

AN EQUALITY FOR 2-SIDED SURFACES WITH A FINITE NUMBER OF WILD POINTS

BY

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Abstract. Let S be a 2-sided surface in a 3-manifold that is wild from one side U at just m points. It is shown that the minimal genus possible for all members of a sequence of surfaces in U converging to S (where these surfaces each separate the same point from S in $U \cup S$) is equal to the sum of the genus of S and a certain multiple of the sum of m special topological invariants associated with the wild points. In this equality, the sum of these invariants is multiplied by just one of the numbers 0, 1, or 2, dependent upon the genus and orientability class of S and the value of m . As an application, an upper bound is given for the number of nonpiercing points that a 2-sided surface has with respect to one side.

1. Introduction and notation. Let S be a 2-sided surface of genus $g(S)$ embedded in a 3-manifold M^3 . There is an open connected neighborhood W of S such that $W - S$ consists of exactly two components, U and V . We define $g(S, U)$ to be n if there exist a point x and a sequence of surfaces S_1, S_2, \dots such that

- (1) the genus of each S_k is n ,
- (2) each S_k separates x from S in $\text{Cl}(U)$,
- (3) $S = \lim S_k$, and
- (4) conditions (1)–(3) cannot be satisfied for any integer smaller than n .

If no such integer n exists we define $g(S, U)$ to be ∞ . This concept coincides with that of *limiting genus of U at S* or *limiting genus of U* which is used by Daverman in [7] and [8]. In [13] it is shown that if S can be locally peripherally collared from U and S is a 2-sphere in S^3 and $g(S, U)$ is positive, then S is wild from U at $g(S, U)$ points at most. More generally, in [7] it is proven that if S can be locally peripherally collared from U , $g(S, U)$ is positive, and S is an orientable surface which separates M^3 , then it follows that $g(S, U) - g(S)$ is an upper bound for the number of points at which S is wild from U . The equality developed in this paper sheds some light on why the above inequalities hold and is of interest in its own right. The authors are indebted to the referee for his constructive suggestions.

Let us suppose that either S is wild from U at p_1, \dots, p_m and nowhere else or else S is tame from U ($m=0$). To each point q of S we can attach a nonnegative integer $g(q, U)$ which is a topological invariant of $\text{Cl}(U)$. This has the property

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that $g(q, U)$ is positive only when q is one of the p_k and otherwise is 0. Our main result is that

$$\begin{aligned} (S, U) &= g(S) && \text{if } m = 0, \\ &= 0 && \text{if } m = 1 \text{ and } g(S) = 0, \\ &= g(S) + \sum_{k=1}^m g(p_k, U) && \text{if } S \text{ is orientable and either } m > 1 \text{ or } g(S) \neq 0, \\ &= g(S) + 2 \sum_{k=1}^m g(p_k, U) && \text{if } m \neq 0 \text{ and } S \text{ is nonorientable.} \end{aligned}$$

Further we are able to show that we can always find arcs A_1, \dots, A_m such that each A_k has p_k for an endpoint and $g(p_k, U) = \text{LEG}(A_k, p_k)$ (where $\text{LEG}(A_k, p_k)$ is the local enveloping genus of A_k at p_k). If S is locally tame from V at p_k , then we can take A_k to be any arc on S which has p_k as an endpoint.

Before attempting to prove anything, however, let us discuss some terms and notation.

It is understood that a manifold and a manifold-with-boundary are both considered to be connected. A manifold may or may not be compact, but it does not have any boundary points. A *surface* is a compact 2-manifold. Our results are still true if M^3 is a manifold-with-boundary, but then we should stipulate that S lies in the interior of M^3 .

Let K be a subset of an n -manifold M^n . Then K is *locally polyhedral at a point* p if there is a neighborhood N of p such that $K \cap \text{Cl}(N)$ is a polyhedron in M^n . We say that K is *locally polyhedral* if it is locally polyhedral at each point.

Let S, M^3, U , and V be as defined earlier. We shall call U and V the (nonunique) *sides* of S . If p is a point of S , then we say that S is *locally tame from* U at p if $\text{Cl}(U)$ is a 3-manifold-with-boundary at p . If $\text{Cl}(U)$ is a 3-manifold-with-boundary at every point of S , then S is *tame from* U . If p is a point of S where S is not locally tame from U , then S is *wild from* U at p .

We shall also use the following notations:

ab —an arc with a and b as endpoints,

Bd —the boundary of a manifold-with-boundary,

Int —the interior of a manifold-with-boundary,

Cl —closure,

d —metric (used for giving the distance between sets as well as points),

diam —diameter,

I — $[0, 1]$,

$N(K, \epsilon)$ —the ϵ -neighborhood of a set K ,

$g(F)$ —the genus of a surface F or of a surface-with-boundary,

$\text{ch}(F)$ —the Euler-Poincaré characteristic of F .

