TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 167, May 1972

HYPERBOLIC LIMIT SETS(1)

BY
SHELDON E. NEWHOUSE

Abstract. Many known results for diffeomorphisms satisfying Axiom A are shown to be true with weaker assumptions. It is proved that if the negative limit set $L^-(f)$ of a diffeomorphism f is hyperbolic, then the periodic points of f are dense in $L^-(f)$. A spectral decomposition theorem and a filtration theorem for such diffeomorphisms are obtained and used to prove that if $L^-(f)$ is hyperbolic and has no cycles, then f satisfies Axiom A, and hence is Ω-stable. Examples are given where $L^-(f)$ is hyperbolic, there are cycles, and f fails to satisfy Axiom A.

1. In [10] and [11], Smale obtained results for a diffeomorphism f of a compact manifold M satisfying Axiom A. Axiom A requires (a) the nonwandering set $\Omega=\Omega(f)$ has a hyperbolic structure, and (b) the periodic points of f are dense in $\Omega(f)$. The purpose of this paper is to point out that many of Smale's results may be obtained under weaker hypotheses. One consequence of our observations is that most of the known results for diffeomorphisms satisfying Axiom A are true for those satisfying Axiom A(a) alone.

Our main result is the following. Let $L^-(f)$ be the closure of the set of α-limit points of f. Then,

THEOREM 4.5. *If $L^-(f)$ is hyperbolic and f has no cycles, then f satisfies Axiom* A (*and has no cycles*).

This gives strengthening of Smale's Ω-stability theorem in two directions. On the one hand, it is not necessary to assume Axiom A(b), and on the other hand, it is not necessary to assume the whole nonwandering set is hyperbolic.

For the theorem to hold, a natural change in the usual definition of cycle is needed (see the definitions preceding (3.9) and Examples 1 and 4 at the end of §3). Our definition reduces to the usual one when f satisfies Axiom A(a).

The basic idea of the proof of Theorem (4.5) is as follows. First we prove that $L^-(f)$ hyperbolic implies that the periodic points of f are dense in $L^-(f)$ and that there is a spectral decomposition theorem. Then, using methods similar to the proof of Smale's Ω-stability theorem, we obtain a filtration theorem for f. In the case of no cycles, this filtration separates the pieces in the spectral decomposition of $L^-(f)$ from which it follows that $L^-(f)$ is the whole nonwandering set.

Received by the editors June 23, 1970 and, in revised form, April 7, 1971.

AMS 1970 *subject classifications*. Primary 58F15, 58F20.

Key words and phrases. Limit set, hyperbolic, periodic point, topologically transitive, filtration, basic set, stable manifold, nonwandering.

(1) Research supported by NSF grant #GP-19815.

Another sufficient condition for Ω-stability is contained in Theorem (4.7).

Although we deal here only with diffeomorphisms, the corresponding results for flows may be obtained by combining slight modifications of the methods used here with the techniques of [7].

At the beginning of §2 we collect several notations and definitions which will be used throughout the paper. Then we prove a spectral decomposition theorem for diffeomorphisms such that the closure of the periodic points is hyperbolic.

In §3 we obtain results when $L^-(f)$ is hyperbolic. In particular, we prove that $L^-(f)$ hyperbolic implies the periodic points of f are dense in $L^-(f)$. We also obtain a filtration theorem and present some examples which show that our results are true generalizations of Smale's results.

The main results of §4 are Theorems (4.5) and (4.7).

I wish to thank J. Palis and R. C. Robinson for some helpful comments and suggestions.

2. Here we obtain some results about periodic points. But we first establish some notation which will be used throughout the paper.

Throughout it is assumed that f is a C^r diffeomorphism, $0<r<\infty$, of a compact C^∞ manifold without boundary. Let $P=P(f)$ be the set of hyperbolic periodic points of f and assume $P\neq\varnothing$. Let $\Omega=\Omega(f)$ denote the nonwandering set of f. For a subset $D\subset M$, $\bar{D}$ or $\mathrm{Cl}\,(D)$ will denote its closure in M, and int D will denote its interior in M.

For $x\in M$, define $\alpha(x)=\alpha(x,f)=\{y\in M$: there is a sequence of integers $n_i\to\infty$ such that $f^{-n_i}(x)\to y$ as $i\to\infty\}$. Let $\omega(x)=\omega(x,f)=\alpha(x,f^{-1})$, $\mathrm{o}(x)=\{f^n(x):-\infty<n<\infty\}$, $L_\alpha=L_\alpha(f)=\{x\in M:\exists y\in M$ such that $x\in\alpha(y)\}$, and $L_\omega=L_\omega(f)=L_\alpha(f^{-1})$. Also, set $L^-=\bar{L}_\alpha$, $L^+=\bar{L}_\omega$, and $L=L^-\cup L^+$. L_α, L^-, L_ω, L^+, and L are called, respectively, the *α-limit set* of f, *negative limit set* of f, *ω-limit set* of f, *positive limit set* of f, *and limit set* of f. $\mathrm{o}(x)$ is called the *orbit* of x.

A compact f invariant set Λ is *hyperbolic* if there are a continuous splitting of the tangent bundle $T_\Lambda M=E^s\oplus E^u$ preserved by the derivative Tf of f, a riemannian metric $|\cdot|$ on M, and a constant $0<\lambda<1$ such that $|Tf(v)|\leqq\lambda|v|$ for $v\in E^s$ and $|Tf(v)|\geqq\lambda^{-1}|v|$ for $v\in E^u$. A metric $|\cdot|$ such as that referred to in the preceding sentence is said to be adapted to Λ.

Let Λ be a hyperbolic set, and $|\cdot|$ be an adapted metric. Let d be the topological metric on M induced by $|\cdot|$. For $x\in\Lambda$, $\varepsilon>0$, let

$$\begin{aligned}
W^u_\varepsilon(x) &= \{y\in M : d(f^{-n}(x),f^{-n}(y)) \leqq \varepsilon \text{ for } n\geqq 0\},\\
W^u(x) &= \{y\in M : d(f^{-n}(x),f^{-n}(y)) \to 0 \text{ as } n\to\infty\},\\
W^s_\varepsilon(x) &= \{y\in M : d(f^{n}(x),f^{n}(y)) \leqq \varepsilon \text{ for } n\geqq 0\},\\
W^s(x) &= \{y\in M : d(f^{n}(x),f^{n}(y)) \to 0 \text{ as } n\to\infty\}.
\end{aligned}$$

We denote that, for $x\in\Lambda$, $\sigma=u,s$, $W^\sigma_\varepsilon(x)\subset W^\sigma(x)$ and $W^\sigma(x)$ is a smooth injectively immersed copy of a Euclidean space. Further, $W^\sigma(x)$ is tangent to E^σ_x at x

(see [2]). For a subset $D \subset \Lambda$, let $W^u_\varepsilon(D) = \bigcup_{x\in D} W^u_\varepsilon(x)$, and make similar definitions for $W^u(d)$, $W^s_\varepsilon(D)$, and $W^s(D)$.

We now proceed to establish some facts about the set P of hyperbolic periodic points of the diffeomorphism f. Recall we have assumed $P \neq \varnothing$.

Let $p_1, p_2 \in P$. We will say that p_1 is *homoclinically related* to p_2 or *h-related* to p_2, denoted $p_1 \sim p_2$, if $W^u(\mathrm{o}(p_1))$ has a point of transversal intersection with $W^s(\mathrm{o}(p_2))$ and $W^u(\mathrm{o}(p_2))$ has a point of transversal intersection with $W^s(\mathrm{o}(p_1))$.

(2.1) PROPOSITION. *The relation $\sim$ on P is an equivalence relation.*

Proof. Reflexivity and symmetry are obvious. For transitivity, suppose $p_1 \sim p_2$ and $p_2 \sim p_3$. Let x_2 be a point of transversal intersection of $W^u(\mathrm{o}(p_2))$ and $W^s(\mathrm{o}(p_3))$, $p_i \in P$, $i = 1, 2, 3$. Since $W^u(\mathrm{o}(p_2))$ is a finite union of injectively immersed cells, there are a point $p_2' \in \mathrm{o}(p_2)$ and a closed disk $D^u_2 \subset W^u(\mathrm{o}(p_2))$ such that p_2' and x_2 are in D^u_2. Since $W^u(\mathrm{o}(p_1))$ has a point of transversal intersection with $W^s(\mathrm{o}(p_2))$, by the Palis λ-lemma [4], $W^u(\mathrm{o}(p_1))$ contains disks D^u_1 arbitrarily C^1 close to D^u_2. Thus we can choose such a D^u_1 which will have a point of transversal intersection with $W^s(\mathrm{o}(p_3))$ near x_2. So $W^u(\mathrm{o}(p_1))$ has a point of transversal intersection with $W^s(\mathrm{o}(p_3))$. Similarly, $W^u(\mathrm{o}(p_3))$ has a point of transversal intersection with $W^s(\mathrm{o}(p_1))$ and so $p_1 \sim p_3$.

We will call the equivalence classes of P, *h-classes*. For $p \in P$, the h-class of p will be denoted by P_p.

The following lemma is essentially due to Birkhoff [1, p. 205].

(2.2) LEMMA. (1) *Let X be a second countable, complete metric space. Let $h: X \to X$ be a continuous map. If for every nonempty open set V in X, $\bigcup_{n\geqq 0} h^n(V)$ is dense in X, then there is an $x_0 \in X$ such that $\omega(x_0) = \omega(x_0, h) = X$.*

(2) *If h is a homeomorphism, and for every nonempty open set V, $\bigcup_{n\geqq 0} h^n(V)$ and $\bigcup_{n\leqq 0} h^n(V)$ are dense, then there is an $x_0 \in X$ such that $\alpha(x_0) = \omega(x_0) = X$.*

Proof. Since for every open V, $\bigcup_{n\geqq 0} h^n(V)$ is dense, we have that for every open V, $\bigcup_{n\leqq 0} h^n(V)$ is a dense open set in X. If $\{V_i\}_{i\in I}$ is a countable basis for the topology of X, then $\bigcap_{i\in I} (\bigcup_{n\leqq 0} h^n(V_i))$ is dense in X by the Baire Category Theorem.

If $x \in \bigcap_{i\in I} (\bigcup_{n\leqq 0} h^n(V_i))$, then $\{h^n(x) : n \geqq 0\}$ is dense in X, so $\omega(x) = X$. (2.2.2) is proved similarly by choosing

$$x \in \bigcap_{i\in I} \left(\bigcup_{n\geqq 0} h^n(V_i) \right) \cap \bigcap_{i\in I} \left(\bigcup_{n\geqq 0} h^n(V_i) \right).$$

If $p \in P$, then a point q of transversal intersection of $W^u(\mathrm{o}(p))$ and $W^s(\mathrm{o}(p))$ such that $q \notin \mathrm{o}(p)$ is called a *transversal homoclinic point* of p. Let H_p be the set of all such points and suppose $H_p \neq \varnothing$.

(2.3) LEMMA. *$\overline{H}_p$ is a closed, f-invariant set such that there is an $x \in \overline{H}_p$ such that $\alpha(x) = \omega(x) = \overline{H}_p$.*

Proof. That $\overline{H}_p$ is closed and f-invariant is obvious. For the last statement we wish to apply Lemma (2.2.2).

To this end, let V_1, V_2 be any nonempty open sets in $\overline{H}_p$. Thus there are $q_i \in H_p$ and open sets U_i in M such that $q_i \in V_i = U_i \cap \overline{H}_p$, $i=1, 2$. Let D_1^u be a disk in $W^u(\mathrm{o}(p))$ of the same dimension as $W^u(\mathrm{o}(p))$ which contains a point $p_1 \in \mathrm{o}(p)$ and q_1 in its interior in $W^u(\mathrm{o}(p))$.

Let D_2^u be a small disk in $W^u(q_2) \cap U_2$ of the same dimension as $W^u(q_2)$ which contains q_2 in its interior in $W^u(q_2)$.

Since D_2^u meets $W^s(q_2) \subset W^s(\mathrm{o}(p))$ transversely, the λ-lemma says that $\bigcup_{n \geqq 0} f^n(D_2^u)$ contains disks arbitrarily C^1 close to D_1^u. Thus $\bigcup_{n \geqq 0} f^n(D_2^u)$ contains points in H_p arbitrarily close to q_1. Thus, $q_1 \in \bigcup_{n \geqq 0} \mathrm{Cl}\,(f^n(D_2^u) \cap H_p)$ $\subset \bigcup_{n \geqq 0} \mathrm{Cl}\,(f^n(U_2) \cap H_p) = \bigcup_{n \geqq 0} \mathrm{Cl}\,(f^n(U_2 \cap H_p)) \subset \bigcup_{n \geqq 0} \mathrm{Cl}\,(f^n(U_2 \cap \overline{H}_p)) =$ $= \bigcup_{n \geqq 0} \mathrm{Cl}\,(f^n(V_2))$. So

$$V_1 \cap \left(\bigcup_{n \geqq 0} f^n(V_2) \right) \neq \varnothing .$$

Similarly, $V_1 \cap (\bigcup_{n \leqq 0} f^n(V_2)) \neq \varnothing$. Since V_1 and V_2 were arbitrary we may apply (2.2.2) to give (2.3).

