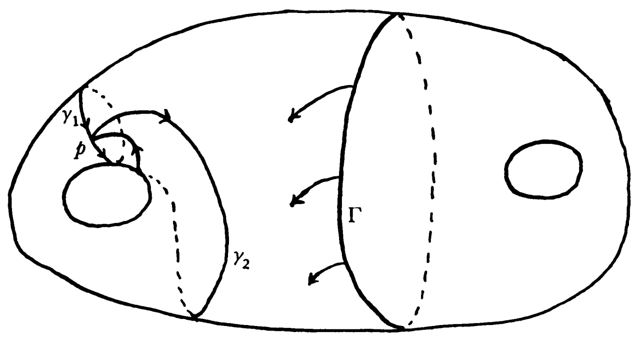


DIAGRAM 4.1a

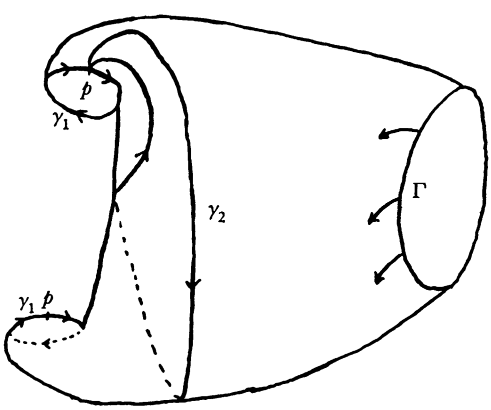


DIAGRAM 4.1b

$\Gamma$. In this example, we see that $\Gamma$ separates $T_2$ into two tori with a 2-cell removed from each.

To see how to fill in the rest of the vector field, we will take the part of $T_2$ to the left of $\Gamma$ (the right side is similar) and cut first along $\gamma_1$ giving Diagram 4.1b and then along $\gamma_2$ giving Diagram 4.1c

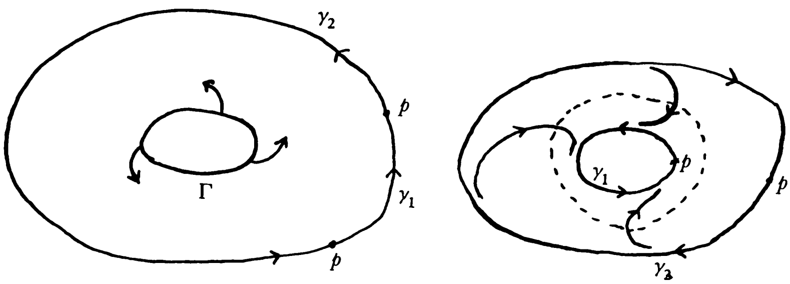


DIAGRAM 4.1c

The dotted line in Diagram 4.1c illustrates a transversal that does not separate $T_2$.

Before proceeding with the proof of Theorem 4.1, it will be helpful to carefully examine the consequences of the construction of the transversal $\Gamma$ of Theorem 3.1 in the case that $\Gamma$ does separate $T_2$.

Suppose that $\Gamma$ separates $T_2$ into two disjoint parts, that $\Gamma$ is not the result of a periodic solution of the perpendicular system (3.1) and that in the curvilinear rectangle $R$ of the proof of Theorem 3.1, trajectories leave $R$ across $CD$ and enter $R$ across $AB$. We had the fact that the solution $p_0(t)$ of equation (3.1) left $R$ at some point $q_1$ on $CD$ at time $t_1$ and entered $R$ at some point $q_2$ on $AB$ at time $t_2 > t_1$. $p_0(t)$ must then leave $R$ again for some first time $t_3 > t_2$ at a point $q_3$ on $CD$. Label the parts that $\Gamma$ separates $T_2$ into $\pi_1$ and $\pi_2$; for example see Diagram 4.2. If $p$ is a point of $\pi_1$, we see that