Let S, M^3, W, U and V be as defined before. We wish to introduce a new term.

DEFINITION. Suppose that S is wild from U at a finite number of points at most. Let p be a point of S and consider this statement which we label $G(p, m)$:

For every positive number ε there are a disc with m handles D and a disc D' such that $p \in \text{Int } (D') \subset S$, $D \cap S = \text{Bd } (D) = \text{Bd } (D')$, $\text{Int } (D) \subset U$, $\text{diam } (D \cup D') < \varepsilon$. If $G(p, m)$ is true and $G(p, m-1)$ is not true, then we define $g(p, U)$ to be m . If $G(p, m)$ is false for $m=0, 1, 2, \dots$, we define $g(p, U)$ to be ∞ . Notice that this is in some sense a generalization of Burgess' concept of local spanning in [4]. The surface S can be locally spanned from U at p if and only if $g(p, U)=0$. If $g(p, U)>0$, then S is wild from U at p . This definition can of course be immediately extended to the situation where S can be locally peripherally collared from U . We wish to use this concept of $g(p, U)$ in proving our equalities and shall show in the last section that it is equivalent to the idea of local enveloping genus that we referred to earlier.

Letting S , M^3 , W , U , and V be as defined earlier, we establish our equalities by breaking them up into inequalities which we establish separately. The following lemma gives us an upper bound for $g(S, U)$.

LEMMA 1. *Let us suppose that S is wild from U at the points p_1, \dots, p_m and nowhere else (this proof is also valid if S is tame from U , i.e., if $m=0$). Then*

$$\begin{aligned} g(S, U) &\leq g(S) + \sum g(p_k, U) && \text{if } S \text{ is orientable,} \\ g(S, U) &\leq g(S) + 2 \sum g(p_k, U) && \text{if } S \text{ is nonorientable.} \end{aligned}$$

Proof. We may suppose that each $g(p_k, U)$ is finite since otherwise the inequality is trivial.

Choose a point x of U not lying in an arbitrary ε -neighborhood $N(S, \varepsilon)$. There exist pairwise disjoint discs D_1, \dots, D_m and pairwise disjoint discs with handles H_1, \dots, H_m such that, for $i=1, \dots, m$, $p_i \in \text{Int } (D_i) \subset S$, $H_i \cap S = \text{Bd } (H_i) = \text{Bd } (D_i)$, $\text{Int } (H_i) \subset U$, H_i has $g(p_i, U)$ handles, and $\text{diam } (H_i \cup D_i) < \varepsilon$.

We may suppose that each surface $H_k \cup D_k$ is contained in some open 3-cell and that U_k is the complementary domain of $H_k \cup D_k$ which lies in that open 3-cell. By [1], we may assume that $M = (S \cup U) - \bigcup_{k=1}^m (U_k \cup \text{Int } D_k)$ is a 3-manifold-with-boundary, and $\text{Bd } M = (S - \bigcup_{k=1}^m D_k) \cup (\bigcup_{k=1}^m H_k)$.

Let R be a regular neighborhood of $\text{Bd } M$ in M such that $R \subset N(S, \varepsilon)$. Call F the component of $\text{Bd } (R)$ that lies in U . By construction, F separates x from S and has genus equal to $g(S) + \sum g(p_k, U)$ if S is orientable or $g(S) + 2 \sum g(p_k, U)$ if S is nonorientable. From the existence of such a surface F in any ε -neighborhood of S , the conclusion of the lemma follows.

2. Some intermediate lemmas. Before we can prove the inequality in the opposite direction (except, as we shall see, for the case when S is a 2-sphere wild from U at a single point), we shall need to establish some lemmas.

LEMMA 2. *Let S be a surface or a surface-with-boundary in a 3-manifold M^3 and let O be a neighborhood of S . Then there are a neighborhood N of S and a homotopy $h: N \times I \rightarrow O$ such that $S \subset N \subset O$, h_0 = the identity map, and h_1 is a retract map of N onto S .*

A proof of this can be found on pp. 97, 121, and 122 of [10].

The next proof amounts to a generalization of certain constructions used by Burgess in [5], but the details are sufficiently involved to make it worth going over.

LEMMA 3. *Suppose that S is a 2-manifold in a 3-manifold M^3 such that U is a complementary domain of S and p is a point of S . Let δ , D , and A be a positive real number, disc, and annulus respectively such that $p \in \text{Int}(D) \subset S$, $A \cap S = \text{Bd}(D) = \text{one of the boundary components of } A$, $A - S \subset U$, $A - S$ is locally polyhedral, and $\text{diam}(A \cup D) < \delta$.*

If δ is sufficiently small, then given δ , D , and A as described above, there exists a positive real number σ such that the following holds:

If F is a locally polyhedral 2-manifold in $U \cap N(S, \sigma)$ which intersects A and is in general position with respect to A , then there exists a component C of $F - A$ such that $\text{diam}(C) < 7\delta$.