(2.4) Theorem. *Let p be a hyperbolic periodic point whose h-class P_p contains more than one orbit, i.e. there is a point $p_1 \in P_p$ such that $p_1 \notin \mathrm{o}(p)$. Then $\overline{H}_p = \overline{P}_p$. In particular, $H_p \neq \varnothing$.*

(2.5) Corollary. *If P_p is the h-class of p, then $\overline{P}_p$ is a closed, f-invariant set such that there is an $x \in \overline{P}_p$ such that $\alpha(x) = \omega(x) = \overline{P}_p$.*

Proof of (2.4). The proof that $P_p \subset \overline{H}_p$ is very similar to the proofs of (2.1) and (2.3). Let $p_1 \in P_p$, $p_1 \notin \mathrm{o}(p)$. Let x be a point of transversal intersection of $W^u(\mathrm{o}(p))$ and $W^s(\mathrm{o}(p_1))$ and let x_1 be a point of transversal intersection of $W^u(\mathrm{o}(p_1))$ and $W^s(\mathrm{o}(p))$. Let D_{x_1} be a disk in $W^u(\mathrm{o}(p_1))$ containing x_1 in its interior in $W^u(\mathrm{o}(p_1))$. By the λ-lemma, there are disks in $W^u(\mathrm{o}(p))$ which are arbitrarily C^1 close to D_{x_1}. Thus $x_1 \in \overline{H}_p$. Hence $p_1 \in \overline{H}_p$, so $P_p \subset \overline{H}_p$.

The fact that $H_p \subset \overline{P}_p$ is a consequence of the following version of Smale's theorem on transversal homoclinic points.

(2.6) Theorem (Smale [9]). *Let p be a hyperbolic periodic point of the diffeomorphism f, and let q be a transversal homoclinic point of p. Then in every neighborhood of q there are infinitely many periodic points which are h-related to p.*

In [9], Smale made use of Sternberg's linearization theorem, and this required eigenvalue assumptions other than hyperbolicity and additional smoothness assumptions. However, he expressed the feeling that Sternberg's theorem was probably not needed for his result. We wish to point out that a proof very close to Smale's original one and avoiding Sternberg's theorem can be given using the tubular family theorems of [6][2]. Taking tubular families for $W^u(\mathrm{o}(p))$ and $W^s(\mathrm{o}(p))$, one can get continuous coordinates on a neighborhood of $\mathrm{o}(p)$ on which there is a continuous splitting of the tangent bundle $T_U M = E^s \oplus E^u$ such that, for $x \in U \cap f^{-1}(U)$,

$$T_x f = \begin{pmatrix} A_x & 0 \\ 0 & D_x \end{pmatrix}$$

with respect to the splitting $E^s \oplus E^u$. Also this can be done so that $\|A_x\| < 1$ and $\|D_x^{-1}\| < 1$ on $f^{-1}(U) \cap U$. Now one can proceed as Smale did in [9]. However, for the reader's convenience we will give a different and more elementary proof of (2.6) in the appendix at the end of the paper.

We will need the following corollary to the proof of (2.4).

(2.7) COROLLARY. *Let P_1 be an h-class. Suppose $p_1, p_2 \in P_1$ and y is a point of transversal intersection of $W^u(p_1)$ and $W^s(p_2)$. Then $y \in \bar{P}_1$.*

Proof. By the first part of the proof of Theorem (2.4), $y \in \bar{H}_{p_1}$. By the second part, $\bar{H}_{p_1} = \bar{P}_1$, so (2.7) is proved.

(2.8) PROPOSITION. *Suppose $\bar{P}$ is a hyperbolic set. Then there are only a finite number of h-classes of P and their closures are pairwise disjoint.*

Proof. If there were infinitely many h-classes, $\{P_i\}$, let $p_1, p_2, \ldots$ be a sequence of points such that $p_i \in P_i$ and $P_i \neq P_j$ for $i \neq j$, $i, j \geqq 1$.

We may assume, by taking a subsequence if necessary that $\dim W^s(p_i) = \dim W^s(p_j)$ for all i, j. Let x be a limit point of $\{p_i\}$. Then by continuous dependence of the stable and unstable manifolds on $\bar{P}$ (see [2]), if p_i and p_j are close to x, then $p_i \sim p_j$. Similarly, if $p_i \nsim p_j$, then $\bar{P}_{p_i} \cap \bar{P}_{p_j} = \varnothing$ where P_{p_i} is the h-class of p_i and P_{p_j} is the h-class of p_j.

The next theorem is the analog of Smale's spectral decomposition theorem [10].

(2.9) THEOREM. *If $\bar{P}$ is hyperbolic, then $\bar{P} = \Lambda_1 \cup \cdots \cup \Lambda_n$ where the Λ_i are the closures of the distinct h-classes. Thus each Λ_i is a closed, invariant, topologically transitive set with periodic points dense. Further, each Λ_i has a local product structure, i.e. for $\varepsilon > 0$ small, $W^u_\varepsilon(\Lambda_i) \cap W^s_\varepsilon(\Lambda_i) \subset \Lambda_i$ (see* [3]).

Proof. All we need prove is the local product structure statement. Thus we need to show if $x, y \in \Lambda_i$ and ε is small then

$$W^u_\varepsilon(x) \cap W^s_\varepsilon(y) \subset \Lambda_i.$$

Choose ε such that $W^u_{2\varepsilon}(x)$ is transverse to $W^s_{2\varepsilon}(y)$ and $W^u_{2\varepsilon}(x) \cap W^s_{2\varepsilon}(y)$ is at most one point. Then if $z \in W^u_\varepsilon(x) \cap W^s_\varepsilon(y)$ and V is a neighborhood of z, there are periodic points $x_1, y_1 \in \Lambda_i$ such that $W^u_{2\varepsilon}(x_1)$ has a point z_1 of transversal intersection with $W^s_{2\varepsilon}(y_1)$ in V. Then $z_1 \in \bar{P}_{x_1} = \Lambda_i$ by Corollary (2.7).

---

[2] It seems that another proof of (2.6) without Sternberg's theorem appears in [14].

We now proceed to state and prove a technical lemma which will be needed for the proof of Theorem (3.1).

We need some notation. By a disk D' we mean a closed ball in some Euclidean space with the usual metric. If D' is a disk, we let $r(D')$ denote its radius and, for a real number $c>0$, we let cD' denote the disk whose center is the same as that of D' and whose radius is $cr(D')$. Let $0<s, u$ be integers and let $D=D^s\times D^u\subset R^{s+u}$ where D^σ is a disk in the Euclidean space R^σ for $\sigma=s, u$. Assume $r(D^s)=r(D^u)$. Let $D_1\subset D$ and let $g\colon D_1\to R^{s+u}$ be a smooth injection. For $z\in D_1$, suppose $T_zg\colon R^s\times R^u\to R^s\times R^u$ is given by

$$T_zg = \begin{pmatrix} A_z & B_z \\ C_z & D_z \end{pmatrix}$$

where $A_z\colon R^s\to R^s$, $B_z\colon R^u\to R^s$, $C_z\colon R^s\to R^u$, and $D_z\colon R^u\to R^u$. Let

$$\begin{aligned} a &= \sup_{z\in D_1} \|A_z\|, \\ c &= \sup_{z\in D_1} \|C_z,\| \\ e &= \sup\{|B_zv|/|D_zv| : z\in D_1,\ v \text{ is a unit vector in } R^u\}, \\ d &= \inf\{|D_zv| : z\in D_1,\ v \text{ is a unit vector in } R^u\}. \end{aligned}$$

Here the norms $\|\cdot\|$ are the usual matrix norms, and e is assumed to be finite.

(2.10) Lemma. *Using the above notation, suppose there is a subdisk $D_1^u\subset D^u$ centered at $y_0\in\frac{1}{4}D^u$ such that if $D_1=D^s\times D_1^u$ then $g\colon D_1\to R^{s+u}$ is a smooth injection such that*

(1) $g(D_1)\subset D$,

(2) $g(D^s\times\{y_0\})\subset D^s\times\frac{1}{4}D^u$,

(3) $a<1$ *and* $d(1-ce(1-a)^{-1})r(D_1^u)>\frac{1}{2}r(D^u)+r(D_1^u)$.

Then g has a unique fixed point in D_1.

Proof. For $z\in D$, let $z=(x, y)$ with $x\in D^s$, $y\in D^u$, and let $\pi^s\colon(x, y)\mapsto x$, $\pi^u\colon(x, y)\mapsto y$ denote the natural projections on D.

For each $y\in D_1^u$, the map $\varphi_1\colon x\mapsto\pi^s g(x, y)$ takes D^s into D^s by (1). Further, $\|T_x\varphi_1\|\leqq a<1$ for all $x\in D^s$. Thus φ_1 is a contraction and, hence for each $y\in D_1^u$, there is a unique $x(y)$ such that $\varphi_1(x(y))=\pi^s g(x(y), y)=x(y)$.

If ψ is the mapping $(x, y)\mapsto\pi^s g(x, y)-x$, then since $a<1$, the partial derivative $\partial\psi/\partial x$ has rank s on D_1, so the implicit function theorem gives that the mapping $y\mapsto x(y)$ is smooth.

Consider the mapping $\varphi_2\colon y\mapsto\pi^u g(x(y), y)$ on D_1^u. We claim $\varphi_2(D_1^u)\supset D_1^u$ and φ_2 is a uniform expansion on D_1^u. Once this is shown, it follows that φ_2 has a unique fixed point y_1 and hence $(x(y_1), y_1)$ is the unique fixed point of g in D_1.

So we first show φ_2 is an expansion on D_1^u. Let v be a unit vector in R^u. Then

$$|T_y\varphi_2(v)| = |Cx'(y)v+Dv|$$

where $T_{(x(y),y)}g=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ and $x'(y)$ is the derivative of $y \mapsto x(y)$ at y. Further, $x'(y) = -(A-I)^{-1}B$ where I is the $s\times s$ identity matrix. Thus,

$$\begin{aligned}|T_y\varphi_2(v)| &\geqq |Dv|-(c(1-a)^{-1})|Bv| \\ &\geqq |Dv|(1-ce(1-a)^{-1}) \geqq d(1-ce(1-a)^{-1}) > 1\end{aligned}$$

where the last inequality follows from (3). So,

$$|T_y\varphi_2(v)| \geqq d(1-ce(1-a)^{-1}) > 1 \tag{4}$$

which shows φ_2 is an expansion.

It remains to show $\varphi_2(D_1^u)\supset D_1^u$. From (3) and (4) it follows that $\varphi_2(D_1^u)$ contains a disk of radius $\frac{1}{2}r(D^u)+r(D_1^u)$ centered at $\varphi_2(y_0)$. But $y_0 \in \frac{1}{4}D^u$ and (2) implies that $\varphi_2(y_0) \in \frac{1}{4}D^u$. Thus $|\varphi_2(y_0)-y_0|<\frac{1}{2}r(D^u)$, so $\varphi_2(D_1^u)\supset D_1^u$.

Several conversations with R. C. Robinson were helpful in working out the proof of Lemma (2.10).

(2.11) Remark. If D_z^{-1} exists for $z \in D_1$, then $e=\sup_{z\in D_1}\{\|B_zD_z^{-1}\|\}$. In the application of Lemma (2.10) to the proof of Theorem (3.1), one cannot use the version of the lemma in which e is replaced by $e'=\sup_{z\in D_1}\{\|B_z\|\cdot\|D_z^{-1}\|\}$. For, in the proof of (3.1), it is essential to keep the appropriate counterpart of e or e' bounded as one takes larger integers N. In general, the counterpart of e' will not remain bounded, whereas that of e will.

3. In this section we establish some properties of diffeomorphisms with hyperbolic negative limit sets.

A slight change in the proof of our first result will also yield a proof of the so-called Anosov closing lemma which says that if f satisfies Axiom A(a), then $\bar{P}=\Omega(f/\Omega)$[3].

(3.1) Theorem. *If L^- is hyperbolic, then $\bar{P}=L^-$.*

Proof. First note that $L^-=L_1\cup\cdots\cup L_{n_0}$ where L_i is closed invariant, the hyperbolic splitting on L_i has constant dimension, and $L_i\neq L_j$ for $1\leqq i<j\leqq n_0$. Secondly, an argument used by Smale [11, p. 782] applied to f^{-1} shows that, for each $y \in M$, there is an i such that $\alpha(y)\subset L_i$.

We show if $x_0 \in L_i$ and V is any neighborhood of x_0, then there is a periodic point in V. Let $x \in V\cap L_\alpha$. We show there is a periodic point in V near x.

Choose a compact neighborhood U of L_i such that there are semi-invariant disk families $\tilde{W}_\delta^s$, $\tilde{W}_\delta^u$, through U (see [3]). If $\tilde{E}_z^s$ ($\tilde{E}_z^u$) is the tangent space to $\tilde{W}_\delta^s(z)$ ($\tilde{W}_\delta^u(z)$) at z, then $\tilde{E}^s\oplus\tilde{E}^u$ is a continuous splitting of T_UM which is preserved by T_xf for $x \in f^{-1}(U)\cap U$.