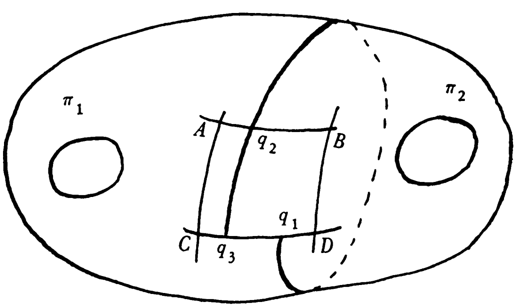


DIAGRAM 4.2

$p(t)$, the solution of (3.1) through $p$, cannot cross into $\pi_2$ for positive time, since by uniqueness of trajectories, the only way it could do so would be across the line segment $q_3q_1$ on $CD$, a situation which cannot occur. Likewise, trajectories of

(3.1) contained in $\pi_2$ cannot cross into $\pi_1$ for negative time. If $\Gamma$ is the result of a periodic solution of (3.1) it is clear that a trajectory cannot cross from $\pi_1$ to $\pi_2$ or from $\pi_2$ to $\pi_1$ for positive or negative time.

The following lemma shows that the picture we have drawn in Diagram 4.2 is in some sense general and hence we will always take $\pi_1$ and $\pi_2$ to be as pictured there; i.e., $\pi_1$ will be that part of $T_2$ that trajectories of (3.1) cannot leave for positive time and $\pi_2$ will be the part of $T_2$ that trajectories cannot leave for negative time. This lemma is easily proven using the Euler characteristic of a manifold.

LEMMA 4.1. *If a non-null-homotopic Jordan curve $C$ separates $T_2$, then each part of $T_2$ is homeomorphic to a torus with a 2-cell removed.*

We are now in a position to prove Theorem 4.1. This will be accomplished in three steps.

THEOREM 4.1a. *If $p^t=\phi(t,p)$ is a $C^k$ flow on $T_2$, $k\geqq 1$, and $p^t$ has one singularity of index $-2$, then no transversal curve $\Gamma$ separates $T_2$.*

THEOREM 4.1b. *Suppose that $p^t=\phi(t,p)$ is a $C^k$ flow on $T_2$, $k\geqq 2$, that has two simple stationary points of index $-1$ each. Further suppose that there is a trajectory $\gamma_1$ of* (3.1) (*the orthogonal system*) *that is negatively base to a stationary point $p$ of* (3.1) *that does not have $p$ in its $\omega$-limit set and also a trajectory $\gamma_2$ positively base to $p$ that does not have $p$ in its $\alpha$-limit set. Then there is a non-null-homotopic transversal $\Gamma$ that does not separate $T_2$.*

THEOREM 4.1c. *Suppose that $p^t=\phi(t,p)$ is a $C^k$ flow on $T_2$, $k\geqq 2$, that has two simple stationary points of index $-1$ each. Further suppose that every non-null-homotopic transversal separates $T_2$. Then the hypotheses of Theorem* 4.1b *hold.*

The application of Theorem 4.1b to Theorem 4.1c shows that not every non-null-homotopic transversal can separate $T_2$ in the case that we have two simple stationary points of index $-1$ each; i.e. some non-null-homotopic transversal must not separate $T_2$. This in conjunction with Theorem 4.1a completes the proof of Theorem 4.1.

**Proof of Theorem 4.1a.** Suppose some transversal $\Gamma$ separates $T_2$. By the proof of Theorem 3.1, we know that $\Gamma$ is non-null-homotopic. Hence by Lemma 4.1, each part that $T_2$ is separated into has Euler characteristic $-1$. Theorem 2.1 (the Poincaré-Hopf theorem) implies that each part of $T_2$ must have stationary points of the flow the sum of whose indices is $-1$. Since we have only one singularity of index $-2$, this is impossible. Q.E.D.

The following lemma will be necessary for the proof of Theorem 4.1b.

LEMMA 4.2. *Let $M$ be an orientable manifold of class $C^2$, $T^t$ be a flow on $M$ of class $C^2$ and $m_0\in M$. Let $\Omega(m_0)$ be the $\omega$-limit set of the trajectory through $m_0$. Suppose that $\Omega(m_0)\neq M$ and that $\Omega(m_0)$ is a nonempty compact set which contains no stationary points. Then $\Omega(m_0)$ is a Jordan curve and $m_0(+t)$ spirals towards $\Omega(m_0)$.*

This lemma appears as a corollary of a theorem due to A. J. Schwartz [6].

