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THE ENVELOPE OF HOLOMORPHY OF RIEMANN DOMAINS OVER A COUNTABLE PRODUCT OF COMPLEX PLANES

BY
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Abstract. This paper deals with the problem of constructing envelopes of holomorphy for Riemann domains over a locally convex space. When this locally convex space is a countable product of complex planes the existence of the envelope of holomorphy is proved and the domains of holomorphy are characterized.

For the Riemann domains over the cartesian product C^N of a countable number of complex planes, the domains of holomorphy are characterized and the existence of the envelope of holomorphy is proved. Also, for Riemann domains over a complex separated locally convex space E such that the closed convex hull of every compact subset is compact, the existence of the normal envelope of holomorphy is proved.

Let $(\boldsymbol{U}, \varphi)$ be a Riemann domain over E. This means that $\boldsymbol{U}$ is a connected separated topological space and φ is a local homeomorphism from $\boldsymbol{U}$ into E. If $\boldsymbol{a} \in \boldsymbol{U}$ and $A \subset E$, $\boldsymbol{a}+A$ is defined by

$$\boldsymbol{a}+A = [\varphi|\boldsymbol{W}]^{-1}(\varphi(\boldsymbol{a})+A),$$

$\boldsymbol{W}$ being an open subset of $\boldsymbol{U}$ where φ is a homeomorphism, $\boldsymbol{a} \in \boldsymbol{W}$, and $\varphi(\boldsymbol{a})+A \subset \varphi(\boldsymbol{W})$. When A has only one element h, $\boldsymbol{a}+h$ denotes the unique element of $\boldsymbol{a}+\{h\}$. If $\boldsymbol{B}$ is a subset of $\boldsymbol{U}$,

$$\boldsymbol{B}+A = \bigcup_{\boldsymbol{b}\in\boldsymbol{B}} (\boldsymbol{b}+A)$$

where $\boldsymbol{b}+A$ has the meaning just stated. A complex mapping f defined in $\boldsymbol{U}$ is holomorphic if, for every $\boldsymbol{u}$ in $\boldsymbol{U}$, there is an open convex balanced neighborhood U of zero in E and a sequence of continuous n-homogeneous polynomials in E: $(n!)^{-1}\,\hat{d}^n f(\boldsymbol{u})$, $n=0, 1, \ldots$, such that $\boldsymbol{u}+U \subset \boldsymbol{U}$ and

$$f(\boldsymbol{u}+h) = \sum_{n=0}^{\infty} (n!)^{-1}\,\hat{d}^n f(\boldsymbol{u})(h),$$

the series converging uniformly for h in U. In the algebra $\mathscr{H}(\boldsymbol{U})$ of all complex

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holomorphic mappings in U it is considered the Nachbin topology generated by all algebra seminorms ported by compact subsets of U [1]. Let $S(U)$ be the spectrum of $\mathscr{H}(U)$, that is, the set of all continuous homomorphisms from $\mathscr{H}(U)$ onto C.

PROPOSITION 1. *If $h \in S(U)$, there is a unique $a_h \in E$ such that $T(a_h) = h(T \circ \varphi)$ for all T in the topological dual E' of E.*

Proof. There is a compact subset K of U such that

$$|h(f)| \leqq \sup\{|f(k)|;\, k \in K\}$$

for every f in $\mathscr{H}(U)$. This fact is denoted by $h \prec K$. Let K be the closed convex hull of $\varphi(K)$. If $\mathscr{T} = \{T_1, T_2, \ldots, T_n\}$ is a finite subset of E' and

$$\alpha_{\mathscr{T}} = \{x \in K;\, T_i(x) = h(T_i \circ \varphi),\, i = 1, 2, \ldots, n\},$$

$\alpha_{\mathscr{T}}$ is nonempty. If this was not the case, $T = (T_1, \ldots, T_n)$ would be a continuous mapping from E into C^n such that $(h(T_1 \circ \varphi), \ldots, h(T_n \circ \varphi)) \notin T(K)$. Hence there would be a complex linear mapping G in C^n such that

$$\operatorname{Re} G(h(T_1 \circ \varphi), \ldots, h(T_n \circ \varphi)) > \sup\{\operatorname{Re} G \circ T(x);\, x \in K\}.$$

This would imply that the holomorphic mapping in $U f = \exp G \circ T \circ \varphi$ would satisfy the inequality

$$|h(f)| > \sup\{|f(u)|;\, u \in K\},$$

against the assumption that $h \prec K$. Since K is compact and the collection $\{\alpha_{\mathscr{T}};\, \mathscr{T} \subset E',\ \mathscr{T}$ finite$\}$ of closed subsets of K satisfies the finite intersection property, the intersection of the whole collection is nonempty. This intersection has a unique point because E' separates the points of E.