Assume U and V are small enough so that

(1) $\tilde{E}^s$ and $\tilde{E}^u$ are defined on $f^{-1}(U)\cap U\cap f(U)$.

(2) $\|Tf|_{\tilde{E}^s}\|<1$ on $U\cap f^{-1}(U)$ and $\|Tf^{-1}|_{\tilde{E}^u}\|<1$ on $U\cap f(U)$.

(3) $V\subset U$ and for $u_1, u_2 \in V$, $\tilde{W}_\delta^u(u_1)\cap\tilde{W}_\delta^s(u_2)$ is a single point.

(3) Another proof of the Anosov closing lemma is in [12].

Let

$$V_1 = \bigcup_{z\in W^s_\delta(x)} \tilde{W}^u_\delta(z), \qquad V_2 = \bigcup_{z\in W^u_\delta(x)} \tilde{W}^s_\delta(z).$$

Then it is proved in [3] that V_1 and V_2 are neighborhoods of x in M. Let $y_0 \in M$ be such that $x \in \alpha(y_0) \subset L_i$. Then there is an integer $n_1 > 0$ such that if $n \geqq n_1$, $f^{-n}(y_0) \in U$. If $\delta > 0$ is small enough, $f^{-n}(\tilde{W}^u_\delta(y_0)) \subset U$ for $n \geqq n_1$ since f^{-1} is a contraction on each $\tilde{W}^u_\delta(z)$.

Let $\exp_x$ denote the exponential map associated to the riemannian metric on M. We claim

(4) there are a disk $D = D^s \times D^u \subset E^s_x \oplus E^u_x$, a subdisk $D_1 \subset D$, integers $N_1 > N_2 > n_1$, and a diffeomorphism $g_1: \exp_x (D) \to D$ such that the map

$$g = g_1 f^{N_1 - N_2} g_1^{-1} | D_1$$

satisfies the hypotheses of (2.10) and $\exp (D) \subset V$.

Once (4) is shown, we can apply (2.10) to get a fixed point z_1 of g in D_1. Then $g_1^{-1}(z_1)$ is a fixed point of $f^{N_1 - N_2}$ in V.

We now prove (4).

For a linear map H from one Euclidean space to another $\|H\|$ denotes its norm, and $m(H)$ denotes its minimum norm which is defined by $m(H) = \inf_{|v|=1} |Hv|$.

Let $\varepsilon > 0$ be small enough such that if E^s_1 and E^u_1 are subspaces of $T_xM = E^s_x \oplus E^x_u$ which are ε-close to E^s_x and E^u_x, respectively, in the induced metric on the Grassmann bundles of M, then the following is true. There is a linear automorphism $H: E^s_x \oplus E^u_x \to E^s_x \oplus E^u_x$ such that $H(E^s_1) = E^s_x$, $H(E^u_1) = E^u_x$, and if

$$H = \begin{pmatrix} I+\sigma_1 & \sigma_2 \\ \sigma_3 & I+\sigma_4 \end{pmatrix} \quad \text{and} \quad H^{-1} = \begin{pmatrix} I+\sigma_5 & \sigma_6 \\ \sigma_7 & I+\sigma_8 \end{pmatrix}$$

with the I's denoting identity operators, then $\|\sigma_i\| < \frac{1}{4}$ for $i = 1, \ldots, 8$ and $m(I+\sigma_i) > \frac{3}{4}$ for $i = 1, 4, 5, 8$.

Choose $D = D^s \times D^u \subset E^s_x \oplus E^u_x$ and $\delta > 0$ small enough such that

(5) $\exp_x (D) \subset U \cap V_1 \cap V_2 \cap V \equiv V_3$,

(6) the manifolds $\{\exp_x (z \times D^u) : z \in D^s\}$ are ε-C^1 close to each other, and the manifolds $\{\exp_x (D^s \times z) : z \in D^u\}$ are ε-C^1 close to each other.

(7) $T_{y_1}\tilde{W}^u_\delta(z_1)$ is ε-close to $T_{y_2}\tilde{W}^u_\delta(z_2)$ for $y_i, z_i \in \exp_x (D)$, $y_i \in \tilde{W}^u_\delta(z_i)$, $i = 1, 2$.

(8) For $z \in \exp_x (D)$, there is a tubular neighborhood retraction $r_z: F_z \to \tilde{W}^u_{\delta/2}(z)$ such that

(a) $F_z \subset \bigcup_{y\in \tilde{W}^u_\delta(z)} \tilde{W}^s_\delta(z) \cap V_3$;

(b) the tangent spaces $T_{z_1}r^{-1}(y_1)$ and $T_{z_2}r^{-1}(y_2)$ are ε-close to those of the $\tilde{W}^s_\delta(z_i)$ for $z_i \in \exp_x (D)$, $y_i \in \tilde{W}^u_\delta(z_i) \cap \exp_x (D)$, $i = 1, 2$;

(c) there is a neighborhood V_4 of x such that $V_4 \subset \operatorname{int} \exp_x (D) \cap V_3$ and such that if $z \in V_4$ then $\operatorname{int} \pi^u \exp_x^{-1} (F_z) \supset D^u$ and $\pi^s \exp_x^{-1} (F_z) \subset \operatorname{int} D^s$;

(d) there is a $\delta_1 > 0$ such that for $y \in \tilde{W}^u_{\delta/4}(z)$ and $z \in \exp_x (D)$, $\tilde{W}^s_{\delta_1}(y) \subset F_z$.

The π^u and π^s in (c) are the natural projections on $E^s_x \oplus E^u_x$.

Let $N>0$. If $z, f^{-N}(z) \in V_4$, let Σ_N be the connected component of $f^{-N}(F_z) \cap \exp_x(D)$ containing $f^{-N}(z)$. If $N>0$ is large enough, $z, f^{-N}(z) \in V_4$, and $y \in \tilde{W}^u_{\delta/2}(z)$, then $f^{-N}(y) \in \exp_x(D)$. In this case, let $\Sigma_{N,y}$ be the connected component of $f^{-N}(r_z^{-1}(y)) \cap \exp_x(D)$ containing $f^{-N}(y)$.

If $\tilde{g}$ is a diffeomorphism of a subset of $\exp_x(D)$ into D, we define its s-submanifolds to be $\{\tilde{g}^{-1}(D^s \times z) : z \in D^u\}$ and its u-submanifolds to be

$$\{\tilde{g}^{-1}(z \times D^u) : z \in D^s\}.$$

To define a diffeomorphism from $\exp_x(D)$ to D it suffices to say what its s-submanifolds and u-submanifolds are. This is what we will do to prove (4).

Since f^{-1} stretches each $\tilde{W}^s_\delta$ and contracts each $\tilde{W}^u_\delta$, there is an integer $N_0 > n_0 > 0$ such that if $N \geqq N_0$, then

(9) if $f^{-n}(z) \in U$ for $0 \leqq n \leqq N$, $z, f^{-N}(z) \in V_4$, and $y \in \tilde{W}^u_{\delta/2}(z)$, then $\pi^s \circ \exp_x^{-1} \circ f^{-N}|_{\Sigma_{N,y}}$ is a diffeomorphism of $\Sigma_{N,y}$ onto D^s.

Assume $N \geqq N_0$ so that (9) holds. Define a diffeomorphism $g_2^N : \Sigma_N \to D$ so that its s-submanifolds are the $\Sigma_{N,y}$ and these submanifolds are ε-C^1 close to each other and to $\exp_x(D^s \times 0)$. Extend g_2^N to $\exp_x(D)$ such that its s-submanifolds are ε-C^1 close to each other and to $\exp_x(D^s \times 0)$.

Now define a diffeomorphism $g_1^N : \exp_x(D) \to D$ such that its s-submanifolds are those of g_2^N and its u-submanifolds are $\{\exp_x(z \times D^u) : z \in D^s\}$. By (6) and the construction of g_1^N, the s-submanifolds of g_1^N are ε-C^1 close to each other and the u-submanifolds of g_1^N are ε-C^1 close to each other.

Now we assert that it is possible to choose $N_1 > N_2 > N_0$ such that $f^{-N_1}(y_0)$, $f^{-N_2}(y_0) \in V_4$ and if $z = f^{-N_2}(y_0)$, $D_1 = g_1^{N_1 - N_2}(\Sigma_{N_1 - N_2})$, and $g_1 = g_1^{N_1 - N_2}$, then g_1 is the diffeomorphism required in (4). That is, $g_1 f^{N_1 - N_2} g_1^{-1}$ satisfies the hypotheses of (2.10).

For $N \geqq N_0$, set

$$T(g_1^N) T f^N T(g_1^N)^{-1} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$$

on $g_1^N(\Sigma_N)$ with respect to the splitting $E^s_x \oplus E^u_x$ on D. To prove the last assertion it suffices to show that, as $N \to \infty$,

$$\|A_0\| \to 0 \tag{10}$$

and

$$m(D_0)\left(1 - \frac{\|C_0\| \sup_{|v|=1} |B_0 v|/|D_0 v|}{1 - \|A_0\|}\right) \to \infty. \tag{11}$$

This will complete the proof of Theorem (3.1).

Let

$$Tf^N = \begin{pmatrix} A_N & 0 \\ 0 & D_N \end{pmatrix}$$

with respect to the splitting $E^s \oplus E^u$ on Σ_N. Then,

$$(12) \qquad \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} = \begin{pmatrix} I+\sigma_1 & \sigma_2 \\ \sigma_3 & I+\sigma_4 \end{pmatrix}\begin{pmatrix} A_N & 0 \\ 0 & D_N \end{pmatrix}\begin{pmatrix} I+\sigma_5 & \sigma_6 \\ \sigma_7 & I+\sigma_8 \end{pmatrix},$$

where $\|\sigma_i\| < \frac{1}{4}$, $i=1,\ldots,8$, and $m(I+\sigma_i) > \frac{3}{4}$, $i=1,4,5,8$. Then if $(0, v) \in \{0\} \times E^u_x$ and $|v| = 1$,

$$\begin{pmatrix} B_0 & v \\ D_0 & v \end{pmatrix} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix},$$

so

$$\frac{|B_0 v|}{|D_0 v|} \leqq \frac{\frac{5}{16}\|A_N\|}{\frac{9}{16}m(D_N) - \frac{1}{16}\|A_N\|} + \frac{\|\sigma_2\|\ |v_1|}{m(I+\sigma_4)|v_1| - \frac{1}{16}\|A_N\|}$$

where $v_1 = D_N(I+\sigma_8)v$. As $N \to \infty$, $\|A_N\| \to 0$ and $m(D_N) \to \infty$, so $|v_1| \to \infty$. Since $\|\sigma_2\| < \frac{1}{4}$ and $m(I+\sigma_4) > \frac{3}{4}$, for N large, $|B_0 v|/|D_0 v| < \frac{1}{3}$. Thus for N large, $\sup_{|v|=1} |B_0 v|/|D_0 v| \leqq \frac{1}{3}$. Further, using the expression in (12), it is easy to see that $m(D_0) \to \infty$ as $N \to \infty$. A similar but easier calculation using the construction of the s-submanifolds on Σ_N shows that $\|A_0\| \leqq e_1\|A_N\|$ where e_1 is a constant independent of N. It also follows from the construction of the s-submanifolds on Σ_N that $\|C_0\| \leqq e_2\|A_0\|$ where e_2 is a constant independent of N. Thus, as $N \to \infty$, $\|C_0\| \to 0$ and $\|A_0\| \to 0$. From these facts (10) and (11) follow.

Combining Theorems (3.1) and (2.9), we obtain

(3.2) THEOREM. *If L^- is hyperbolic, then $L^- = \Lambda_1 \cup \cdots \cup \Lambda_n$ where the Λ_i are pairwise disjoint closed invariant topologically transitive sets with periodic points dense. Further, each Λ_i has a local product structure.*

(3.3) PROPOSITION. *For $L^- = \Lambda_1 \cup \cdots \cup \Lambda_n$ as in* (3.2), *and $x \in M$, $\alpha(x)$ meets at most one Λ_i.*

Proof. Use the argument at the bottom of p. 782 of [10] for f^{-1}.

(3.4) COROLLARY. $M = W^u(\Lambda_1) \cup \cdots \cup W^u(\Lambda_n)$.

Proof. If $x \in M$, there is an i such that $\alpha(x) \subset \Lambda_i$. Since Λ_i has a local product structure, there is a $y \in \Lambda_i$ such that $x \in W^u(y)$ by Theorem (1.1) of [3] applied to f^{-1}.

Following Smale's convention, if $\bar{P}$ is hyperbolic we will call the sets Λ_i of Theorem (2.9) *basic sets.*

A sequence $M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = \varnothing$ of compact submanifolds with boundary such that $f(M_i) \subset \operatorname{int} M_i$ is called a *filtration* for f. In [11] Smale constructs a filtration which "separates" basic sets if f satisfies Axiom A and has no cycles. In this case, if Λ_i is a basic set, then $W^s(\Lambda_i) \cap W^u(\Lambda_i) = \Lambda_i$ is, of course, the smallest closed invariant set containing Λ_i. We show below that if the negative limit set $L^-(f)$ is hyperbolic, even in the presence of cycles, one can obtain a filtration for

f which "separates" certain closed invariant sets. In case the limit set $L(f)$ is hyperbolic, these sets turn out to be the intersections of the stable and unstable manifolds of c-loop classes of basic sets (see definition below). Our filtration will be constructed by modifying the methods in [11].