Proof. Choose a disc D' in S so that p is a point in the interior of D' . There is a positive real number ϵ such that the intersection of $N(p, 5\epsilon)$ and S is contained in the interior of D' . Suppose that δ , A , and D are as described in the hypothesis and $\delta < \epsilon$. Clearly D is a subset of $\text{Int}(D')$. Let J be the name of that boundary component of A which lies in U . By using Lemma 2 and looking at collections of 3-cells which contain D' in the unions of their interiors, we see that there are a positive real number σ' , a neighborhood N of D' , and a homotopy h_t of N in M^3 such that $h_0 = \text{the identity map on } N$, h_1 is a retract map of N onto D' , $h_t(N)$ does not intersect J for $0 \leq t \leq 1$, $\sigma' < \epsilon$, $N(D', \sigma')$ is a subset of N , and if x and y are points of N and the distance from x to y is less than σ' , then there exists an arc xy in N such that $\text{diam}(xy) < \delta$.

Then we can find a positive real number σ and a subannulus A' of A such that $\sigma < \sigma'$, $\text{Bd}(D)$ is one of the components of $\text{Bd}(A')$, and $N(D', \sigma) \cap A \subset A' \subset N(D', \sigma')$.

Now let F be as in the hypothesis. It follows that F intersects A only at points of A' . By using the methods of Wilder [14, p. 66] and "walking over" D and "up the side" of A' , we may construct an arc pa such that a is a point in F , $\text{Int}(pa)$ is contained in $U - A$, pa is contained in $N(S, \sigma')$, and $\text{diam}(pa) < \delta$.

It follows that pa is contained in $N(D', \sigma')$. Let $C = \text{the component of } F - A \text{ containing } a$. Now suppose that $\text{diam}(C)$ is greater than or equal to 7δ . Then there is an arc ab in C such that the distance from a to $b = 3\delta$ and for z in $\text{Int}(ab)$ the distance from a to z is less than 3δ . Since ab is in $N(S, \sigma)$, there must be a point c in S which is a distance less than σ from b . It can be easily shown that b is in N and c is in $\text{Int}(D')$ and hence also in N . By the way we constructed N , there exists an arc bc in N such that $\text{diam}(bc) < \delta$. It follows, in addition, that ab is contained in N and that bc does not intersect A or D . Let cp be an arc in N such that $\text{Int}(cp)$ does not intersect A or D , and let J' be the continuous image of a simple closed curve which can be obtained by considering the union of pa , ab , bc and cp . Since J' is contained in N , it can be homotoped into D' and from there into U without

intersecting J . On the other hand, J' intersects the disc $D \cup A$ at the point p only and there it pierces it. But this shows that the mod 2 linking number of J' with J , calculated with respect to the fixed surface-with-boundary $D \cup A$, must be both 0 and 1, an impossibility. Therefore $\text{diam}(C) < 7\delta$.

LEMMA 4. *Let S , M^3 , W , U , and V be as in the Introduction and let S be wild from U at a finite number of points at most. Then there is an $\varepsilon > 0$ such that if F is a surface in U with the property that F separates S in $\text{Cl}(U)$ from the set of points of U which are more than a distance ε from S , then F is of the same orientability class as S .*

Proof. Suppose that S is orientable. By Theorem 2 of [12], there exists a positive number ε_1 such that $N(S, \varepsilon_1)$ can be embedded in S^3 . Let $\varepsilon = \varepsilon_1/2$. This automatically ensures that F can be embedded in S^3 and hence is orientable.

Now suppose that S is nonorientable. We can find two Moebius bands M and M^* on S such that neither one contains any of the points at which S is wild from U and $M \subset \text{Int}(M^*)$. By Theorem 1 of [3], there is a homeomorphism h such that $h: \text{Int}(M^*) \times [0, 2] \rightarrow \text{Cl}(U)$, $h(x, 0) = x$ for every $x \in \text{Int}(M^*)$ and

