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LOCALLY B^*-EQUIVALENT ALGEBRAS(1)

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Abstract. Let A be a Banach *-algebra. A is locally B^*-equivalent if, for every selfadjoint element $t \in A$, the closed *-subalgebra of A generated by t is *-isomorphic to a B^*-algebra. In this paper it is shown that when A is locally B^*-equivalent, and in addition every selfadjoint element in A has at most countable spectrum, then A is *-isomorphic to a B^*-algebra.

1. **Introduction.** Assume that A is a Banach *-algebra. We say that A is B^*-equivalent if there exists a *-isomorphism of A onto a B^*-algebra. When $t \in A$ is selfadjoint, $C(t)$ denotes the closed *-subalgebra of A generated by t. We say that A is locally B^*-equivalent if $C(t)$ is B^*-equivalent for every selfadjoint element t in A. It does not seem unlikely that every locally B^*-equivalent Banach *-algebra A is B^*-equivalent. This is true when A is commutative; see Proposition 2.2. In this paper we prove this for certain noncommutative algebras. Specifically we prove (Theorem 4.1) that when A is a Banach *-algebra which is locally B^*-equivalent, and in addition has the property that every selfadjoint element has at most a countable spectrum, then A is B^*-equivalent.

This paper is motivated in part by some theorems of Y. Katznelson which hold for commutative Banach algebras. Let A be a semisimple Banach *-algebra with identity and hermitian involution. For the present assume that A is commutative. When $a \in A$, we denote the Gelfand transform of a by $\hat{a}$. Given a complex function ϕ defined on a subset $\mathscr{D}$ of the complex plane, we say ϕ operates on A if $\phi \circ \hat{a} \in \hat{A}$ whenever the range of $\hat{a}$ is in $\mathscr{D}$. When A is noncommutative, we say ϕ operates on A if ϕ operates on $C(t)$ for every selfadjoint element t in A. Let $\surd$ denote the positive square root function, with domain $\mathscr{D}$ the nonnegative reals. Katznelson proves in [5] that if A is commutative and $\surd$ operates on A, then A is B^*-equivalent. Therefore when A is noncommutative and $\surd$ operates on A, then A is locally B^*-equivalent. If this implies that A is B^*-equivalent, then Katznelson's Theorem extends to the noncommutative algebra A. In fact the question of whether Katznelson's Theorem holds for noncommutative algebras is equivalent to the question of whether local B^*-equivalence implies B^*-equivalence.

2. **Locally B^*-equivalent algebras.** In this section we establish those properties of locally B^*-equivalent algebras which we need in the subsequent sections.

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Throughout this section A is a Banach *-algebra. When A has no identity, let A_1 denote the usual Banach *-algebra formed by adjoining an identity to A. Then it is easy to verify that A is B^*-equivalent (locally B^*-equivalent) if and only if A_1 is B^*-equivalent (locally B^*-equivalent). This means that to show that a particular locally B^*-equivalent algebra A is B^*-equivalent, we may assume without loss of generality that A has an identity.

Now we establish some basic properties of locally B^*-equivalent algebras.

PROPOSITION 2.1. *Assume that A is a Banach *-algebra which is locally B^*-equivalent. Then*

(1) *A is semisimple.*

(2) ** is continuous on A.*

(3) ** is symmetric on A.*

(4) *There is a unique norm $|\cdot|$ on A which has the B^*-property, $|a^*a|=|a|^2$ for all $a\in A$.*

Proof. The radical R of A is a closed *-ideal of A. If $t\in R$ and $t=t^*$, then $C(t)\subset R$. Since $C(t)$ is B^*-equivalent and t has zero spectrum, then $t=0$. Therefore $R=0$. This proves (1). (2) follows from (1) by Johnson's Theorem [3, Theorem 2, p. 539].