We will also examine some consequences of the filtration theorem.

For the remainder of this section we assume $L^-=L^-(f)$ is hyperbolic. Thus $L^-=\bar{P}=\Lambda_1\cup\cdots\cup\Lambda_n$ as in Theorem (3.2) and $M=W^u(\Lambda_1)\cup\cdots\cup W^u(\Lambda_n)$.

We define two relations on $\{\Lambda_i\}$.

1. $\Lambda_i \geqq_1 \Lambda_j$ if there is a sequence $\Lambda_i=\Lambda_{i_1},\ldots,\Lambda_{i_r}=\Lambda_j$ such that $\mathrm{Cl}\,(W^u(\Lambda_{i_k}))\cap W^s(\Lambda_{i_{k+1}})\neq\varnothing$ for $1\leqq k<r$.

2. $\Lambda_i \geqq_2 \Lambda_j$ if there is a sequence $\Lambda_i=\Lambda_{i_1},\ldots,\Lambda_{i_r}=\Lambda_j$ such that $\mathrm{Cl}\,(W^u(\Lambda_{i_k}))\cap \mathrm{Cl}\,(W^s(\Lambda_{i_{k+1}}))\neq\varnothing$ for $1\leqq k<r$.

We will show that these relations are the same, i.e. $\Lambda_i\geqq_1\Lambda_j$ if and only if $\Lambda_i\geqq_2\Lambda_j$. To each of these relations there is a corresponding equivalence relation $\sim_k$ defined by $\Lambda_i\sim_k\Lambda_j$ if $\Lambda_i\geqq_k\Lambda_j$ and $\Lambda_j\geqq_k\Lambda_i$, $k=1, 2$. It will follow that the equivalence classes of these two relations are the same.

Let $\gamma_1',\ldots,\gamma_m'$ be the distinct equivalence classes of $\{\Lambda_i\}$ under $\sim_1$. Note that $\{\gamma_i'\}$ is partially ordered by $\gamma_i'\geqq_1\gamma_j'$ if there are $\Lambda_i\in\gamma_i'$, $\Lambda_j\in\gamma_j'$ such that $\Lambda_i\geqq_1\Lambda_j$.

Let $\geqq$ be a simple ordering on $\{\gamma_i'\}$ such that if $\gamma_i'\geqq\gamma_j'$, then $\gamma_j'\not>_1\gamma_i'$ (i.e., γ_j' does not strictly precede γ_i' in the $\geqq_1$ ordering). We will call such a simple ordering a *filtration ordering* for $\{\gamma_i'\}$.

For $i=1,\ldots,m$, $\sigma=u, s$, define $W^\sigma(\gamma_i')=\bigcup\{W^\sigma(\Lambda) : \Lambda\in\gamma_i'\}$. Write $\bigcup\gamma_i' =\bigcup\{\Lambda : \Lambda\in\gamma_i'\}$.

We will need a lemma due to Smale.

(3.5) Lemma (See [11]). *Suppose F is a compact f-invariant subset of M and Q is a compact neighborhood of F such that $\bigcap_{n\geqq 0}f^n(Q)=F$. Then there is a compact neighborhood V of F such that $V\subset Q$ and $f(V)\subset\mathrm{int}\,V$.*

Proof (due to Smale). Let $A_r=Q\cap f(Q)\cap\cdots\cap f^r(Q)$, $r\geqq 0$.

Then $A_0\supset A_1\supset\cdots$ and $\bigcap_{i\geqq 0}A_i=F$. Since $f(F)\subset F$, there is an integer $r>0$ such that $A_r\subset\mathrm{int}\,Q$ and $f(A_r)\subset\mathrm{int}\,Q$. But then $f(A_r)=A_{r+1}\subset A_r$ and $f^j(A_r)=A_{r+j}$, $j\geqq 0$. Thus there is an integer $r_1>0$ such that $f^{r_1}(A_r)\subset\mathrm{int}\,A_r$. If $r_1=1$, we are done, so suppose $r_1>1$. Let $W_0\subset\mathrm{int}\,Q$ be a compact neighborhood of A_r such that $f^{r_1}(W_0)\subset\mathrm{int}\,A_r$. Let $W_1=(W_0\cap f^{r_1-1}(W_0))\cup A_r\subset\mathrm{int}\,Q$. Since $r_1-1\geqq 1$, $f^{r_1-1}(A_r) =A_{r_1-1+r}\subset A_r\cap\mathrm{int}\,f^{r_1-1}(W_0)\subset\mathrm{int}\,W_0\cap\mathrm{int}\,f^{r_1-1}(W_0)\subset\mathrm{int}\,(W_0\cap f^{r_1-1}(W_0))$ and $f^{r_1-1}(W_0\cap f^{r_1-1}(W_0))\subset f^{2r_1-2}(W_0)\subset\mathrm{int}\,A_r$. Thus $f^{r_1-1}(W_1)\subset\mathrm{int}\,W_1$. Continue by downward induction to prove the lemma.

(3.6) Theorem. *Let $\geqq$ be any filtration ordering for $\{\gamma_i'\}$ and label $\{\gamma_i'\}$ such that $\gamma_m'>\gamma_{m-1}'>\cdots>\gamma_1'$ where $\gamma_i'>\gamma_j'$ is taken to mean $\gamma_i'\geqq\gamma_j'$ but $\gamma_i'\neq\gamma_j'$. Then there is a filtration for f, $M=M_m\supset M_{m-1}\supset\cdots\supset M_1\supset M_0=\varnothing$, such that for $1\leqq i\leqq m$*

(1) $\bigcup\gamma_i'\subset\mathrm{int}\,(M_i-M_{i-1})$,

(2) $\bigcap_{n\geqq 0} f^n \operatorname{Cl}((M_i - M_{i-1})) \subset W^u(\gamma_i')$,
(3) $\bigcup_{j\leqq i} \operatorname{Cl}(W^u(\gamma_j')) \subset \operatorname{int} M_i$,
(4) $\bigcap_{n\geqq 0} f^n(M_i) = \bigcup_{j\leqq i} W^u(\gamma_j') = \bigcup_{j\leqq i} \operatorname{Cl}(W^u(\gamma_j'))$,
(5) *if* $\gamma_j' > \gamma_i'$, *then* $\operatorname{Cl}(W^s(\gamma_j')) \cap M_i = \varnothing$.

Proof. Say that f satisfies $(*)_k$ if there is a sequence $M_k \supset M_{k-1} \supset \cdots \supset \varnothing$ of compact manifolds with boundary such that $f(M_i) \subset \operatorname{int} M_i$ and (1)–(5) hold for $1 \leqq i \leqq k$.

To begin, take V to be a compact neighborhood of $W^u(\gamma_1')$ such that $V \cap \bigcup_{j>1}(\bigcup \gamma_j') = \varnothing$. Then if $x \in \bigcap_{n\geqq 0} f^n(V)$, then $\alpha(x) \subset V$, so $x \in W^u(\gamma_1')$. Thus $\bigcap_{n\geqq 0} f^n(V) = W^u(\gamma_1') = \operatorname{Cl}(W^u(\gamma_1'))$. By Lemma (3.5), there is a compact neighborhood $\tilde{V}$ of $\operatorname{Cl}(W^u(\gamma_1'))$ such that $\tilde{V} \subset V$ and $f(\tilde{V}) \subset \operatorname{int} V$. Further, we may suppose $\tilde{V}$ is a compact manifold with boundary. Taking $M_1 = \tilde{V}$, we see that f satisfies $(*)_1$.

Now suppose f satisfies $(*)_k$. Then $\operatorname{Cl}(W^u(\gamma_{k+1}')) \cup M_k$ is a closed set which does not meet $\bigcup_{j>k+1}(\bigcup \gamma_j')$. Let V be a compact neighborhood of $\operatorname{Cl}(W^u(\gamma_{k+1}')) \cup M_k$ such that $V \cap \bigcup_{j>k+1}(\bigcup \gamma_j') = \varnothing$. Then if $x \in \bigcap_{n\geqq 0} f^n(V)$, $\alpha(x) \subset V$, so $x \in \bigcup_{1\leqq j\leqq n} W^u(\gamma_j') \cap V = \bigcup_{j\leqq k+1} W^u(\gamma_j')$. Thus $\bigcap_{n\geqq 0} f^n(V) = \bigcup_{j\leqq k+1} W^u(\gamma_j') = \bigcup_{j\leqq k+1} \operatorname{Cl}((W^u(\gamma_j'))$ is closed and in the interior of V. Again applying (3.5) there is a compact submanifold with boundary M_{k+1} such that $\bigcap_{n\geqq 0} f^n(V) \subset f(M_{k+1}) \subset \operatorname{int} M_{k+1} \subset V$. Now clearly properties (1), (3), and (4) hold for $1 \leqq i \leqq k+1$. If $x \in \bigcap_{n\geqq 0} f^n \operatorname{Cl}((M_{k+1} - M_k))$, then $\alpha(x) \subset (M_{k+1} - M_k) \cap L^- = \bigcup \gamma_{k+1}'$, so (2) holds. Finally, since $f(M_{k+1}) \subset \operatorname{int} M_{k+1}$, if $\operatorname{Cl}(W^s(\gamma_j')) \cap M_{k+1} \neq \varnothing$, then $W^s(\gamma_j') \cap M_{k+1} \neq \varnothing$, so $\bigcup (\gamma_j') \cap V \neq \varnothing$. Hence, by the construction of V, $\gamma_{k+1}' \geqq \gamma_j'$ which shows that (5) holds. Thus f satisfies $(*)_{k+1}$ and we are done.

REMARK. For basic sets Λ_i and Λ_j, call a sequence from Λ_i to Λ_j as in the definition of $\geqq_1$ a *c-path* from Λ_i to Λ_j. Let $\mathscr{P}(\Lambda_i) = \{\Lambda_j : \text{there is a } c\text{-path from } \Lambda_i \text{ to } \Lambda_j\}$. The techniques in the proof of (3.6) can be used to show that for any Λ_i, there is a compact neighborhood V of $\operatorname{Cl}(W^u(\mathscr{P}(\Lambda_i)))$ such that $f(V) \subset \operatorname{int} V$ and

$$\bigcap_{n\geqq 0} f^n(V) = \operatorname{Cl}(W^u(\mathscr{P}(\Lambda_i))) = W^u(\mathscr{P}(\Lambda_i)).$$

(Of course, $W^u(\mathscr{P}(\Lambda_i)) = \bigcup_{\Lambda_j \in \mathscr{P}(\Lambda_i)} W^u(\Lambda_j)$.)

(3.7) THEOREM. *If* $\Lambda_i \geqq_2 \Lambda_j$, *then* $\Lambda_i \geqq_1 \Lambda_j$.

(3.8) COROLLARY. *The equivalence relations* $\sim_1$ *and* $\sim_2$ *give the same equivalence classes.*

Proof of (3.7). We prove that if $\Lambda_i \ngeqq_1 \Lambda_j$, then $\Lambda_i \ngeqq_2 \Lambda_j$. If $\Lambda_i \ngeqq_1 \Lambda_j$, there is a filtration ordering $\geqq$ such that $[\Lambda_j]^1 > [\Lambda_i]^1$ where $[\Lambda]^1$ is the equivalence class of Λ under $\sim_1$, $\Lambda = \Lambda_i$ or Λ_j.

Let $\{M_i\}$ be a filtration corresponding to $\geqq$ as in Theorem (3.6). Then if $\Lambda_i \geqq_2 \Lambda_k$, an easy induction on the length of a sequence from Λ_i to Λ_k as in the definition of $\geqq_2$ shows that $\Lambda_k \subset M_i$. But since $\Lambda_j \cap M_i = \varnothing$, we get $\Lambda_i \ngeqq_2 \Lambda_j$.

We will use the notation $\Lambda_i \sim \Lambda_j$ to mean $\Lambda_i \sim_1 \Lambda_j$ or $\Lambda_i \sim_2 \Lambda_j$ which is justified by Corollary (3.8).

By analogy with the usual definition of cycles in the case of Axiom A (see [5] and [11]), we define an *r-cycle* to be a sequence $\Lambda_{i_0}, \dots, \Lambda_{i_r}$ such that $\Lambda_{i_0} = \Lambda_{i_r}$ and $\hat{W}^u(\Lambda_{i_k}) \cap \hat{W}^s(\Lambda_{i_{k+1}}) \neq \varnothing$ for $0 \leqq k < r$ where $\hat{W}^\sigma(\Lambda) = W^\sigma(\Lambda) - \Lambda$, $\sigma = u, s$. A *cycle* will mean an r-cycle for some r. The reason we need to use $\hat{W}^\sigma(\Lambda)$ instead of $W^\sigma(\Lambda)$ as in the case of Axiom A is that 1-cycles (our definition) can occur for L^- hyperbolic (see Examples 1 and 4 at the end of this section), whereas they cannot occur for diffeomorphisms satisfying even Axiom A(a).