$$h(\text{Int}(M^*) \times (0, 2]) \subset U.$$

Choose ε so that $h(M \times 1)$ is a distance more than ε from S . Let F be as in the statement of the lemma. By [2] we may assume F to be polyhedral. There is a positive number δ such that $F \cap h(M \times I) \subset h(M \times (\delta, 1))$. Since $M \times [\delta, 1]$ is locally tame in $\text{Int}(M^*) \times [0, 2]$, by Theorem 9 of [1] we may assume that $h(M \times [\delta, 1])$ is polyhedral (we can adjust it by a very small homeomorphism of M^3 onto itself) and satisfies all the earlier requirements put on it. We may then further assume that F and $h(\text{Bd}(M) \times I)$ are in general position with respect to one another. It follows that $F \cap h(M \times I)$ consists of a finite number of polyhedral surfaces-with-boundary F_1, \dots, F_n such that for $k = 1, \dots, n$, $\text{Bd}(F_k) \subset h(\text{Bd}(M) \times I)$,

$$\text{Int}(F_k) \subset h(\text{Int}(M) \times I),$$

and $\bigcup_{k=1}^n F_k$ separates $h(M \times 0)$ and $h(M \times 1)$ in $h(M \times I)$.

From Lemma 4 of [9] we see that at least one of the F_k must be nonorientable. Thus F is nonorientable.

3. The main theorem. Since we have Lemma 1, all we have to do is find an appropriate lower bound for $g(S, U)$. Then in the next section we shall show how to replace $g(p, U)$ by the local enveloping genus of a suitable arc at an endpoint.

THEOREM 1. *Let S be a 2-sided surface in a 3-manifold M^3 and let W be an open connected neighborhood of S such that $W - S$ consists of two components, U and V (the two sides of S). We suppose that S is wild from U at the points p_1, \dots, p_m and nowhere else. Then*

$$g(S, U) = 0 \quad \text{if } g(S) = 0 \text{ and } m = 1,$$

$g(S, U) = g(S) + \sum_{k=1}^m g(p_k, U)$ if S is orientable and either $g(S) \neq 0$ or $m \neq 1$,
and

$$g(S, U) = g(S) + 2 \sum_{k=1}^m g(p_k, U) \text{ if } S \text{ is nonorientable.}$$

Prior to giving a proof of Theorem 1, we first offer a rough outline of how it goes. We essentially start with one collection (out of infinitely many collections) of m annuli each sticking into U except for a boundary component on S circling a wild point of S , and we suppose we have a surface of genus $g(S, U)$ in U which is sufficiently close to S , separates the boundary components of each annulus in the collection, and is in general position with respect to these annuli. We then surgically alter this surface until we ultimately obtain at the end of the operation a surface with genus no greater than the original surface and with all the special properties of the original one except that the new surface intersects each annulus nicely in exactly one simple closed curve provided, as is eventually shown, each annulus is small enough. This new surface is made up of two types of pieces: (1) m small discs with handles, each cut out by an annulus, and (2) a large connected piece which is what is left when the small discs with handles are removed. The inequality is then established by showing that the sum of the genera of the small pieces must be no smaller than the sum (or twice the sum) of the local enveloping genera of the wild points of S and that the genus of the large piece must be no smaller than the genus of S .

Proof of Theorem 1. Before getting into the thick of difficulties, we consider the case in which S is a 2-sphere which is wild from U at a single point. It follows from the fact that a neighborhood of S may be embedded in S^3 [12] and from [6] and [1] that $g(S, U) = 0$. So we may suppose that either $g(S) \neq 0$ or $m \geq 2$.

Step 1. We need first of all to construct a number of objects which will be used in the proof. We suppose W chosen so close to S that W can be homotoped onto S and any point of S will go onto itself (guaranteed by Lemma 2). For $i, j = 1, \dots, m$ and $k = 1, 2, \dots$, we can find discs D_{ik} and annuli A_{ik} such that $p_i \in \text{Int}(D_{ik}) \subset S$; $\text{Bd}(D_{ik}) = \text{one of the boundary components of } A_{ik}$; $A_{ik} - S \subset U$; J_{ik} is the name we give to the boundary component of A_{ik} in U ; $\text{diam}(A_{ik} \cup D_{ik}) < \delta_k$, where δ_k is a positive number chosen sufficiently small to make Lemma 3 work for D_{ik} and A_{ik} ($i = 1, \dots, m$) and such that $\delta_k < 1/(7 \times 2^k)$; and $(A_{ik} \cup D_{ik}) \cap (A_{jk} \cup D_{jk}) = \emptyset$ if $i \neq j$.

We then know that for every k we can find σ_k such that $0 < \sigma_k < \delta_k$ and the conclusions of Lemma 3 hold for σ_k with respect to δ_k . By [2] we may suppose each $A_{ik} - S$ is locally polyhedral. For $k = 1, 2, \dots$, let G_k be a connected polyhedral 3-manifold-with-boundary in U such that G_k contains all points of U which are a distance greater than or equal to $\sigma_k/2$, G_k contains J_{1k}, \dots, J_{mk} , and any surface F in U which separates S from G_k in $\text{Cl}(U)$ must have the same orientability class as S (Lemma 4).