Given $t\in A$, $t=t^*$, then the spectrum of t in $C(t)$ is real by [6, Lemma (4.8.1)(i), p. 240]. Then the spectrum of t in A is real. Therefore * is a hermitian involution on A. By Shirali's Theorem [2], * is symmetric on A. Therefore there exists a norm $|\cdot|$ on A with the B^*-property, $|a^*a|=|a|^2$ for all $a\in A$, by [6, Corollary (4.7.16), p. 237]. Assume that $|\cdot|_1$ is another norm on A with the B^*-property. Given $a\in A$, $C(a^*a)$ is a B^*-algebra in some norm. Then by [6, Corollary (4.8.6), p. 241], $|a^*a|=|a^*a|_1$. Therefore $|a|=|a|_1$. This completes the proof of the proposition.

In the next proposition we prove that commutative locally B^*-equivalent algebras are B^*-equivalent. Clearly any closed *-subalgebra of a locally B^*-equivalent algebra is locally B^*-equivalent. Therefore the proposition implies that every maximal commutative *-subalgebra of a locally B^*-equivalent algebra is B^*-equivalent.

PROPOSITION 2.2. *Assume that A is a commutative Banach *-algebra which is locally B^*-equivalent. Then A is B^*-equivalent.*

Proof. We may assume that A has an identity. Y. Katznelson has shown that when every continuous complex function operates on a commutative semisimple Banach algebra B, then B is B^*-equivalent [4, Theorem 2]. When B is a commutative semisimple Banach *-algebra with hermitian involution, then Katznelson's proof establishes that B is B^*-equivalent if every continuous real function of a real variable operates on B. Now assume that ϕ is a continuous real function with real domain $\mathscr{D}$. If $t\in A$ and $\hat{t}$ has range in $\mathscr{D}$, then $t=t^*$ by (1) and (3) of Proposition 2.1. Then $C(t)^\wedge$ is a sup norm complete *-subalgebra of $\hat{A}$. It follows that $\phi\circ\hat{t}\in\hat{A}$ by [6, Theorem (4.8.7), p. 241]. Therefore A is B^*-equivalent by Katznelson's Theorem.

The next result shows that the property of local B^*-equivalence is preserved under continuous $*$-homomorphisms. If B is any $*$-algebra, we denote by B_s the set of selfadjoint elements of B.

PROPOSITION 2.3. *Assume that A is locally B^*-equivalent and that I is a closed $*$-ideal of A. Then A/I is locally B^*-equivalent.*

Proof. Assume that $R \in A/I$ and $R = R^*$. $R = s + I$ for some $s \in A$. Set $t = (s+s^*)/2$. Since $s - s^* \in I$, then $s - t = (s-s^*)/2 \in I$. Therefore $R = t + I$ where $t \in A_s$. Now set $B = C(t) + I$. Define a map ϕ: $C(t)/(I \cap C(t)) \to B/I$ by

$$\phi(a + I \cap C(t)) = a + I$$

for $a \in C(t)$. ϕ is a $*$-isomorphism of $C(t)/I \cap C(t)$ onto B/I. It follows that B/I is B^*-equivalent. Then $R \in B/I$ which is a closed $*$-subalgebra of A/I. Therefore $C(R) \subset B/I$, and since B/I is B^*-equivalent, then $C(R)$ is B^*-equivalent. It follows that A/I is locally B^*-equivalent.

Next we prove a sequence of three lemmas. These lemmas are the basic ingredients in the proof of our main result, Theorem 4.1.

LEMMA 2.4. *Assume that A is locally B^*-equivalent. Assume that B is a $*$-subalgebra of A, and that I is a closed $*$-ideal of A such that $I \subset B$. Then if I is B^*-equivalent and B/I is B^*-equivalent, we have B is B^*-equivalent.*

Proof. Let $|\cdot|$ be the unique norm on A with the B^*-property (Proposition 2.1(4)). We prove that $|\cdot|$ is a complete norm on B. Let B^c be the completion of B with respect to $|\cdot|$. I is complete in the norm $|\cdot|$ by hypothesis, so that I is a closed $*$-ideal of B^c. Define the usual quotient norm $|a+I| = \inf_{b \in I} |a-b|$ on B^c/I. By [6, Theorem (4.9.2), p. 249], $|\cdot|'$ is a norm with the B^*-property on B^c/I. Since B/I is a B^*-algebra in some norm, B/I is complete in the norm $|\cdot|'$ by [6, Corollary (4.8.6), p. 241]. Assume now that $\{b_n\} \subset B$ and $|b_n - b_m| \to 0$. Then $|(b_n - b_m) + I|' \to 0$. Therefore there exists $b \in B$ such that $|(b_n - b) + I|' \to 0$. Then we can choose $\{a_n\} \subset I$ such that $|(b_n - b) - a_n| \to 0$. Then $|a_n - a_m| \to 0$, and since $|\cdot|$ is complete on I, there exists $a \in I$ such that $|a_n - a| \to 0$. Finally $|b_n - (b+a)| \to 0$, so that $|\cdot|$ is complete on B.