Define a *c-cycle* (for closure cycle) to be a sequence $\Lambda_{i_0}, \dots, \Lambda_{i_r}$ such that $\Lambda_{i_0} = \Lambda_{i_r}$ and $(\mathrm{Cl}\,(W^u(\Lambda_{i_k})) - \Lambda_{i_k}) \cap \hat{W}^s(\Lambda_{i_{k+1}}) \neq \varnothing$ for $0 \leqq k < r$.

A sequence $\Lambda_{i_0}, \dots, \Lambda_{i_r}$ such that $\Lambda_{i_0} = \Lambda_{i_r}$ and $\mathrm{Cl}\,(W^u(\Lambda_{i_k})) \cap W^s(\Lambda_{i_{k+1}}) \neq \varnothing$ for $0 \leqq k < r$ will be called a *c-loop*. Also we will call the equivalence classes of $\{\Lambda_i\}$ under $\sim$, *c-loop classes*.

The proof of the following lemma was worked out with the aid of J. Palis.

(3.9) Lemma. *Suppose* $\Lambda_1 \neq \Lambda_2$ *are basic sets such that* $(\mathrm{Cl}\,(W^u(\Lambda_1)) - \Lambda_1) \cap \hat{W}^u(\Lambda_2) \neq \varnothing$. *Then* $(\mathrm{Cl}\,(W^u(\Lambda_1)) - \Lambda_1) \cap \hat{W}^s(\Lambda_2) \neq \varnothing$.

Proof. Since Λ_2 has a local product structure, there is a proper fundamental neighborhood V for $W^s(\Lambda_2)$ (see [3]). Moreover, we may choose V to be arbitrarily close to a proper fundamental domain $D \subset W^s(\Lambda_2) - \Lambda_2$. By Theorem (1.1) of [3], $V' = \bigcup_{n \geqq 0} f^n(V) \cup W^u(\Lambda_2)$ is a neighborhood of Λ_2 in M. But then for ε small enough, V' is a neighborhood of $W^u_\varepsilon(\Lambda_2)$. Since $(\mathrm{Cl}\,(W^u(\Lambda_1)) - \Lambda_1) \cap \hat{W}^u(\Lambda_2) \neq \varnothing$, $(\mathrm{Cl}\,(W^u(\Lambda_1)) - \Lambda_1) \cap (W^u_\varepsilon(\Lambda_2) - \Lambda_2) \neq \varnothing$. But then $(W^u(\Lambda_1) - \Lambda_1) \cap \bigcup_{n \geqq 0} f^n(V) \neq \varnothing$ so $(W^u(\Lambda_1) - \Lambda_1) \cap V \neq \varnothing$. Since V was arbitrarily close to D,

$$(\mathrm{Cl}\,(W^u(\Lambda_1)) - \Lambda_1) \cap \hat{W}^s(\Lambda_2) \neq \varnothing.$$

(3.10) Proposition. *A c-loop class contains a cycle if and only if it contains a c-cycle.*

Proof. Suppose γ is a c-loop class which contains a c-cycle. We prove that γ contains a cycle. The converse is obvious.

For $\Lambda_1, \Lambda_2 \in \gamma$, call a sequence $\Lambda_{i_0}, \dots, \Lambda_{i_s}$ a proper sequence from Λ_1 to Λ_2 if $\Lambda_{i_0} = \Lambda_1$, $\Lambda_{i_s} = \Lambda_2$ and $\hat{W}^u(\Lambda_{i_k}) \cap \hat{W}^s(\Lambda_{i_{k+1}}) \neq \varnothing$ for $0 \leqq k < s$. Let $(\Lambda_{i_0}, \dots, \Lambda_{i_r})$ be a c-cycle in γ.

If for each $0 \leqq j < r$ there is a proper sequence from Λ_{i_j} to Λ_{i_r}, taking $j = 0$, we get a cycle in γ. If there is a $0 \leqq j < r$ such that there is no proper sequence from Λ_{i_j} to Λ_{i_r}, let j_0 be the largest such integer. Then there is no proper sequence from $\Lambda_{i_{j_0}}$ to $\Lambda_{i_{j_0+1}}$.

Let $x \in (\mathrm{Cl}\,(W^u(\Lambda_{i_{j_0}})) - \Lambda_{i_{j_0}}) \cap \hat{W}^s(\Lambda_{i_{j_0+1}})$. Let Λ_{j_1} be the basic set such that $x \in W^u(\Lambda_{j_1})$. Then since $x \in \hat{W}^s(\Lambda_{i_{j_0+1}})$, $x \in \hat{W}^u(\Lambda_{j_1})$, so we may apply Lemma (3.9) to conclude that $\Lambda_{i_{j_0}} >_1 \Lambda_{j_1}$. Now assume $\Lambda_{j_k} \neq \Lambda_{i_{j_0}}$ is defined for $k \leqq \nu$ such that $\Lambda_{i_{j_0}} >_1 \Lambda_{j_k}$ and $\hat{W}^u(\Lambda_{j_k}) \cap \hat{W}^s(\Lambda_{j_{k-1}}) \neq \varnothing$.

If any two of the Λ_{j_k}'s are equal we have a cycle so we may assume they are all distinct. Let $x \in (\mathrm{Cl}\,(W^u(\Lambda_{i_{j_0}})) - \Lambda_{i_{j_0}}) \cap \hat{W}^s(\Lambda_{j_v})$. Since there is no proper sequence from $\Lambda_{i_{j_0}}$ to Λ_{j_v}, if $\Lambda_{j_{v+1}}$ is the basic set such that $x \in W^u(\Lambda_{j_{v+1}})$, then $\Lambda_{j_{v+1}} \neq \Lambda_{i_{j_0}}$, and $\Lambda_{i_{j_0}} >_1 \Lambda_{j_{v+1}}$. So we get a sequence of distinct basic sets $(\Lambda_{j_{v+1}}, \Lambda_{j_v}, \ldots, \Lambda_{j_1})$ such that $\hat{W}^u(\Lambda_{j_e}) \cap \hat{W}^s(\Lambda_{j_{e-1}}) \neq \varnothing$ for $2 \leqq e < v+1$. Continuing as above, since there are only finitely many basic sets, we eventually get a cycle.

A basic set Λ is called a source if $W^s(\Lambda) = \Lambda$.

If Λ is a *source*, there is a compact neighborhood V of Λ such that $f^{-1}(V) \subset \mathrm{int}\, V$.

(3.11) Proposition. *If γ is a c-loop class which is maximal with respect to the partial ordering $>_1$ (hence $>_2$), then γ has only one element, and that element is a source.*

Proof. Suppose γ is maximal with respect to $\geq_1$. Choose a filtration ordering with γ as its largest element. Let $M = M_m \supset M_{m-1} \supset \cdots \supset M_1 \supset \varnothing$ be the corresponding filtration for f so that $\bigcup \gamma \subset \mathrm{int}\,(M_m - M_{m-1})$. By Theorem (3.6) if Λ is a basic set such that $W^u(\Lambda) \cap (M_m - M_{m-1}) \neq \varnothing$, then $[\Lambda]^1 \geqq \gamma$. Thus $\Lambda \in \gamma$ since γ is maximal. Thus $M_m - M_{m-1} \subset W^u(\gamma)$. So $W^u(\gamma) - M_{m-1}$ is an open neighborhood of $W^s(\gamma)$. Thus there is a $\Lambda \in \gamma$ such that $W^u(\Lambda)$ contains an open subset of M. Let $\varepsilon > 0$. Then $\bigcup_{n \geqq 0} f^n(W^u_\varepsilon(\Lambda)) = W^u(\Lambda)$ so $W^u_\varepsilon(\Lambda)$ contains an open subset of M, say V. Since the periodic points of Λ are dense in Λ, there is a periodic point $p \in \Lambda$ such that $W^u_\varepsilon(p) \cap V \neq \varnothing$. By an application of the λ-lemma, $W^s(\mathrm{o}(p)) \subset \mathrm{Cl}\,(\bigcup_{n \geqq 0} f^{-n}(V))$. Further, as observed by Smale, if Λ is a basic set, then $W^s(\Lambda) \subset \mathrm{Cl}\,(W^s(\mathrm{o}(p)))$. Thus, $W^s(\Lambda) \subset \mathrm{Cl}\, W^s(\mathrm{o}(p)) \subset \mathrm{Cl}\,(\bigcup_{n \geqq 0} f^{-n}(V)) \subset W^u_\varepsilon(\Lambda)$. Here the last inclusion follows since $W^u_\varepsilon(\Lambda)$ is closed and f^{-1}-invariant. Thus $W^s(\Lambda) \subset W^u_\varepsilon(\Lambda)$ for all $\varepsilon > 0$. But $\bigcap_{\varepsilon > 0} W^u_\varepsilon(\Lambda) = \Lambda$, so $W^s(\Lambda) \subset \Lambda$ and we are done.

(3.12) Remark. 1. The preceding results are true with L^+ replacing L^- where the obvious changes are made; e.g., if L^+ is hyperbolic, then $\bar{P} = L^+ = \Lambda_1 \cup \cdots \cup \Lambda_n$, $M = W^s(\Lambda_1) \cup \cdots \cup W^s(\Lambda_n)$, there is a filtration for f^{-1} as in Theorem (3.6), etc.

2. If $L = L^- \cup L^+$ is hyperbolic and $M_m \supset M_{m-1} \supset \cdots \supset M_1 \supset \varnothing$ is the filtration for f as in Theorem (3.6), then $\bigcap_{-\infty < n < \infty} f^n(\mathrm{Cl}\,(M_i - M_{i-1})) = W^s(\gamma_i) \cap W^u(\gamma_i)$. For, if $x \in \bigcap_{n \leqq 0} f^n(\mathrm{Cl}\,(M_i - M_{i-1}))$ then $\omega(x) \in M_i - M_{i-1}$, so $x \in W^s(\gamma_i)$.

3. Clearly, if f satisfies Axiom A(a), then L is hyperbolic. In [5], Palis shows that if f satisfies Axiom A and has a cycle, then f may be perturbed to given an Ω-explosion. Notice that Remark (3.12.2) can be used to give some kind of control on the size of the Ω-explosion. That is, if g is close to f, then $\Omega(g)$ is close to $\bigcup_{1 \leqq i \leqq m} (W^u(\gamma_i) \cap W^s(\gamma_i))$.

Before proceeding, we consider some examples. All of these examples will be diffeomorphisms on the two-sphere S^2.

1. This example is such that L^- is hyperbolic, but L^+ is not hyperbolic. It also shows that minimal elements of a filtration ordering do not have to consist of

single basic sets (see Proposition (3.10)). We take f to be the time-one map φ_1 of the flow φ_t pictured below.

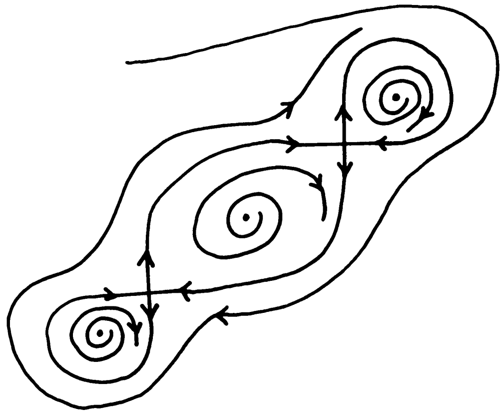

FIGURE 1

Here, for φ_t, there is an expanding spiral source at infinity. There are two hyperbolic saddle points and three expanding spiral fixed points as in the picture. The saddle points taken together form a minimal element in the filtration ordering and all orbits except the saddle connections and the fixed points spiral in to the saddle connections.

In this case $L^-(f)$ is the set of fixed points of f so it is hyperbolic. $L^+(f)$ is the set of fixed points together with the saddle connections (which are the stable and unstable manifolds of the saddle points).

2. Here $\bar{P}(f)$ is hyperbolic and finite, but $L^-(f)$ and $L^+(f)$ are neither. Again f is the map φ_1 for a flow φ_t. This flow is described as follows. On a two disk D_1, let ψ_t be a flow transversal to the boundary whose ω-limit set is two spiral sources and a figure eight as in Figure 2a.

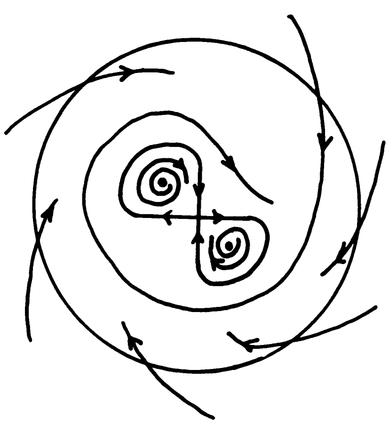

FIGURE 2a

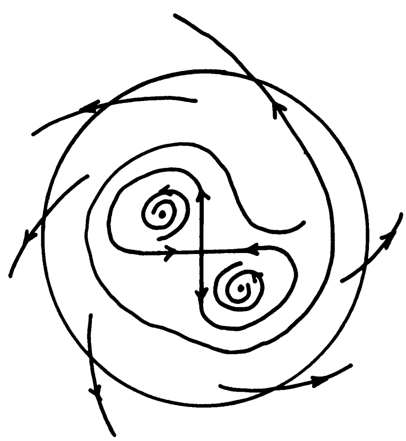

FIGURE 2b

Here ψ_t has only three critical points, so $P(f)=\bar{P}(f)$ is the set of those three points. Let η_t be the inverse of the flow ψ_t on another copy D_2 of D_1, i.e. $\eta_t(x)=\psi_{-t}(x)$ (see Figure 2b). Then η_t has three critical points and its α-limit set is not hyperbolic. Now glue the two disks D_1 and D_2 together along their boundaries and fit ψ_t and η_t together to give a flow as required.