Next for $k=1, 2, \dots$ and $i=1, \dots, m$, we choose A'_{ik} such that A'_{ik} is a subannulus of A_{ik} , $\text{Bd}(D_{ik}) = \text{one of the boundary components of } A'_{ik}$, and $A'_{ik} \subset N(S, \sigma_k)$.

Now we describe a condition which will be useful in the rest of the proof. We say that a polyhedral surface F in U satisfies condition (C_k) if and only if

F separates S from G_k in $\text{Cl}(U)$,

F is in general position with respect to each A_{ik} , and

$F \cap A_{ik} \subset \text{Int}(A'_{ik})$ for $i=1, \dots, m$.

Note that F must have the same orientability class as S , that $F \cap A_{ik}$ must contain a component which is an essential simple closed curve in A_{ik} , and that F is contained in $N(S, \sigma_k)$. Finally, for $k=1, 2, \dots$, we choose M_k to be a polyhedral surface in U such that it satisfies condition (C_k) and $g(M_k) = g(S, U)$.

Step 2. For $i=1, \dots, m$ and $k=1, 2, \dots$ we know that $M_k \cap A_{ik}$ consists of a finite number of simple closed curves in $\text{Int}(A'_{ik})$. Suppose that for fixed i and k there exists a component L of $M_k \cap A_{ik}$ such that L bounds a disc D in A'_{ik} . We wish to show that we may so modify M_k as to avoid this situation.

We may assume that M_k has no points in common with $\text{Int}(D)$. By cutting M_k apart along L and attaching two new discs to the cut ends of M_k , we can manufacture a set N such that N satisfies condition (C_k) (though N might not be connected), $N \cap A_{ik} = (M_k \cap A_{ik}) - L$, $\text{ch}(N) = \text{ch}(M_k) + 2$, and N consists of a single 2-manifold or two disjoint 2-manifolds.

Now if N is a single 2-manifold, then $g(N) < g(M_k) = g(S, U)$.

Suppose that N consists of two disjoint 2-manifolds N_1 and N_2 . We know that N separates G_k from S in $\text{Cl}(U)$ and we wish to show that either N_1 or N_2 alone does this. Suppose not. There are disjoint arcs B_1 and B_2 such that B_i has endpoints q_i and r_i in G_k and S respectively, $\text{Int}(B_i) \subset U$, and $B_i \cap N_i = \emptyset$. There exist arcs C_1 and C_2 such that C_1 goes from q_1 to q_2 in G_k and C_2 goes from r_1 to r_2 in S . Let J be the simple closed curve obtained by taking the union of B_1 , B_2 , C_1 , and C_2 . Clearly B_2 must pierce N_1 an odd number of times to get from S to G_k , so the mod 2 intersection number of J with N_1 is one. But by the way W was chosen, we can homotopy J into S and thus prove that the mod 2 intersection number of J with N_1 is zero. Contradiction. Thus we may assume that N_1 separates G_k from S in $\text{Cl}(U)$. Hence N_1 satisfies condition (C_k) . Now using the facts that $\text{ch}(N) = \text{ch}(M_k) + 2$, that N_1 and M_k are of the same orientability class, and that $\text{ch}(N_2)$ is less than or equal to 2, it clearly follows that $g(N_1) \leq g(M_k) = g(S, U)$.

It is clear from the above description that we can replace each M_k by a 2-manifold P_k such that P_k satisfies condition (C_k) , $g(P_k) \leq g(S, U)$, and for $i=1, \dots, m$ each component of the intersection of A_{ik} and P_k is a polyhedral simple closed curve which is essential in A_{ik} .

Step 3. We should like to show that by a process similar to that of Step 2 we can so modify P_k as to reduce the number of components of $P_k \cap A_{ik}$ to one and ensure that P_k still satisfies condition (C_k) and has genus not greater than $g(S, U)$. Unfortunately we cannot always do this, but in those cases where we cannot, we

will be able to show the existence of a disc on P_k the boundary of which is one of the components of $P_k \cap A_{ik}$.

Let us consider a choice of k and i such that the intersection of P_k and A_{ik} contains at least three components. Then there are two such components F_1 and F_2 such that F_1 and F_2 bound an annulus A^* on A_{ik} and $\text{Int}(A^*)$ misses P_k . There exists a solid polyhedral torus T in U such that $A_{ik} \cap \text{Bd}(T) = F_1 \cup F_2$, $A_{ik} \cap T = A^*$, T is contained in $U - G_k$, $T \cap P_k = \text{Bd}(T) \cap P_k =$ the union of two disjoint polyhedral annuli Q_1 and Q_2 , and F_j is essential in $\text{Int}(Q_j)$ for $j=1, 2$.