We will always denote the given norm in A by $\|\cdot\|$, and $\nu(a)$ is the spectral radius of an element a in A, $\nu(a) = \inf_n \|a^n\|^{1/n}$. When E is a subset of A, $\bar{E}$ is the closure of E in the norm $\|\cdot\|$.

LEMMA 2.5. *Assume that A is locally B^*-equivalent. Assume that D is a $*$-subalgebra of A, and that there exists $K > 0$ such that $K\nu(t) \geqq \|t\|$ for all $t \in D_s$. Then $\bar{D}$ is B^*-equivalent.*

Proof. $*$ is symmetric on A by Proposition 2.1(3). It follows by [6, Lemma 4.7.10, p. 234] that $\nu(h+k) \leqq \nu(h) + \nu(k)$ whenever $h, k \in A_s$. Given $h \in (\bar{D})_s$ and

$\varepsilon > 0$, we can choose $k \in D_s$ such that $\|h-k\| < \varepsilon$. Then

$$\begin{aligned} \|h\| &\leqq \|h-k\| + \|k\| \leqq \varepsilon + K\nu(k) \\ &\leqq \varepsilon + K(\nu(h) + \nu(k-h)) \leqq (1+K)\varepsilon + K\nu(h). \end{aligned}$$

Since ε was an arbitrary positive number, then $\|h\| \leqq K\nu(h)$. It follows that $\bar{D}$ is B^*-equivalent by Theorem 2.4 and Lemma 2.6 of [7].

Given $a, b \in A$, let $a \circ b = a + b - ab$. When a and b are commuting selfadjoint idempotents, then $a \circ b$ is a selfadjoint idempotent which in some sense is the least upper bound of a and b.

LEMMA 2.6. *Assume that A is locally B^*-equivalent. Assume that B is a *-subalgebra of A with the properties*:

(1) *For any $t \in B_s$, $C(t) \subset B$ and $C(t)$ is the closed linear span of the selfadjoint idempotents in $C(t)$.*

(2) *When f is a selfadjoint idempotent of B, then $(1-f)B(1-f) \neq 0$.*

Then there exists a selfadjoint idempotent $e \in B$ such that $(1-e)\bar{B}(1-e)$ is B^*-equivalent.

Proof. Assume that the lemma is false. Then $\bar{B}$ is not B^*-equivalent. Therefore by Lemma 2.5 with $K=2$, there exists $h_1 \in B_s$ such that $\nu(h_1) < \frac{1}{2}\|h_1\|$. Then by (1) there exists an element $s_1 \in C(h_1)$ such that $s_1 = \lambda_1 g_1 + \cdots + \lambda_n g_n$, where the λ_k are real scalars and the g_k are selfadjoint idempotents in $C(h_1)$, and such that $\nu(s_1) < \frac{1}{2}\|s_1\|$. Set $e_1 = g_1 \circ g_2 \circ \cdots \circ g_n$. Note that $s_1(1-e_1) = s_1(1-g_1)(1-g_2)\cdots(1-g_n) = 0$ and $(1-e_1)s_1 = 0$. By (2) $(1-e_1)B(1-e_1) \neq 0$. By assumption, $(1-e_1)\bar{B}(1-e_1)$ is not B^*-equivalent, and then, by Lemma 2.5 with $K=4$, there exists $h_2 \in (1-e_1)B \cdot (1-e_1)$ such that $h_2 = h_2^*$ and $\nu(h_2) < \frac{1}{4}\|h_2\|$. As before we can choose $s_2 \in C(h_2)$ such that $s_2 = \mu_1 f_1 + \cdots + \mu_m f_m$, where the μ_k are real scalars and the f_k are selfadjoint idempotents in $C(h_2)$, and such that $\nu(s_2) < \frac{1}{4}\|s_2\|$. Let $e_2 = e_1 \circ f_1 \circ \cdots \circ f_m$. Note that $s_1 s_2 = (s_1 e_1 s_2) = 0$ and $s_2 s_1 = 0$. Continuing in this fashion, we can choose a sequence $\{s_k\}$ such that each s_k is selfadjoint, $s_k s_j = s_j s_k = 0$ for $k \neq j$, and $\nu(s_k) < (\frac{1}{2})^k \|s_k\|$ for all $k \geqq 1$. Let C be a maximal commutative *-subalgebra of A which contains the sequence $\{s_k\}$. By Proposition 2.2, C is B^*-equivalent. This contradicts the inequalities $\nu(s_k) < (\frac{1}{2})^k \|s_k\|$, $k \geqq 1$. Therefore the lemma holds. ·