**Proof of Theorem 4.1b.** Suppose that every transversal separates $T_2$; we will arrive at a contradiction. We construct a transversal $\Gamma$ as in the proof of Theorem 3.1 using a trajectory of (3.1) not base to $p$. Maintaining the conventions established in the discussion accompanying Diagram 4.2, we call the parts that $T_2$ is divided into, $\pi_1$ and $\pi_2$. We also assume that $p \in \pi_1$ and that the other singular point is in $\pi_2$. The proof is analogous if this situation is reversed.

We know that the $\omega$-limit set $\Omega(\gamma_1)$ must lie in $\pi_1$. By Lemma 4.2, $\Omega(\gamma_1)$ is a periodic orbit and hence is also a closed transversal curve. As such, $\Omega(\gamma_1)$ separates $T_2$ into two parts. Label these $\pi_1(\gamma_1)$ and $\pi_2(\gamma_1)$ such that $\pi_1(\gamma_1)$ is contained in $\pi_1$. Then $p$ is contained in $\pi_1(\gamma_1)$ and as above, $A(\gamma_2)$ separates $T_2$ into two parts $\pi_1(\gamma_2)$ and $\pi_2(\gamma_2)$. $\pi_1(\gamma_2) \subseteq \pi_1(\gamma_1)$ and $p \in \pi_1(\gamma_2)$.

There are two cases, either $A(\gamma_2) \neq \Omega(\gamma_1)$ or $A(\gamma_2) = \Omega(\gamma_1) = L$.

If $A(\gamma_2) \neq \Omega(\gamma_1)$, $\pi_1(\gamma_2) \subset \pi_1(\gamma_1)$. Then $A(\gamma_2) \subset \pi_1(\gamma_1)$ and hence $p$ and $\Omega(\gamma_1)$ lie on opposite sides of $A(\gamma_2)$. But this implies that $\gamma_1$ crosses $A(\gamma_2)$ to reach its $\omega$-limit set. This contradicts uniqueness of trajectories.

If $A(\gamma_2) = \Omega(\gamma_1) = L$, Lemma 4.2 implies that $\gamma_1$ spirals towards $L$ and $\gamma_2$ spirals away from $L$ with increasing time. By uniqueness of trajectories they both spiral on the same side of $L$. However, this is impossible for if $l$ is a small line segment transverse to $L$, $\gamma_1$ and $\gamma_2$ both cross $l$ but in opposite directions, which violates differentiability of our vector field.

These two contradictions complete the proof of Theorem 4.1b. Q.E.D.

In order to proceed with the proof of Theorem 4.1c, it is helpful to have the following lemma.

LEMMA 4.3. *Let $M$ be a 2-manifold of class $C^k$ with a $C^k$ flow $p^t = \phi(t, p)$ on it, $k \geqq 1$. Suppose the flow has a saddle point $p$ and that a trajectory $\gamma$ not positively base to $p$ has $p$ in its $\omega$-limit set. Then at least one of the trajectories positively base to $p$ and one of the trajectories negatively base to $p$ belong to the $\omega$-limit set of $\gamma$.*

**Proof.** Let $N$ be a small closed 2-cell containing $p$ in its interior. The trajectories base to $p$ divide $N$ into four quadrants. Let $\{p_i\}$ be a sequence of points in $\Omega(\gamma)$ monotonically approaching $p$ in the closure of one of the quadrants $Q$. A curve segment $C_j = \{p_j^t = \phi(t, p_j) \cap N\}$ and the portion of $\partial N$ contained in quadrant $Q$ bounds a simply connected open set $S_j$ and $\{S_i\}_{i=1}^{\infty}$ satisfies $S_i \subset S_{i+1} \subset \bar{Q}$ for all $i$. Hence $S = \bigcup S_i$ as an open set has a connected boundary in $\bar{Q}$. It is easy to verify that $\partial S \subset \Omega(\gamma)$, $p \in \partial S$ and that $\partial S$ consists of a finite union of segments of trajectories. Hence $\partial S$ contains segments of trajectories negatively base and positively base to $p$, thus so does $\Omega(\gamma)$. Q.E.D.