Let $N = (P_k \cup \text{Bd}(T)) - (\text{Int}(Q_1) \cup \text{Int}(Q_2))$. Then N consists of either one or two new 2-manifolds and $\text{ch}(N) = \text{ch}(P_k)$.

Case 1. Suppose N is connected. Then N satisfies condition (C_k) and $g(N) = g(P_k) \leq g(S, U)$. The only important difference between N and P_k is that $N \cap A_{ik}$ has two components fewer than $P_k \cap A_{ik}$.

Case 2. Suppose N consists of two disjoint 2-manifolds, N_1 and N_2 . From the proof given in Step 2, we see that we can assume that N_1 separates G_k from S in $\text{Cl}(U)$. Clearly N_1 satisfies condition (C_k) .

Subcase i. Suppose $g(N_2) = 0$. We know there is an annulus A on N_2 such that $N_2 \cap \text{Bd}(T) = A$ and $N_2 - A$ is a subset of P_k . Then there exists a disc H on N_2 such that H is a subset of P_k and $\text{Bd}(H)$ is one of the components of $\text{Bd}(A)$. We can find an annulus Q^* such that Q^* lies on either Q_1 or Q_2 and the boundary components of Q^* are $\text{Bd}(H)$ and one of the simple closed curves F_1, F_2 . Then $H \cup Q^*$ is a disc on P_k the boundary of which is an essential simple closed curve on A_{ik} .

Subcase ii. Suppose N_2 is nonorientable and $g(N_2) = 1$. Taking A as defined in Subcase i, a few simple genus calculations show that A must separate N_2 and that one of the components of $N_2 - \text{Int}(A)$ must be a disc H . We then go on to the same conclusion as in Subcase i.

Subcase iii. Now consider any possibility other than the two subcases discussed above. Then $\text{ch}(N_2) \leq 0$ and hence $g(N_1) \leq g(P_k)$, so that we obtain a situation like that of Case 1.

Now let F be a polyhedral surface in U satisfying condition (C_k) and consider the following statements:

$F - i$: $F \cap A_{ik}$ consists of a single simple closed curve which is essential in A_{ik} , and $g(F) \leq g(S, U)$.

$F * i$: Every component of $F \cap A_{ik}$ is essential in A_{ik} , and one of these components bounds a disc on F .

It follows from the above considerations that for $k=1, 2, \dots$ we may replace P_k with a surface S_k satisfying condition (C_k) such that either $S_k - i$ is true for each i or $S_k * i$ is true for some i .

Step 4. We shall now show that for all but a finite number of k the statement $S_k - i$ is true for $i=1, \dots, m$.

Suppose not. Then for an infinite number of k we know that $S_k * i$ is true for

some i . We may as well suppose $S_k * 1$ is true for an infinite number of k . If $S_k * 1$ holds, then there is a disc D' on S_k such that $\text{Bd } (D')$ is one of the components of $S_k \cap A_{1k}$. There exists a subdisc D of D' such that $\text{Bd } (D)$ is a component of $S_k \cap A_{ik}$ for some i (say $i=1$) and A_{jk} does not intersect $\text{Int } (D)$ for any j . By "walking" up the side of A_{1k} , we can construct an arc p_1a such that

- $p_1a - p_1 \subset (U - A_{1k}) \cap N(S, \sigma_k)$,
- p_1a intersects S_k a finite number of times,
- a is a point in G_k , and
- $\text{diam } (p_1a) < \delta_k$.

Case 1. Suppose $m \geq 2$. Then there are arcs ac , cd , and dp_1 such that ac is in G_k and does not intersect S_k , d is a point of S , $cd \subset A_{2k}$, and $\text{Int } (dp_1) \subset V$.

Let J be the union of p_1a , ac , cd , and dp_1 (we may assume J to be a simple closed curve). Now let A^* be the subannulus of A'_{1k} bounded by $\text{Bd } (D_{1k})$ and $\text{Bd } (D)$, and consider the 2-sphere S' obtained by taking the union of D_{1k} , A^* , and D . We can homotopy J into S (by virtue of our choice of W) and then into $S - D_{1k}$. Hence the mod 2 intersection number of J with S' is zero. Since J pierces S' at p_1 , there must be another point at which J intersects S' . From our construction it follows that there is a point b in the interior of p_1a such that b is a point of D . By letting p_1b (the obvious subarc of p_1a) take the place of pa in the proof of Lemma 3, we can show as in that lemma that $\text{diam } (D) < 7\delta_k < 1/2^k$. Now let D_k^* be the disc $D \cup A^*$. Since $\text{diam } (D_k^*) < 1/2^{k-1}$ and D_k^* must exist for an infinite number of k , it follows that S can be locally spanned from U at p_1 and hence that S is locally tame from U at p_1 [4, Theorem 10]. But this contradicts the hypotheses of our theorem.