3. **Algebras with dense socle which are locally B^*-equivalent.** The socle of a semisimple algebra A is the sum of the minimal left ideals of A, or 0 if A has no minimal left ideals. In this section we prove that a Banach *-algebra with dense socle which is locally B^*-equivalent is B^*-equivalent. We need several facts about minimal ideals and the socle. Assume that A is a semisimple (complex) Banach algebra. An idempotent e of A is a minimal idempotent of A if $eAe = \{\lambda e \mid \lambda \text{ complex}\}$. When e is a minimal idempotent of A, then Ae is a minimal left ideal of A. Conversely every minimal left ideal of A is of the form Ae for some minimal idempotent e of A; see [6, pp. 45–46]. If in addition A has an involution * with the property that $a^*a = 0$ implies $a = 0$, then the idempotent e above may be chosen selfadjoint

by [6, Lemma (4.10.1), p. 261]. We denote the socle of A as S_A. Although defined as the sum of the minimal left ideals of A, S_A is also the sum of the minimal right ideals of A. Therefore given $s, t \in S_A$, there exist minimal idempotents of A, e_k, f_j, $1 \leqq k \leqq n$, $1 \leqq j \leqq m$, such that $s \in e_1A + \cdots + e_nA$ and $t \in Af_1 + \cdots + Af_m$. By [7, Lemma 5.1, p. 358], e_kAf_j is either one dimensional or 0 for all k and j. Therefore, $sAt \subset \sum_{k,j} e_kAf_j$ which is finite dimensional. Now assume that $\bar{S}_A = A$. In this case A has no primitive ideal which contains S_A. Therefore, A/S_A is a radical algebra. Then given an idempotent g in A, the residue class $g + S_A$ is an idempotent in the radical algebra A/S_A. Since radical algebras contain no nonzero idempotents, $g \in S_A$. Thus when $\bar{S}_A = A$, every idempotent of A is in S_A.

When X is a normed linear space, $\mathscr{F}(X)$ denotes the algebra of bounded operators on X which have finite dimensional range, and $\mathscr{C}(X)$ denotes the algebra of compact operators on X.

THEOREM 3.1. *Assume that A is a primitive Banach *-algebra with dense socle and that A is locally B^*-equivalent. Then A is *-isomorphic to $\mathscr{C}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.*

Proof. Assume that M is a minimal left ideal of A. Then there exists a selfadjoint minimal idempotent f of A such that $M = Af$. We introduce an inner product $(\cdot, \cdot)$ on Af by the rule $(xf, yf)f = fy^*xf$. That $(\cdot, \cdot)$ is an inner product on Af is verified in [6, Theorem (4.10.3), p. 261]. When $a \in A$, define T_a on Af by $T_a(xf) = axf$, $xf \in Af$. $a \to T_a$ is a *-representation of A into the bounded operators on the inner-product space Af, again by [6, Theorem (4.10.3)]. Set $K = \{a \in A \mid aAf = 0\}$. K is the kernel of the representation $a \to T_a$. Since $KAf = 0$ and 0 is a primitive ideal of A by hypothesis, then $K = 0$ by [6, Theorem (2.2.9)(iv), p. 54]. Therefore the representation $a \to T_a$ is faithful.