3. This example was shown to me by J. Palis. In it the limit set $L(f)=L^-(f) \cup L^+(f)$ is finite and hyperbolic, but $\Omega(f)$ is neither.

Start with a flow φ_t having two sources, two sinks and two saddle points x_1, x_2 connected by trajectories as in Figure 3a.

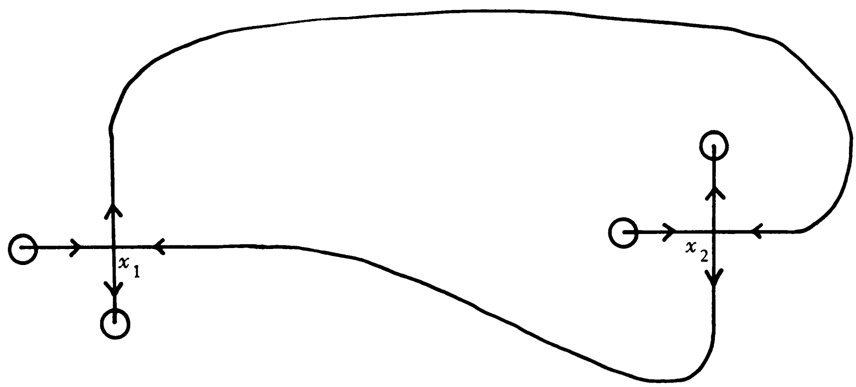

FIGURE 3a

The circles represent the sources and sinks and all the critical elements are assumed hyperbolic. Thus $\varphi_1=f$ satisfies Axiom A and has a 2-cycle. Now by a slight change of $\varphi_1=f$ in the space of diffeomorphisms we make one component of $W^u(x_1)-\{x_1\}$ have nonempty transversal intersection with one component of $W^s(x_2)-\{x_2\}$ as in Figure 3b.

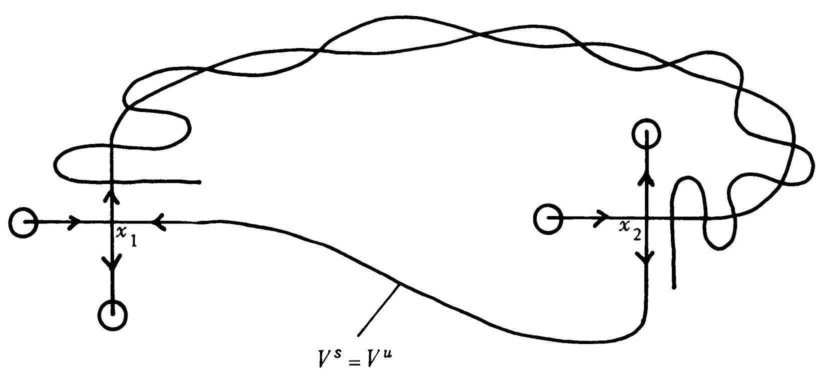

FIGURE 3b

This can be done so as not to change the other components V^s (V^u) of $W^s(x_1)-\{x_1\}$ ($W^u(x_2)-\{x_2\}$). Of course, $V^s=V^u$. For the new diffeomorphism g, $\Omega(g)$ will consist of $P(g) \cup V^s$.

We can also make Ω countable and keep L finite by making V^s and V^u intersect nontransversely in an appropriate way. An example is depicted in Figure 3c.

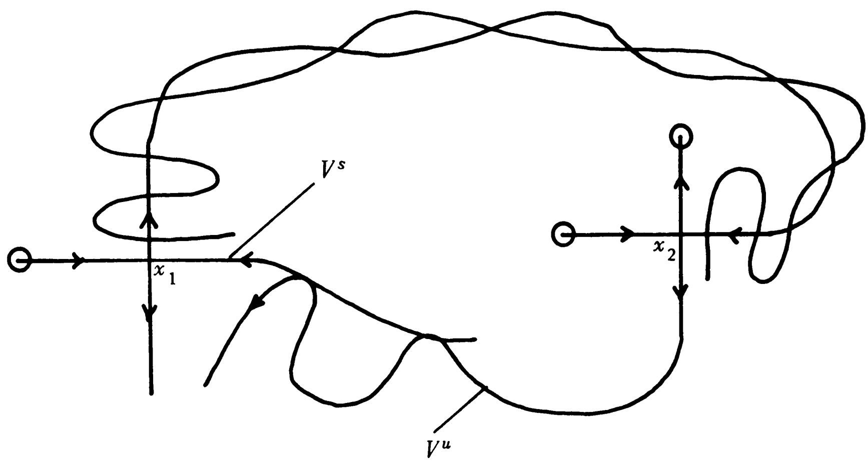

FIGURE 3c

4. Here we have L hyperbolic, Axiom A(a) is not satisfied and there is a 1-cycle. Start with the familiar horseshoe example of Smale on S^2. Thus $\Omega(f)$ consists of a source p_0, a sink p_1, and a Cantor set Λ on which f is topologically conjugate to a shift automorphism on two symbols. This is pictured in Figure 4a.

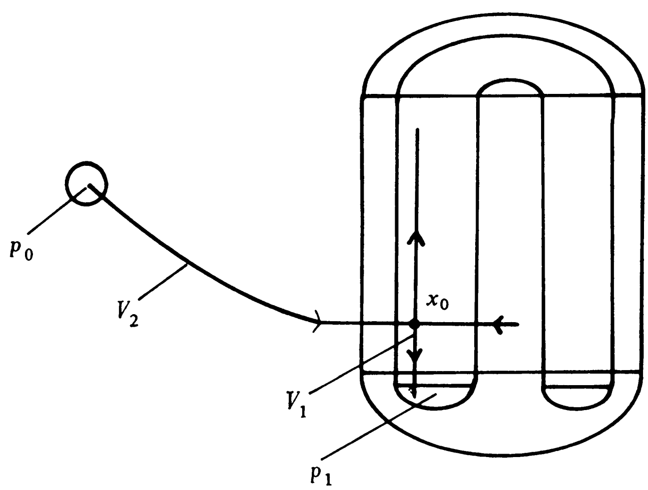

FIGURE 4a

There is a single fixed point $x_0 \in \Lambda$ such that one component V_1 of $W^u(x_0)-\{x_0\}$ is contained in $W^s(p_1)$ and one component V_2 of $W^s(x_0)-\{x_0\}$ is contained in $W^u(p_0)$.

There are open intervals V^s and V^u in the other components of $W^s(x_0)-\{x_0\}$ and $W^u(x_0)-\{x_0\}$ such that $\mathrm{Cl}\,(V^s)$ and $\mathrm{Cl}\,(V^u)$ are closed intervals bounded on one side by x_0. We suppose V^s and V^u are as depicted in Figure 4b, so that $V^s \cap V^u$ consists of four points.

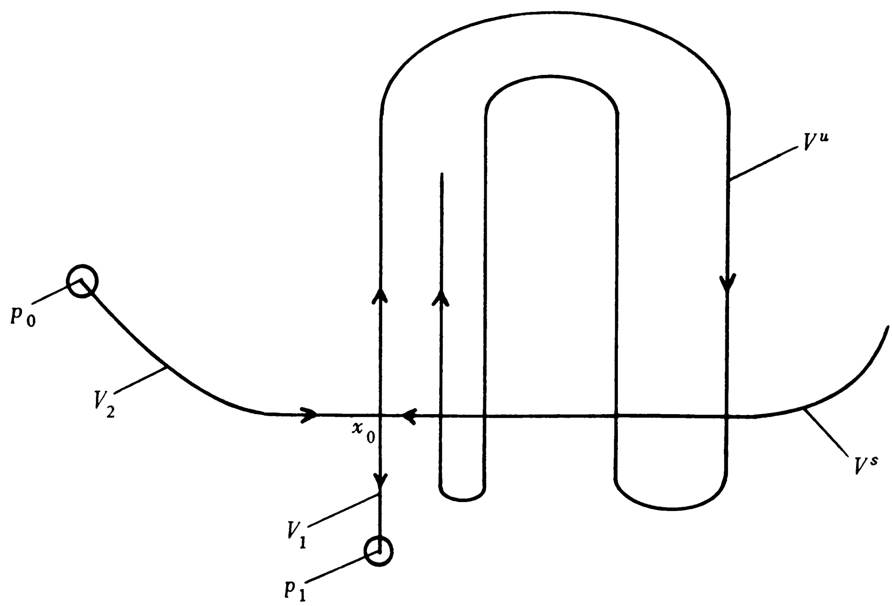

FIGURE 4b

Now one can modify the diffeomorphism away from Λ so as to produce a unique tangency y off Λ of V^s and V^u. This is depicted in Figure 4c.

The modification can be done so that one gets a diffeomorphism g such that

(a) $\Omega(g)=\Omega(f) \cup o(y)$.

(b) For each $x \in M$ and each small neighborhood U of y, $o(x) \cap U$ has at most two points.

It follows from (a), (b) and the construction of such a g, that $L(g)=\Omega(f)$ and g has a 1-cycle (Λ_0, Λ_1) with $\Lambda_0=\Lambda_1=\Lambda$.

4. In this section we complete the proof that if $L^-(f)$ is hyperbolic and there are no cycles, then f satisfies Axiom A. We also give another sufficient condition for Ω-stability. The latter result is that if $\bar{P}$ is hyperbolic, $L_\alpha^N \subset \bar{P}$ (see definition before (4.7)), and there are no c-cycles, then f satisfies Axiom A (and has no cycles).

(4.1) THEOREM. *Suppose $L^-(f)$ is hyperbolic, and f has no c-cycles. Then $L^- = \bar{P} = \Omega$, so f satisfies Axiom* A. *Further, f has no cycles, so f is Ω-stable.*

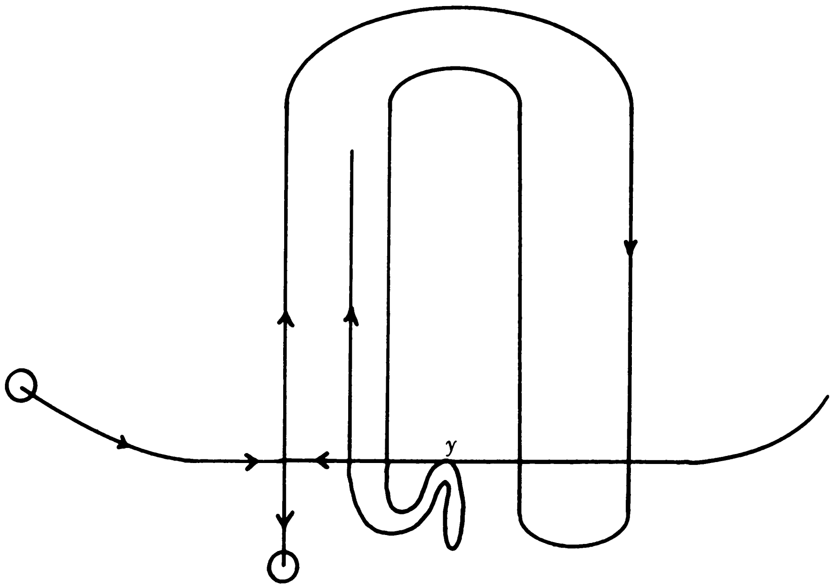

FIGURE 4c

Proof. Let $L^-(f)=\Lambda_1 \cup \cdots \cup \Lambda_n$ as in Theorem (3.2). Since f has no c-cycles, each c-loop class has only one element. Let $\{\Lambda_n\} \geqq \{\Lambda_{n-1}\} \geqq \cdots \geqq \{\Lambda_1\}$ be a filtration ordering for f with filtration $M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset \varnothing$ as in Theorem (3.6). We claim

$$\text{(4.2)} \qquad \text{for each } 1 \leqq i \leqq n, \quad \bigcap_{-\infty<n<\infty} f^n(\mathrm{Cl}\,(M_i - M_{i-1})) = \Lambda_i.$$

This will prove that $L^- = \Omega$, so $L^- = \bar{P} = \Omega$. Thus we will have that f satisfies Axiom A. Clearly, the hypotheses imply that f has no cycles. Further, once (4.2) is proved we will, in fact, have a filtration for f exactly the same as the filtration Smale obtains in [11]. Then Ω-stability will follow as in his proof by proving Ω-stability on each $M_i - M_{i-1}$. This last step follows from Theorem (7.3) of [2].

So we need only prove (4.2). Fix i. By (3.6.2), $\bigcap_{n\geqq 0} f^n(\mathrm{Cl}\,(M_i - M_{i-1})) \subset W^u(\Lambda_i)$. We claim

$$\text{(4.3)} \qquad \bigcap_{n\leqq 0} f^n(\mathrm{Cl}\,(M_i - M_{i-1})) \subset W^s(\Lambda_i).$$

Once this is shown (4.2) follows since we will have

$$\Lambda_i \subset \bigcap_{-\infty<n<\infty} f^n(\mathrm{Cl}\,(M_i - M_{i-1})) \subset W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$$

where the last equality holds since there are no 1-cycles.