Case 2. Suppose S is not a 2-sphere and $m=1$. There is a simple closed curve L on S such that L does not intersect p_1 and L is not nullhomotopic in S . We can find an annulus A satisfying the requirements that L is one of its boundary components, $A-L$ is contained in U , and $A-L$ is locally polyhedral and in general position with respect to D . We may assume that we are considering k so large that A_{1k} and D_{1k} do not intersect A and $\text{Bd } (A) - L$ is contained in G_k . We can construct an arc ac so that ac does not intersect S_k and c is a point of A . Now suppose $D \cap A$ contains a simple closed curve L' which is essential in A . Let H = the disc on D bounded by L' and B = the subannulus of A bounded by L and L' . Using Lemma 2 and our choice of W , we can shove the singular disc $H \cup B$ into S and then use it to show that L is nullhomotopic on S . Since this contradicts our choice of L , we see that every component of $D \cap A$ must bound a disc in A . Hence we can construct an arc cd in A such that d is a point in S and cd does not intersect D , A_{1k} , or D_{1k} . The rest of the proof is a copy of Case 1.

We therefore conclude that $S_k * i$ can be true for at most a finite number of k and i .

Step 5. For this last step, we shall consider only values of k for which $S_k - i$ is true for $i=1, \dots, m$.

We see that $g(S_k) \leq g(S, U)$ and that since each S_k separates S from G_k in $\text{Cl}(U)$ we have $\lim S_k = S$. For sufficiently large k , we see from Lemma 3 that there are pairwise disjoint discs with handles C_{1k}, \dots, C_{mk} , components of $S_k - \text{Int}(A_{1k}), \dots, S_k - \text{Int}(A_{mk})$ respectively, such that $\text{diam}(C_{ik}) < 7\delta_k < 1/2^k$. Let A_{ik}^* be the subannulus of A_{ik} between $\text{Bd}(C_{ik})$ and $\text{Bd}(D_{ik})$. Since $C_{ik} \cup A_{ik}^*$ is a disc with $g(C_{ik})$ handles which is contained in U except for its boundary, is "cupped" over p_i , and is of diameter less than $1/2^{k-1}$, we then see that we may assume we are working with a value of k for which $g(C_{ik})$ is greater than or equal to $g(p_i, U)$. Let $C_{0k} = \text{Cl}(S_k - \bigcup_{i=1}^m C_{ik})$.

We consider first the case in which S is orientable. We have

$$g(S, U) \geq g(S_k) = \sum_{i=0}^m g(C_{ik}) \geq g(C_{0k}) + \sum_{i=1}^m g(p_i, U).$$

It is now necessary only to show that $g(C_{0k})$ is greater than or equal to $g(S)$. This is trivial if $g(S) = 0$, so suppose $g(S) \neq 0$. There exist in S pairwise disjoint discs with one handle H_1, \dots, H_N (where $N = g(S)$) containing none of the points p_1, \dots, p_m . Let H be the union of H_1, \dots, H_N . Since S is locally tame from U at every point of H , there is a homeomorphism h from $H \times I$ into $\text{Cl}(U)$ such that $h(x, 0) = x$ and $h(x, t)$ is a point in U for $t > 0$ and $x \in H$. We may assume k chosen so large that $h(H \times I)$ intersects S_k only in $\text{Int}(C_{0k})$ and S_k separates H and $h(H \times I)$ in $\text{Cl}(U)$. Furthermore we may suppose by [1] that $h(H \times (0, 1))$ is locally polyhedral and thus may be assumed to be in general position with respect to C_{0k} . It follows then from Lemma 2 of [9] that for $i = 1, \dots, N$ we can find a handle on some component of $C_{0k} \cap h(H_i \times I)$. These are clearly distinct handles for distinct values of i , and thus $g(C_{0k}) \geq g(S)$.

The case in which S is nonorientable is treated in the same way except that H_1, \dots, H_N are Moebius strips and one uses Lemma 4 of [9]. The 2 in front of $\sum_{i=1}^m g(p_i, U)$ arises from the fact that $g(p_i, U)$ is obtained by counting handles on a disc while $g(S, U)$ is found by computing the genera of nonorientable surfaces approaching S . This finishes the proof of the theorem.

4. An application. Daverman in [8] defines a concept, the local enveloping genus of a crumpled cube at a point, which we shall denote $g'(p, U)$ and generalize to 2-manifolds in 3-manifolds in the following way:

DEFINITION. Let S be a 2-sided surface in a 3-manifold M^3 such that S has U for one of its two sides. Let W' be the union of U , S , and $S \times I$ where every x in S is identified with $(x, 0)$. We know from [11] that W' is a 3-manifold at each point of S . Choose a point p of S and let A be the arc $p \times I$. We then define $g'(p, U)$ to be the local enveloping genus of A at p . Now we need to define local enveloping genus of A at p .