If A is finite dimensional, then Af is a finite-dimensional Hilbert space, and $a \to T_a$ is a faithful *-representation of A onto $\mathscr{C}(Af)$. In what follows we assume that A is not finite dimensional. Now we verify that A satisfies (1) and (2) of Lemma 2.6. Suppose that g is a selfadjoint idempotent of A such that $(1-g)A(1-g) = 0$. Let $|\cdot|$ be the unique norm on A with the B^*-property (Proposition 2.1(4)). Then $|x(1-g)|^2 = |(1-g)x|^2 = |(1-g)x^*x(1-g)| = 0$ for all $x \in A$. It follows that $A = gAg$ which is finite dimensional since $g \in S_A$. This contradiction proves that, for any selfadjoint idempotent g of A, $(1-g)A(1-g) \neq 0$. This is (2) of Lemma 2.6. Now assume that $t \in A_s$. $C(t)$ is B^*-equivalent, and therefore, from [1, Corollary, p. 517] and [8, Theorem 4.1, p. 42], $C(t)$ has dense socle. It follows that $C(t)$ is the closed linear span of its selfadjoint idempotents. This verifies (1) of Lemma 2.6. Therefore Lemma 2.6 implies that there exists a selfadjoint idempotent $e \in A$ such that $(1-e)A(1-e)$ is B^*-equivalent. $(1-e)A(1-e)$ is complete in the norm $|\cdot|$. $(1-e)A(1-e)$ is a nonzero semisimple Banach *-algebra with dense socle. Choose f a selfadjoint minimal idempotent of $(1-e)A(1-e)$. Then $ef = fe = 0$ so that, for any $x \in A$, $fxf = f(1-e)x(1-e)f = \lambda f$ for some scalar λ. Therefore f is a minimal idempotent of A. Let $(\cdot, \cdot)$ be the inner-product introduced on Af as in the first paragraph of the proof. Let $|\cdot|_2$ be the corresponding norm on Af. Then for any $x \in A$, $|xf|^2 = |fx^*xf| = (xf, xf)|f| = |xf|_2^2$. Therefore, $|\cdot|$ and $|\cdot|_2$ are identical on Af. Also $(1-e)A(1-e)f = (1-e)Af$ is a minimal left ideal of $(1-e)A(1-e)$ by the choice of f. Then $(1-e)Af$ is a closed, and hence complete, subspace of $(1-e)A(1-e)$ in the norm $|\cdot|$. Therefore $|\cdot|_2$ is a complete norm on $(1-e)Af$. But eAf is finite dimensional, hence complete. Therefore $Af = eAf \oplus (1-e)Af$ is a Hilbert space. We denote the Hilbert space Af by $\mathscr{H}$.

As in the first paragraph of the proof, $a \to T_a$ is a faithful *-representation of A into the bounded operators on $\mathscr{H}$. When $t \in S_A$, then tAf is finite dimensional, so that $T_t \in \mathscr{F}(\mathscr{H})$. Let $B = \{T_a \mid a \in A\}$. When T is a bounded operator on $\mathscr{H}$, denote the operator norm of T by $\|T\|_{\text{op}}$. By Proposition 2.1(4), $|a| = \|T_a\|_{\text{op}}$ for all $a \in A$. The norm $\|\cdot\|$ on A dominates the norm $|\cdot|$ by [6, Corollary (4.1.16), p. 187]. A has dense socle in the norm $\|\cdot\|$, and therefore A has dense socle in the norm $|\cdot|$. It follows that $\{T_s \mid s \in S_A\}$ is dense in B in the norm $\|\cdot\|_{\text{op}}$. Then every operator T_a in B is the operator norm limit of operators with finite-dimensional range, so that $B \subset \mathscr{C}(\mathscr{H})$.