**Proof of Theorem 4.1c.** Under the hypothesis of the theorem every transversal separates $T_2$. We saw that this implies that there must be one stationary point of the flow in each part. Following the conventions already established, we will do the proof for the part we called $\pi_1$ and label the stationary point in this part $p$. The proof proceeds in two parts

(i) not every trajectory of the orthogonal flow (3.1) negatively (positively) base to $p$ can be positively (negatively) base to $p$.

We use this to prove,

(ii) not every trajectory of (3.1) negatively base to $p$ can have $p$ in its $\omega$-limit set and not every trajectory positively base to $p$ can have $p$ in its $\alpha$-limit set.

Part (ii) is the conclusion of Theorem 4.1c.

For part (i), suppose that every trajectory of (3.1) negatively base to $p$ is also positively base to $p$. Let $\gamma_1$ be a trajectory negatively and positively base to $p$. $\gamma_1 \cup p$ is a closed curve and must be non-null-homotopic. We claim that $\gamma_1 \cup p$ cannot separate $T_2$. Suppose $\gamma_1 \cup p$ does separate $T_2$ into two disjoint parts $\pi_1(\gamma_1)$ and $\pi_2(\gamma_1)$ such that $\pi_1(\gamma_1) \subset \pi_1$. $p \in \partial\pi_1(\gamma_1)$. Let $\gamma_2$ be a trajectory of (3.1) not positively base to $p$ such that $\gamma_2 \subset \pi_1(\gamma_1)$. Construct a transversal using $\gamma_2$ and call it $\Gamma_2$. $\Gamma_2$ separates $T_2$, by hypothesis into two disjoint parts $\pi_1(\gamma_2)$ and $\pi_2(\gamma_2)$ such that one part, say $\pi_1(\gamma_2)$, is contained in $\pi_1(\gamma_1)$. Hence $p \in \pi_1(\gamma_2)$. But this is impossible since $p \in \partial\pi_1(\gamma_1)$. Hence, $\gamma_1 \cup p$ cannot separate $T_2$.

The problem thus reduces to the following problem on a torus with a 2-cell removed. Can all the trajectories negatively base to $p$ be positively base to $p$ and together with $p$ be non-null-homotopic curves such that none of the trajectories together with $p$ form a closed curve surrounding the removed 2-cell? In this problem the boundary of the 2-cell is $\Gamma$. That none of the closed curves formed by considering trajectories base to $p$ together with $p$ can surround the removed 2-cell corresponds to the fact that such a trajectory together with $p$ cannot separate $T_2$, as proven in the above paragraph. A negative answer to this problem will complete the proof of part (i).

Let $\gamma_1$ be a trajectory positively and negatively base to $p$. By assumption, $\gamma_1 \cup p$ is a closed non-null-homotopic curve on the torus with the disc removed that does not surround the disc. Hence, $\gamma_1 \cup p$ can be taken as a generator of the first homotopy group of the torus and as such can be taken as a pair of opposite boundaries of the following familiar rectangular representation of the torus with a disc removed (Diagram 4.3). Because of the configuration of a simple singularity of

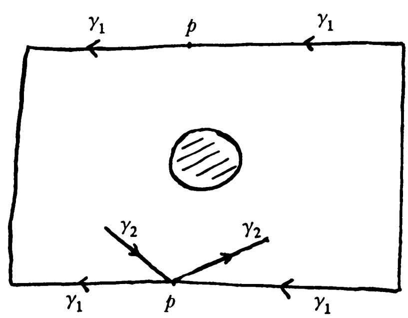


DIAGRAM 4.3

index $-1$, all the remaining trajectories base to $p$ will have to be base at the same representation of $p$. Let $\gamma_2$ be negatively base to $p$, $\gamma_2 \neq \gamma_1$. By assumption, $\gamma_2$ is also positively base to $p$. $\gamma_2$ cannot intersect the boundaries of the rectangle that $\gamma_1$ lies on and hence, can only intersect the other boundaries. Since $\gamma_2 \cup p$ cannot surround the disc by cutting and piecing operations, it is easily seen that $\gamma_1 \cup p$ and $\gamma_2 \cup p$ will have to bound a region homeomorphic to a disc. Hence, there will have to be a singular point other than $p$ inside the region; a contradiction of the fact that $p$ is the only singular point inside $\pi_1$. Thus, $\gamma_1$ and $\gamma_2$ cannot both be positively and negatively base to $p$, completing the proof of the lemma.