DEFINITION. Let A be an arc in E^3 with endpoints p and q . Consider the statement $H(p, n)$: For every positive real number ε there is a sphere with n handles K such that K separates p and q , the intersection of K and A is a single point in $\text{Int}(A)$,

and K is contained in $N(p, \epsilon)$. If $H(p, n)$ is true and $H(p, n-1)$ is not, then we define the *local enveloping genus of A at p* , $\text{LEG}(A, p)$, to be n .

Not surprisingly, this turns out to be the same as our $g(p, U)$, and we sketch here a proof of the equivalence of the concepts.

LEMMA 5. *Let S be a 2-sided surface in a 3-manifold M^3 such that S has U for one of its sides. Let p be a point in S such that S is locally tame from U at every point except possibly p . Then $g(p, U) = g'(p, U)$.*

Proof. Let W' be the union of U , S , and $S \times I$ where all the points x in S are identified with $(x, 0)$. Let A be the arc $p \times I$. Then $g'(p, U) = \text{LEG}(A, p)$, the local enveloping genus being calculated in the 3-manifold W' .

Let H be a disc with $g(p, U)$ handles which is "cupped" over p in the manner described in the definition of $g(p, U)$. We can attach a disc D to H in such a way that $\text{Int}(D)$ is contained in $S \times (0, 1)$, the intersection of A and D consists of a single point, A pierces D at that point, and $\text{Bd}(D) = \text{Bd}(H)$. Since the sphere with $g(p, U)$ handles which is formed by taking the union of H and D can be constructed in an arbitrarily small neighborhood of p , we conclude that $g(p, U)$ is greater than or equal to $g'(p, U)$.

Now we have to prove that $g(p, U)$ is less than or equal to $g'(p, U)$. We see that we can find a sphere with handles Q , a simple closed curve C , and an annulus B such that A intersects Q in a single point r and pierces it there, Q can be assumed to lie in some arbitrarily small neighborhood of p which has been chosen beforehand, the genus of $Q = g'(p, U)$, $B = C \times I \subset S \times I$, $C \times 0$ bounds a disc D' on S such that p lies in $\text{Int}(D')$, $C \times \frac{1}{2}$ bounds a disc D'' on Q such that r lies in $\text{Int}(D'')$, and B intersects Q in precisely the set $C \times \frac{1}{2}$.

Let Q' be the disc with $g'(p, U)$ handles and boundary $C \times 0$ which is obtained by taking the union of $Q - D''$ and $C \times [0, \frac{1}{2}]$. Since $S - \text{Int}(D')$ can be collared from U by [3] and that part of Q' which intersects $S \times I$ lies in $(S - \text{Int}(D')) \times I$, it is not very difficult to construct a homeomorphism h on Q' such that h is the identity on $\text{Bd}(Q')$ and $h(\text{Int}(Q'))$ is contained in U . Because $h(Q')$ corresponds to a disc with handles as described in the definition of $g(p, U)$ and $h(Q')$ can be constructed in an arbitrarily small neighborhood of p , it follows that $g'(p, U)$ is greater than or equal to $g(p, U)$.

The concept of $g'(p, U)$ has the advantage that it can be applied to any 2-manifold embedded in a 3-manifold. Abandoning the temporary notation $g'(p, U)$ in favor of the symbol $g(p, U)$, we give the definition of nonpiercing point and generalize Theorem 5 of [8].

DEFINITION. Let S be a 2-sided surface in a 3-manifold M^3 such that S has U for one of its sides, and let p be a point of S . We say that p is a *piercing point of S with respect to U* if and only if there is a neighborhood N of p in $\text{Cl}(U)$ and a homeomorphism h from N into E^3 such that $h(S \cap N)$ can be pierced at p by a tame arc.

THEOREM 2. *Let S be a 2-sided surface in a 3-manifold M^3 such that S has U for one of its sides. Then the number of points of S which are nonpiercing points with respect to U is less than or equal to*

$$\begin{aligned} 1 & \quad \text{if } g(S) = g(S, U) = 0, \\ g(S, U) - g(S) & \quad \text{if } S \text{ is orientable and either } g(S, U) \neq 0 \text{ or } g(S) \neq 0, \text{ and} \\ \frac{1}{2}(g(S, U) - g(S)) & \quad \text{if } S \text{ is nonorientable.} \end{aligned}$$

Proof. The proof is the same as that of Theorem 5 of [8] except that where Daverman uses [7] and [13], we use Theorem 1 and Lemma 5 of this paper.

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