It remains to be shown that $B = \mathscr{C}(\mathscr{H})$. Given F an operator with 1-dimensional range on $\mathscr{H}$, then there exist $\phi, \psi \in \mathscr{H}$ such that $F(\gamma) = (\gamma, \phi)\cdot\psi$ for all $\gamma \in \mathscr{H}$. $\phi = uf$ and $\psi = vf$ for some $u, v \in A$. Then $T_{vfu^*}(xf) = vfu^*xf = (xf, uf)vf = F(xf)$ for all $xf \in Af$. Therefore $F = T_{vfu^*} \in B$, and it follows that $\mathscr{F}(\mathscr{H}) \subset B$. Given $T = T^* \in \mathscr{C}(\mathscr{H})$, then by the Spectral Theorem for compact operators, there exists a sequence of real scalars $\{\lambda_k\}$, and a corresponding orthogonal sequence of selfadjoint projections with finite-dimensional range $\{E_k\}$, such that $T = \sum_{k=1}^{+\infty} \lambda_k E_k$, convergence being in operator norm. We have shown that $\{E_k\} \subset B$. Let C be a maximal commutative *-subalgebra of B containing $\{E_k\}$. By Proposition 2.2, C is complete in the operator norm. Therefore $T \in C \subset B$, and the proof is complete.

Now we consider the general case when A is a locally B^*-equivalent algebra with dense socle.

THEOREM 3.2. *Assume that A is a Banach *-algebra with dense socle, and that A is locally B^*-equivalent. Then A is B^*-equivalent.*

Proof. Just as in the proof of Theorem 3.1, there exists a selfadjoint idempotent e in A such that $(1-e)A(1-e)$ is B^*-equivalent. Also since A has dense socle, $e \in S_A$. If f is a minimal idempotent of A, then $(AfA)^-$ is a minimal closed two-sided ideal of A. It follows that every element of the socle of A is contained in a finite sum of minimal closed two-sided ideals of A. Furthermore, if M and N are distinct minimal closed two-sided ideals of A, then $M \cdot N \subset M \cap N = 0$. Therefore there exist minimal closed two-sided ideals of A, M_k, $1 \leqq k \leqq n$, such that $e \in M_1 + M_2 + \cdots + M_n$ and $M_k M_j = 0$ when $k \neq j$. Also setting $D = \{x \in A \mid xM_k = 0, 1 \leqq k \leqq n\}$ then $S_A \subset M_1 \oplus M_2 \oplus \cdots \oplus M_n \oplus D$. Denote this sum as B. Each M_k is a closed *-ideal of A with dense socle and a primitive algebra. Therefore by Theorem 3.1,

M_k is B^*-equivalent for $1 \leqq k \leqq n$. Since $e \in M_1 \oplus \cdots \oplus M_n$, then $D \subset (1-e)A(1-e)$. Therefore the closed *-algebra D is B^*-equivalent. Then B is B^*-equivalent, hence complete in the norm $|\cdot|$. But since $S_A \subset B$, B is dense in A in the norm $|\cdot|$. Then $A = B$ which completes the proof.

4. **The main result.** In this section we assume that A is locally B^*-equivalent, and that whenever $t \in A_s$, the spectrum of t is at most countable. Then the spectrum of t is totally disconnected, and it follows that $C(t)$ is the closed linear span of its selfadjoint idempotents; see [6, p. 293]. We prove now that A is B^*-equivalent.

THEOREM 4.1. *Assume that A is a Banach *-algebra which is locally B^*-equivalent and which has the property that whenever $t \in A_s$, then the spectrum of t is at most countable. Then A is B^*-equivalent.*