For the proof of (4.3) suppose $x \in \bigcap_{n\leqq 0} f^n(\mathrm{Cl}\,(M_i - M_{i-1}))$. Then $\omega(x) \subset \mathrm{Cl}\,(M_i - M_{i-1})$, so $\omega(x) \subset \bigcap_{n\geqq 0} f^n(\mathrm{Cl}\,(M_i - M_{i-1})) \subset W^u(\Lambda_i)$ where the first inclusion follows from the f-invariance of $\omega(x)$ and the second one follows from (3.6.2).

Since there are no 1-c-cycles, $W^u_\varepsilon(\Lambda_i)$ is a neighborhood of Λ_i in $W^u(\Lambda_i)$ for $\varepsilon>0$. But $\omega(x)$ is a compact subset of $W^u(\Lambda_i)$ and $\bigcup_{n\geqq 0} f^n(W^u_\varepsilon(\Lambda_i))=W^u(\Lambda_i)$. So there is an integer $n_1>0$ such that $\omega(x)\subset f^{n_1}(W^u_\varepsilon(\Lambda_i))$. Since $\omega(x)$ is f^{-1}-invariant, $\omega(x)\subset\bigcap_{n\geqq 0} f^{-n}(f^{n_1}(W^u_\varepsilon(\Lambda_i)))=\Lambda_i$. But Λ_i has a local product structure, so $x\in W^s(\Lambda_i)$ and (4.3) is proved.

(4.4) Remark. Example 4 in §3 gives a diffeomorphism such that $L=L^-\cup L^+$ is hyperbolic, each c-loop class has only one element, and there is a single 1-cycle. Also, $L\neq\Omega$ and Axiom A(a) does not hold.

We now come to our main result.

(4.5) Theorem. *Suppose $L^-(f)$ is hyperbolic and f has no cycles. Then f satisfies Axiom* A.

Proof. Since f has no cycles, it has no c-cycles by Proposition (3.10). Now (4.5) follows from (4.1).

(4.6) Remark. 1. Using Theorem (4.5) one can obtain, of course, that Axiom A(a) and no cycles imply Axiom A(b). Also using Theorem (3.6) and a result similar to the theorem in §1 of [13], one can prove that if f satisfies Axiom A(a) and every c-loop $(\Lambda_{i_0},\ldots,\Lambda_{i_r})$ is 2-related in the sense that for $0\leqq j,\ l\leqq r$, $\mathrm{Cl}\,(W^u(\Lambda_{i_j}))\cap W^s(\Lambda_{i_l})\neq\varnothing$, then f satisfies Axiom A(b). However, it is still unknown if Axiom A(a) implies Axiom A(b) in general.

2. J. Robbin has recently proved that if f is C^2 and satisfies Axiom A and the strong transversality condition, then f is structurally stable [8], thus confirming part of a conjecture of Smale. Using Theorem (4.5) and some other well-known results, Robbin's theorem may be restated in the following way. If f is C^2, $L^-(f)$ is hyperbolic, and for $x,y\in L^-(f)$, $W^u(x)$ is transverse to $W^s(y)$, then f is structurally stable.

For V a closed subset of M, let $L^0_\alpha(V)=V$ and $L^N_\alpha(V)=\{y\in V:\exists x\in L^{N-1}_\alpha(V)$ such that $y\in\alpha(x)$ and $\alpha(x)\subset V\}$, for $N>0$. Note that, for $N>0$, $L^N_\alpha(V)=\{y\in V:$ there is a sequence $x_0,x_1,\ldots,x_N$ in V such that $x_N=y$ and $x_i\in\alpha(x_{i-1})\subset V$ for $1\leqq i\leqq N\}$. Let $L^N_\alpha(M)=L^N_\alpha$ so that $L^1_\alpha=L_\alpha$ as defined earlier. Notice also that $P\subset L^N_\alpha\subset\cdots\subset L_\alpha\subset\Omega$ for all $N>0$.

Our final result is the following.

(4.7) Theorem. *If $\bar P$ is hyperbolic, $L^N_\alpha\subset\bar P$ for some $N>0$, and there are no c-cycles for the basic sets in the spectral decomposition of $\bar P$, then f satisfies Axiom* A.

Proof. Let $N>0$ be such that $L^N_\alpha\subset\bar P$. We prove that $L_\alpha\subset\bar P$ and then (4.7) follows from (4.1).

Let $\bar P=\Lambda_1\cup\cdots\cup\Lambda_n$ as in Theorem (2.9). Since there are no c-cycles, each c-loop class has only one element, so there is a simple ordering $\Lambda_n\geqq\Lambda_{n-1}\geqq\cdots\geqq\Lambda_1$ such that if $\Lambda_i\geqq\Lambda_j$, then $\mathrm{Cl}\,(W^u(\Lambda_j))\cap\hat W^s(\Lambda_i)=\varnothing$, i.e. $\Lambda_j\not>_1\Lambda_i$, as defined before.

Say that f satisfies $(*)_k$ if there is a sequence of compact submanifolds with boundary $M_k \supset M_{k-1} \supset \cdots \supset M_1$ such that for $1 \leqq i \leqq k$

$(a)_k$ $\Lambda_i \subset \operatorname{int}(M_i - M_{i-1})$,

$(b)_k$ $f(M_i) \subset \operatorname{int} M_i$,

$(c)_k$ $\bigcap_{n \geqq 0} f^n(M_i) = \bigcup_{j \leqq i} W^u(\Lambda_j) = \bigcup_{j \leqq i} \operatorname{Cl}(W^u(\Lambda_j))$,

$(d)_k$ $L_\alpha(M_k) \subset \bigcup_{1 \leqq j \leqq k} \Lambda_j$.

Then one proves by induction on k that f satisfies $(*)_k$ for $1 \leqq k \leqq n$. It follows from $(d)_n$ that $L_\alpha \subset \bar{P}$.

We will prove $(*)_1$. The induction step for $(*)_{k+1}$ from $(*)_k$ is similar, so we omit its proof.

Let V be a compact neighborhood of $\operatorname{Cl}(W^u(\Lambda_i))$ such that $V \cap \Lambda_i = \varnothing$ for $i > 1$. Now $L_\alpha^N \subset \bar{P}$, so $L_\alpha^N(V) \subset \Lambda_1$. We claim that $L_\alpha^N(V) \subset \Lambda_1$ implies that $L_\alpha^{N-1}(V) \subset \Lambda_1$. For, if $x \in L_\alpha^{N-1}(V)$, then $\alpha(x) \subset L_\alpha^N(V) \subset \Lambda_1$, so $x \in W^u(\Lambda_1)$. But there is a $y \in L_\alpha^{N-2}(V)$ such that $x \in \alpha(y)$ and $\alpha(y) \subset V$. Also $\alpha(y) \subset L_\alpha^{N-1}(V) \subset W^u(\Lambda_1)$. Now, as in the proof of (4.3), $\alpha(y) \subset \Lambda_1$ since there are no 1-c-cycles. But $x \in \alpha(y)$, so $x \in \Lambda_1$. Thus $L_\alpha^{N-1}(V) \subset \Lambda_1$.

Proceeding by downward induction we get that $L_\alpha(V) \subset \Lambda_1$. But then $\bigcap_{n \geqq 0} f^n(V) = W^u(\Lambda_1) = \operatorname{Cl}(W^u(\Lambda_1))$. Now, as in the proof of Theorem (3.6), by Smale's lemma, there is a compact submanifold with boundary M_1 such that $f(M_1) \subset \operatorname{int} M_1$ and $\bigcap_{n \geqq 0} f^n(M_1) = W^u(\Lambda_1)$. Again as in the proof of (4.3), $M_1 = \bigcap_{n \leqq 0} f^n(M_1) \subset W^s(\Lambda_1)$. So $W^u(\Lambda_1) \subset W^s(\Lambda_1)$ and hence $W^u(\Lambda_1) = \Lambda_1$. This proves $(*)_1$.

Questions and Remarks. 1. Does (4.7) remain true if one replaces the no c-cycles assumption by the assumption that there are no cycles?

2. Let $F_1 = \bar{P}$, $F_2 = L^-$, $F_3 = L$, and $F_4 = \Omega$. Say that f is F_i-stable, $i = 1, \ldots, 4$, if there is a neighborhood $\mathcal{N}$ of f in $\operatorname{Diff}(M)$ such that if $g \in \mathcal{N}$, there is a homeomorphism $h: F_i(f) \to F_i(g)$ such that $hf = gh$. Is F_i-stability equivalent to F_j-stability for $i, j = 1, \ldots, 4$? Is it true that f is L^- stable if and only if L^- is hyperbolic and there are no cycles? Palis has shown (unpublished) that if f is F_i-stable and F_i is hyperbolic, then there are no cycles. Therefore, the main part of the last question is: does L^--stability imply L^- is hyperbolic? By the closing lemma this can be reduced to: does $\bar{P}$-stability imply that $\bar{P}$ is hyperbolic?

3. There are easily constructed examples where $\bar{P}$ is hyperbolic, there are no cycles, and $\bar{P} \subsetneq L^-$. Hence, by the closing lemma, it follows that $\bar{P}$ hyperbolic and no cycles is not sufficient for $\bar{P}$-stability.

Appendix: The homoclinic point theorem. Here we give a fairly elementary proof of Theorem (2.6) based on the stable manifold theory for a hyperbolic fixed point and Lemma (2.10).

We first need a lemma due to Hirsch and Pugh [2].

Let E, F be Banach spaces and give $E \times F$ the norm $|(x, y)| = \max\{|x|, |y|\}$.

Lemma 1. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a linear map from* $E \times F$ *to itself such that* D^{-1} *exists.*

Let $\eta < 1$. *If*

(1) $\|A\| \cdot \|D^{-1}\| + \|BD^{-1}\| + \|C\| \cdot \|D^{-1}\| < \eta$ *and*

(2) $\|A\| \cdot \|D^{-1}\| + 2\|C\| \cdot \|D^{-1}\| < 1$,

then there is a unique linear map $P\colon F \to E$ *such that* $\|P\| < \eta$ *and*

(3) $T(\text{graph } P) = \text{graph } P$.

Further, if $m(D) - \|C\| > 1$, *then* $T|_{\text{graph } P}$ *is expanding (recall* $m(D) = \inf_{|v|=1} |Dv|$).

Proof. Condition (3) can be written $T(Py, y) = (Pu, u)$ where $u = CPy + Dy$ or $APy + By = PCPy + PDy$ for $y \in F$. Thus as linear maps, $AP + B = PCP + PD$, or P is a fixed point of the map $H\colon P \mapsto APD^{-1} + BD^{-1} - PCPD^{-1}$.

Let $\mathscr{H}_\nu$ be the complete metric space of bounded linear maps from F to E with norm less than or equal to ν where $\nu = \|A\| \cdot \|D^{-1}\| + \|BD^{-1}\| + \|C\| \cdot \|D^{-1}\|$.

For $P \in \mathscr{H}_\nu$, $\|H(P)\| \leqq \|A\| \cdot \|D^{-1}\| + \|BD^{-1}\| + \|C\| \cdot \|D^{-1}\| = \nu < \eta$ by (1). Thus, H maps $\mathscr{H}_\nu$ into itself.

Similarly, by (2), H is a contraction and so has a unique fixed point P.

Also, if $(Py, y) \in \text{graph } P$, then, since $\|P\| < 1$, $|T(Py, y)| = |CPy + Dy| \geqq (m(D) - \|C\|)|y| > |y| = |(Py, y)|$.

This proves $T|_{\text{graph } P}$ is expanding and completes the proof of Lemma 1.

Now returning to the notation of Lemma (2.10), suppose

$$T_z g = \begin{pmatrix} A_z & B_z \\ C_z & D_z \end{pmatrix} \quad \text{for } z \in D_1, \quad \text{and}$$

$$T_w g^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{for } w \in g(D_1)$$

where $T_w g^{-1}$ is assumed to exist.

By analogy with the definitions of a, c, e, d before Lemma (2.10), define

$$\begin{aligned} a_1 &= \inf \{|A_{1w} v| : w \in g(D_1),\ v \text{ is a unit vector in } R^u\}, \\ b_1 &= \sup_{w \in g(D_1)} \|B_{1w}\|, \qquad d = \sup_{w \in g(D_1)} \|D_{1w}\|, \\ e_1 &= \sup \{|C_{1w} v|/|A_{1w} v| : w \in g(D_1),\ v \text{ is a unit vector in } R^s\}. \end{aligned}$$

Notice if A_{1w} is invertible for all w, then $e_1 = \sup_{w \in g(D_1)} \|C_{1w} A_{1w}^{-1}\|$, and if D_z is invertible for all z, then $e = \sup_{z \in D_1} \|B_z D_z^{-1}\|$.