To prove part (ii), suppose that both trajectories $\gamma_1$ and $\gamma_2$ of (3.1) that are negatively base to $p$ have $p$ as an $\omega$-limit point and (by part (i)) that $\gamma_1$ is not positively base to $p$. Then, by Lemma 4.3 there is a point $q$ on a trajectory $\alpha$ positively base to $p$ that is an $\omega$-limit point of $\gamma_1$. This implies that all of $\alpha$ is contained in the $\omega$-limit set of $\gamma_1$. Let $R$ be a small curvilinear rectangle about $q$ one pair of whose opposite sides are arcs of solutions of (3.1) and whose other pair of opposite sides are arcs of solutions of (3.2). Since $R$ is small, $\alpha$ crosses $R$ entering and leaving across the opposite boundaries that are arcs of solutions of (3.2) and $\alpha$ divides $R$ into two parts.

We can find an increasing sequence of times $\{t_i\}$ such that $\gamma_1$ leaves $R$ for each $t_i$ on the same side of $\alpha$ and enters on that side for some time $t_i' > t_i$. By using this sequence of times $\{t_i\}$ and connecting $\gamma_1(t_i')$ to $\gamma(t_i)$ by a $C^k$ curve for each $i$ as in Theorem 3.1, we construct a sequence of transversal curves $\{\Gamma_i\}$ such that $\Gamma_i$ cuts $T_2$ into two disjoint parts $\pi_1^i$ and $\pi_2^i$, where $\pi_1^i \subset \pi_1$.

Let $t_i''$ be the first time that $\gamma_1$ leaves $R$ after $t_i'$. $\gamma_1(t_i'')$ will be closer to $\alpha$ than $\gamma_1(t_i)$, for otherwise, by uniqueness of trajectories and the fact that transversals must separate $T_2$, $\gamma_1(t)$ cannot get close to $\alpha$ which is in its $\omega$-limit set. Hence, the sequence of times $\{t_i\}$ can be taken such that the distance between $\Gamma_i$ and $p$ approaches 0 as $i$ approaches $\infty$, $\pi_1^{i+1} \subset \pi_1^i$, for all $i$, and $\Gamma_{i+1} \subset \pi_1^i$.

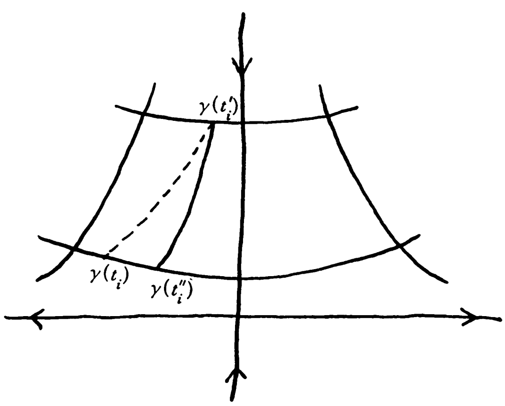


DIAGRAM 4.4

Let $P_i = \pi_1^i \cup \Gamma_i$. $P_i$ is closed, hence $\pi = \bigcap_i P_i$ is closed. $p \in \partial\pi$. Since no trajectories can leave $\pi_1^i$, for any $i$, for large time, the trajectories negatively base to $p$ are contained in $\pi$ also. This is impossible; $\gamma_1$ cannot be contained in $\pi$ since, by the previous construction, part of it forms the boundary $\Gamma_i$ of each $\pi_1^i$ and $\pi_1^i$ is not in $\pi_1^k$ for $k > i$; a contradiction.

We have shown that not every trajectory of (3.1) negatively base to $p$ can have $p$ in its $\omega$-limit set. To show that not every trajectory positively base to $p$ can have $p$ in its $\alpha$-limit set, we can carry out the same construction of a sequence of transversal curves using a decreasing sequence of times and a positively base trajectory. Q.E.D.

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