Proof. We assume that $A \neq 0$. When E is a subset of A, let $L[E] = \{a \in A \mid aE = 0\}$. By [1, Theorem 2.3, p. 513], $L[\bar{S}_A] = 0$. Also $\bar{S}_A$ is B^*-equivalent by Theorem 3.2. Let $\mathscr{I}$ be the set of all closed *-ideals I of A which are B^*-equivalent and have the property that $L[I] = 0$. $\mathscr{I}$ is partially ordered by inclusion and nonempty since $\bar{S}_A \in \mathscr{I}$. Let $\mathscr{C}$ be any chain in $\mathscr{I}$. Set $K = \bigcup_{J \in \mathscr{C}} J$. Suppose for some selfadjoint idempotent f in K, $(1-f)K(1-f) = 0$. Then $(1-f)xx^*(1-f) = 0$ for all $x \in K$, and therefore $(1-f)x = 0$ for all $x \in K$. Then $A(1-f) \subset L[K] = 0$, so that f is a right identity for A. But $f \in J$ for some J in $\mathscr{C}$. In this case $J = A$, and we are done. Therefore we may assume that $(1-f)K(1-f) \neq 0$ for every selfadjoint idempotent $f \in K$. Assume now that $t \in K_s$. Then $t \in J$ for some $J \in \mathscr{C}$, and then $C(t) \subset J \subset K$, since J is closed. Also $C(t)$ is the closed linear span of its selfadjoint idempotents by the remarks preceding the statement of the theorem. This verifies (1) and (2) of Lemma 2.6. Therefore there exists a selfadjoint idempotent $e \in K$ such that $(1-e)\bar{K}(1-e)$ is B^*-equivalent. $e \in J$ for some J in $\mathscr{C}$ and J is B^*-equivalent. Let $|\cdot|$ be the unique norm on A with the B^*-property. J and $(1-e)\bar{K}(1-e)$ are complete with respect to $|\cdot|$. Assume $\{x_n\} \subset \bar{K}$ and $|x_n - x_m| \to 0$. Set $y_n = (1-e)x_n e + e x_n$. Note that $\{y_n\} \subset J$. Then for all $n \geqq 1$, $x_n = (1-e)x_n(1-e) + y_n$. Then $\{y_n\}$ is Cauchy in J and $(1-e)x_n \cdot (1-e)$ is Cauchy in $(1-e)\bar{K}(1-e)$ in the norm $|\cdot|$. It follows that $\{x_n\}$ converges in $\bar{K}$ in the norm $|\cdot|$. Therefore $\bar{K}$ is a closed B^*-equivalent *-ideal of A in $\mathscr{I}$, and $\bar{K}$ is an upper bound for $\mathscr{C}$. By Zorn's Lemma $\mathscr{I}$ has a maximal element I. Suppose that $I \neq A$. A/I is locally B^*-equivalent by Proposition 2.3. A/I is semisimple, and every selfadjoint element in A/I has at most countable spectrum. Let T be the closure of the socle of A/I in A/I. Note that $T \neq 0$ by [1, Theorem 2.3, p. 513]. Let $M = \{x \in A \mid x + I \in T\}$. M is a closed *-ideal of A which properly contains I. Also $M/I = T$. I is B^*-equivalent, and by Theorem 3.2, M/I is B^*-equivalent. Therefore by Lemma 2.4, M is B^*-equivalent. This contradicts the maximality of I. Therefore $A = I$, so that A is B^*-equivalent.

COROLLARY 4.2. *Assume that A is a Banach *-algebra which is locally B^*-equivalent and such that A has a separable dual space. Then A is B^*-equivalent.*

Proof. Let $|\cdot|$ be the unique norm on A with the B^*-property. Given $t \in A_s$, there exists a number $m > 0$ such that $m|a| \geqq \|a\|$ for all $a \in C(t)$. Since the dual space of A is separable, then the dual space of $C(t)$ is separable. If ϕ and ψ are any two distinct nonzero multiplicative linear functionals on $C(t)$, then

$$\begin{aligned}\|\phi-\psi\| &= \sup\{|(\phi-\psi)(a)| \mid a \in C(t),\ \|a\| \leqq 1\}\\ &\geqq (1/m)\sup\{|(\phi-\psi)(a)| \mid a \in C(t),\ |a| \leqq 1\} = (2/m).\end{aligned}$$

It follows that the carrier space of $C(t)$ is at most countable. Therefore the spectrum of t is at most countable. Then the corollary follows from Theorem 4.1.

COROLLARY 4.3. *Assume that A is a Banach *-algebra with the property that, whenever $t \in A_s$, then the spectrum of t is at most countable. Then if $\surd$ operates on A, we have A is B^*-equivalent.*

The corollary follows from Theorem 4.1 and [5, Corollaire, p. 169].

COROLLARY 4.4. *Assume that A is a Banach *-algebra with the property that, whenever $t \in A_s$, then the spectrum of t is at most countable. Then if every orthogonal sequence of selfadjoint idempotents of A is bounded, we have A is B^*-equivalent.*

The corollary follows from Theorem 4.1 and the results of Katznelson [5, pp. 167–169].

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