LEMMA 2. *Suppose the hypotheses of Lemma* (2.10) *are satisfied and* $g(D_1) \subset (\frac{1}{2} D^s) \times D^u$. *Let* $\eta < 1$, *and let* $z_1 = (x_1, y_1)$ *be the fixed point of* g *in* D_1. *Suppose,*

(1) $a/d + e + c/d < \eta$,

(2) $a/d + 2c/d < 1$,

(3) $d - c > 1$,

and

(1)′ $d_1/a_1 + e_1 + b_1/a_1 < \eta$,

(2)′ $d_1/a_1 + 2b_1/a_1 > 1$,

(3)′ $a_1 - b_1 > 1$.

Then,

(4) *z_1 is a hyperbolic fixed point of g,*

(5) $W^u(z, g, D_1)=\{z \in D_1 : g^{-n}(z) \in D_1$ *for all* $n \geqq 0$ *and* $g^{-n}(z) \to z_1$ *as* $n \to \infty\}$, *and* $W^s(z, g, D_1)=\{z \in D_1 : g^n(z) \in D_1$ *for all* $n \geqq 0$ *and* $g^n(z) \to z_1$ *as* $n \to \infty\}$ *are smooth manifolds,*

(6) *if* $(d-c)\,\mathrm{dist}\,(z_1, \text{boundary } D_1) > \mathrm{dist}\,(z_1, D^s \times 0)$, *then* $gW^u(z_1, g, D_1)$ *has a unique point of transversal intersection with* $D^s \times 0$,

(7) $W^s(z_1, g, D_1)$ *has a point of transversal intersection with* $0 \times D^u$.

Proof. Applying Lemma 1 to $T_{z_1}g$ and $T_{z_1}g^{-1}$, we see that there are unique $T_{z_1}g$ invariant subspaces E^u, E^s in R^{s+u} such that

(a) E^u is the graph of a linear function $P^u: R^u \to R^s$ such that $\|P^u\| < \eta$ and E^s is the graph of a linear function $P^s: R^s \to R^u$ such that $\|P^s\| < \eta$, and

(b) $\|T_{z_1}g^{-1}|E^u\| < 1$, $\|T_{z_1}g|E^s\| < 1$.

Now (4) follows easily from (a) and (b). Also (5) follows from the stable and unstable manifold theorem for the hyperbolic fixed point z_1.

Let $\pi^\sigma: D \to D^\sigma$, $\sigma = s, u$, denote the natural projections on D.

To prove (6), we will show that

(c) $gW^u(z_1, g, D_1)$ is the graph of a smooth function $\varphi^u: \pi^u gW^u(z_1, g, D_1) \to D^s$ such that if $\mathscr{L}(\varphi^u)$ is the Lipschitz constant of φ^u, then $\mathscr{L}(\varphi^u) < \eta < 1$, and

(d) the center 0 of D^s is in $\pi^u gW^u(z_1, g, D_1)$.

We first prove (c). Let $z_1 = (x_1, y_1)$. From the unstable manifold theorem for the point z_1, if D_2^s and D_2^u are small disks in R^s and R^u centered at x_1 and y_1, then $W^u(z_1, g, D_2^s \times D_2^u) = \bigcap_{n \geq 0} g^n(D_2^s \times D_2^u) = \{z \in D_2^s \times D_2^u : g^{-n}(z) \to z\}$ is a smooth manifold tangent to E^u at z_1. Thus, if D_2^s and D_2^u are small enough, $gW^u(z_1, g, D_2^s \times D_2^u)$ is the graph of a smooth function $\varphi_0: \pi^u gW^u(z_1, g, D_2^s \times D_2^u) \to D^s$ such that $\mathscr{L}(\varphi_0) < \eta$.

Let $W_0^u = W^u(z_1, g, D_2^s \times D_2^u)$ and let $W^u = W^u(z_1, g, D_1)$.

For $i > 0$, define $W_i^u = g(W_{i-1}^u) \cap D_1$. We claim, for each $i > 0$, gW_i^u is the graph of a smooth function $\varphi_i: \pi^u gW_i^u \to D^s$ such that $\mathscr{L}(\varphi_i) < \eta$, and there is an integer $n_0 > 0$ such that $W_j^u = W_{n_0}^u = W^u$ for $j \geqq n_0$. Once the claim is proved, (c) follows by taking the function φ_{n_0}.

Suppose $\varphi_i: \pi^u gW_i^u \to D^s$ has been defined such that $\mathrm{graph}\,(\varphi_i) = gW_i^u$ and $\mathscr{L}(\varphi_i) < \eta < 1$.

By (3), the mapping $\psi_i: y \mapsto \pi^u g(\varphi_i(y), y)$ is a uniform expansion and hence a diffeomorphism on $\pi^u gW_i^u \cap D_1^u$ (recall $D_1^u = \pi^u D_1$). Further, $\psi_i(\pi^u gW_i^u \cap D_1^u) = \pi^u gW_{i+1}^u$.

On $\pi^u gW_{i+1}^u$, define $\varphi_{i+1}(y) = g(\varphi_i(\psi_i^{-1}(y)), \psi_i^{-1}(y))$ and observe that

$$\mathrm{graph}\,(\varphi_{i+1}) = gW_{i+1}^u.$$

To prove $\mathscr{L}(\varphi_{i+1}) < \eta$, it suffices to prove the following fact.

Let $v=(v^s, v^u)$ be a tangent vector to $gW_i^u \cap D_1$ at $z=(\varphi_i(y), y)$ (so $v^s=T_y\varphi_i(v^u)$ and $|v^s|/|v^u|<\eta<1$). Then, letting $T_zg(v)=(v_1^s, v_1^u)$, we have $|v_1^s|/|v_1^u|<\eta$.

Indeed, $T_zg(v)=(v_1^s, v_1^u)=(A_zT_y\varphi_i(v^u)+B_zv^u, C_zT_y\varphi_i(v^u)+D_zv^u)$, so if $v^u=D_z^{-1}w$, then

$$\frac{|v_1^s|}{|v_1^u|} \leqq \frac{(\|A_z\|\cdot\|D_z^{-1}\|+\|B_zD_z^{-1}\|)|w|}{(1-\|C_z\|\cdot\|D_z^{-1}\|)|w|} \leqq \frac{a/d+e}{1-c/d}$$

since $\|T_y\varphi_i\|<1$.

But, since $\eta<1$, (1) implies that $(a/d+e)/(1-c/d)<\eta$. Thus $\mathscr{L}(\varphi_{i+1})<\eta$.

Now each map ψ_i is a uniform expansion (in fact

$$\inf\{|T_y\psi_i(v^u)| : y\in\pi^u gW_i^u \cap D_1,\ |v^u| = 1\} > d-c > 1),$$

$\pi^u D_1 \subset \frac{1}{4}D^u$, $\pi^s gD_1 \subset \frac{1}{2}D^s$, and $\mathscr{L}(\varphi_i)<1$. Thus there is an integer $n_0>0$ such that $\pi^u W_{n_0}^u = D_1^u$. Further, if n_0 is the least such integer, then $W_j^u = W_{n_0}^u$ for $j\geqq n_0$. Notice also that $W_0^u \subset W_1^u \subset \cdots$. Since $W^u \subset \bigcup_{i\geqq 0} W_i^u$, we have that $W^u = W_{n_0}^u$. This completes the proof of (c).

For the proof of (d), since ψ_{n_0} expands everywhere more than $d-c$ and $(d-c)$ dist $(z_1$, boundary $D_1^u) >$ dist $(z_1, D^s\times 0)$, it follows that the center 0 of D^u is in the image of ψ_{n_0} which equals $\pi^u gW^u$. This completes the proof of (6). The proof of (7) is similar.

We now apply Lemma 2 to prove Theorem (2.6).

Let p be the hyperbolic periodic point of the diffeomorphism f. Let q be a transversal homoclinic point of p. Let $D=D^s\times D^u$ be a disk in T_qM such that there is a diffeomorphism $g_1: D\to M$ such that

(1) $g_1(0)=q$,

(2) $g_1(D^s\times 0)\subset W^s(q)$, $\quad g_1(0\times D^u)\subset W^u(q)$,

(3) the manifolds $g_1(D^s\times y)$, $y\in D^u$, are C^1 close to each other, and the manifolds $g_1(x\times D^u)$, $x\in D^s$, are C^1 close to each other.

We claim if D is small enough, there is a subdisk $D_1^u \subset D^u$, an integer $N>0$, and a diffeomorphism $g_2: D\to M$ such that

(4) $g_2(D^s\times 0)\subset W^s(q)$, $g_2(0\times D^u)\subset W^u(q)$,

(5) if $D_1=D^s\times D_1^u$ and $g=g_2^{-1}f^Ng_2|_{D_1}$, then g satisfies the hypotheses of Lemma 2.

After this has been shown, Theorem (2.6) will be proved, for $g_2(z_1)$ will be a hyperbolic point of f which is h-related to p.

Once D is chosen small enough, the disk D_1^u, diffeomorphism g_2, and integer N are constructed in a way similar to the analogous constructions in Theorem (3.1). One defines the u-submanifolds of g_2 $(\{g_2(x\times D^u) : x\in D^s\})$ and then the s-submanifolds $(\{g_2(D^s\times y) : y\in D^u\})$ of g_2. We will indicate how to define the u-submanifolds of g_2.

Using the λ-lemma one can show that, given $\varepsilon>0$, there is a small real number μ and an integer $N>0$ such that there is a connected component Σ_N of $f^N(g_1(D^s\times\mu D^u))$ $\cap\, g_1(D)$ satisfying

(6) the manifolds $\Sigma_{N,x} \equiv f^N(g_1(x \times \mu D^u)) \cap \Sigma_N$ for $x \in D^s$ are ε-C^1 close to $g_1(0 \times D^u)$,

(7) the manifolds $f^{-N}(g_1(D^s \times y) \cap \Sigma_N)$ for $y \in D^u$ are ε-C^1 close to $g_1(D^s \times 0)$,

(8) f^N expands a large amount on the manifolds $g_1(x \times \mu D^u) \cap f^{-N}(\Sigma_N) = f^{-N}(\Sigma_{N,x})$, $x \in D^s$,

(9) f^{-N} expands a large amount on the manifolds $g_1(D^s \times y) \cap \Sigma_N$, $y \in D^u$.

Now define a diffeomorphism $g_2' : D \to g_1(D)$ whose u-submanifolds are $\{g_1(x \times \mu D^u)\}$ and such that there is a subdisk $D_1^u \subset D^u$ such that $g_2'(D^s \times D_1^u) = f^{-N}(\Sigma_N)$.

Then define $g_2 : D \to g_1(D)$ as follows. Take the u-submanifolds of g_2 to be those of g_2'. In $f^{-N}(\Sigma_N)$, take the s-submanifolds of g_2 to be

$$\{f^{-N}(g_1(D^s \times y) \cap \Sigma_N) : y \in D^u\},$$

and off $f^{-N}(\Sigma_N)$ take them so that all of the s-submanifolds of g_2 are ε-C^1 close to $g_1(D^s \times 0)$.

The computations needed to prove that ε, D_1, N, and g_2 can be taken so that the hypotheses of Lemma 2 are satisfied for $g = g_2^{-1} f^N g_2|_{D_1}$ are similar to those in the proof of Theorem (3.1) and will be left to the reader.

REMARK. Here again, as in the proof of (3.1), one cannot use the weaker version of Lemma (2.10) in which e is replaced by $\sup_{z \in D_1} \{\|B_z\| \cdot \|D_z^{-1}\|\}$ (see Remark (2.11)).

REFERENCES

1. G. Birkhoff, *Dynamical systems*, Amer. Math. Soc. Colloq. Publ., vol. 9, Amer. Math. Soc., Providence, R. I., 1927.

2. M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970. MR **41** #7686.

3. M. Hirsch, J. Palis, C. Pugh and M. Shub, *Neighborhoods of hyperbolic sets*, Invent. Math. **9** (1969/70), 121–134. MR **41** #7232.

4. J. Palis, *On Morse-Smale dynamical systems*, Topology **8** (1968), 385–404. MR **39** #7620.

5. ———, *A note on Ω-stability*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970. MR **41** #7686.

6. J. Palis and S. Smale, *Structural stability theorems*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970. MR **41** #7686.

7. C. Pugh and M. Shub, *Ω-stability for flows*, Invent. Math. **11** (1970), 150–158.

8. J. Robbin, *A structural stability theorem* (to appear).

9. S. Smale, *Diffeomorphisms with many periodic points*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Univ. Press, Princeton, N. J., 1965, pp. 63–80. MR **31** #6244.

10. ———, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR **37** #3598.

11. ———, *The Ω-stability theorem*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970. MR **41** #7686.

12. Z. Nitecki, *On semi-stability for diffeomorphisms*, Invent. Math. **14** (1971), 83–122.

13. R. Abraham and S. Smale, *Non-genericity of* Ω*-stability*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970. MR **41** #7686.

14. L. P. Šil′nikov, *Structure of the neighborhood of a homoclinic tube of an invariant torus*, Dokl. Akad. Nauk SSSR **180** (1968), 286–289 = Soviet Math. Dokl. **9** (1968), 624–628. MR **40** #2111.

INSTITUTO DE MATEMÁTICA PURA E APLICADA, RIO DE JANEIRO, BRAZIL

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201