Let $h \prec K$ and U be an open balanced convex neighborhood of zero in E such that $K + U \subset U$ and $K + L$ is compact for every compact subset L of U. If $u \in U$ and $f \in \mathscr{H}(U)$,

$$h_u(f) = \sum_{n=0}^{\infty} (n!)^{-1} h(\hat{d}^n f(\cdot)(u)),$$

defines an element h_u of $S(U)$ such that $a_{h_u} = a_h + u$. For all h, K and U as above, the collection of all

$$N_{h,U} = \{h_u;\, u \in U\}$$

defines a topology in $S(U)$ such that the mapping $\pi\colon h \in S(U) \mapsto a_h \in E$ is a local homeomorphism which maps the connected open set $N_{h,U}$ onto $\pi(h) + U$ homeomorphically. The proofs of these facts are the same as those of the case in which E is a Banach space [2]. It is supposed now that $\mathscr{H}(U)$ separates the points of U. If $i(u) \in S(U)$ is the evaluation homomorphism associated to $u \in U$, then $i\colon U \to S(U)$ is a biholomorphism from U onto an open subset U_S of $S(U)$. Let $(E(U), \pi)$ be the Riemann domain over E, where $E(U)$ is the connected component of $S(U)$ containing U_S. If $f \in \mathscr{H}(U)$, its extension $f' \in \mathscr{H}(E(U))$ is defined by $f'(h) = h(f)$ for all h in $E(U)$.

Let $(\boldsymbol{E}, \psi)$ and $(\boldsymbol{D}, \varphi)$ be Riemann domains over E such that $\boldsymbol{D}$ is canonically identified to an open subset of $\boldsymbol{E}$ by means of a biholomorphism. $(\boldsymbol{E}, \boldsymbol{D})$ is an extension pair if for each f in $\mathscr{H}(\boldsymbol{D})$ there is $f' \in \mathscr{H}(\boldsymbol{E})$ such that $f'|\boldsymbol{D}=f$. If the mapping $f \in \mathscr{H}(\boldsymbol{D}) \mapsto f' \in \mathscr{H}(\boldsymbol{E})$ is a homeomorphism, the extension pair $(\boldsymbol{E}, \boldsymbol{D})$ is normal.

$(E(\boldsymbol{U}), \boldsymbol{U})$ is an extension pair. It is normal by the following result.

THEOREM 1. *Let $(\boldsymbol{E}, \boldsymbol{D})$ be an extension pair. If, for every $\boldsymbol{x}$ in $\boldsymbol{E}$, the mapping $f \in \mathscr{H}(\boldsymbol{D}) \mapsto f'(\boldsymbol{x}) \in \boldsymbol{C}$ is linear and continuous, then $(\boldsymbol{E}, \boldsymbol{D})$ is normal.*

Proof. Let π and φ be the local homeomorphisms defining the Riemann domains $\boldsymbol{E}$ and $\boldsymbol{D}$ respectively. To prove the theorem it is enough to show that for every algebra seminorm p in $\mathscr{H}(\boldsymbol{E})$, ported by a compact subset $\boldsymbol{L}$ of $\boldsymbol{E}$, there is an algebra seminorm q in $\mathscr{H}(\boldsymbol{D})$, ported by some compact subset $\boldsymbol{K}$ of $\boldsymbol{D}$, such that $p(f') \leqq q(f)$ for all f in $\mathscr{H}(\boldsymbol{D})$. By the assumptions of the theorem, for each $\boldsymbol{x}$ in $\boldsymbol{L}$ there is a compact subset $\boldsymbol{K}_x$ of $\boldsymbol{D}$ such that $|f'(\boldsymbol{x})| \leqq \sup \{|f(\boldsymbol{t})|\,;\, \boldsymbol{t} \in \boldsymbol{K}_x\}$ for every f in $\mathscr{H}(\boldsymbol{D})$. Let V_x be a closed convex balanced neighborhood of zero in E such that V_x is contained in the interior $(2V_x)^\circ$ of $2V_x$ and (i) $\boldsymbol{x}+2V_x \subset \boldsymbol{E}$, (ii) $\boldsymbol{K}_x+4V_x \subset \boldsymbol{D}$, (iii) $\boldsymbol{K}_x+L$ is compact for all compact subsets L of $4V_x$. Since $\boldsymbol{L}$ is compact, there is a finite cover $\{\boldsymbol{x}_i+V_{x_i}=\boldsymbol{x}_i+V_i\,;\, i=1, 2, \dots, n\}$ of $\boldsymbol{L}$. For $i=1, 2, \dots, n$ we set:

$$\boldsymbol{L}_i = \boldsymbol{K}_{x_i} + \bigcup_{|\lambda| \leqq 2} \lambda\{[\pi(\boldsymbol{L} \cap (\boldsymbol{x}_i + V_i)) - \pi(\boldsymbol{x}_i)] \cap V_i\}.$$

Hence $\boldsymbol{L}_i \subset \boldsymbol{K}_{x_i}+2V_i \subset \boldsymbol{D}$ and $\boldsymbol{L}_i$ is compact. Let $\boldsymbol{K}$ be the union of the $\boldsymbol{L}_i$, $i=1, 2, \dots, n$, which is a compact subset of $\boldsymbol{D}$. For every open subset $\boldsymbol{W}$ of $\boldsymbol{D}$ containing $\boldsymbol{K}$, there is an open balanced neighborhood A of zero in E such that

$$\boldsymbol{W}_i = \boldsymbol{K}_{x_i} + \bigcup_{|\lambda| \leqq 2} \lambda\{[\pi((\boldsymbol{L}+A) \cap (\boldsymbol{x}_i+(2V_i)^\circ)) - \pi(\boldsymbol{x}_i)] \cap (2V_i)^\circ\}$$

is an open subset of $\boldsymbol{D}$ containing $\boldsymbol{L}_i$ for all $i=1, 2, \dots, n$. The union of all $\boldsymbol{W}_i$ contains $\boldsymbol{K}$. The set A may be chosen in such a way that this union is a subset of $\boldsymbol{W}$. Now

$$B(\boldsymbol{W}) = \bigcup_{i=1}^{n} [(\boldsymbol{L}+A) \cap (\boldsymbol{x}_i+(2V_i)^\circ)]$$

is an open subset of $\boldsymbol{E}$ containing $\boldsymbol{L}$. If $\boldsymbol{x} \in B(\boldsymbol{W})$, then $\boldsymbol{x}$ is in $\boldsymbol{x}_i+(2V_i)^\circ$ for some i, $\pi(\boldsymbol{x})-\pi(\boldsymbol{x}_i)$ belongs to $(2V_i)^\circ$ and, for $|\lambda| \leqq 1$, $\boldsymbol{x}_i+\lambda(\pi(\boldsymbol{x})-\pi(\boldsymbol{x}_i))$ is in $\boldsymbol{E}$. The function $g(\lambda)=f'(\boldsymbol{x}_i+\lambda(\pi(\boldsymbol{x})-\pi(\boldsymbol{x}_i)))$ defined in a neighborhood of the set

$$\{\lambda \in \boldsymbol{C};\, |\lambda| \leqq 1\}$$

is such that:

$$g(\lambda) = \sum_{m=0}^{\infty} \lambda^m (m!)^{-1} g^{(m)}(0) \quad \text{and} \quad f'(\boldsymbol{x}) = g(1) = \sum_{m=0}^{\infty} (m!)^{-1} g^{(m)}(0).$$

Therefore

$$\begin{aligned} |f'(\boldsymbol{x})| &\leqq \sum_{m=0}^{\infty} (m!)^{-1} |\hat{d}^m f'(\boldsymbol{x}_i)(\pi(\boldsymbol{x})-\pi(\boldsymbol{x}_i))| \\ &\leqq \sum_{m=0}^{\infty} (m!)^{-1} \sup_{\boldsymbol{k} \in \boldsymbol{K}_{x_i}} |\hat{d}^m f(\boldsymbol{k})(\pi(\boldsymbol{x})-\pi(\boldsymbol{x}_i))|. \end{aligned} \tag{1}$$

If $k \in K_{x_i}$, then $k+\lambda(\pi(x)-\pi(x_i))$ is in D for all $|\lambda| \leqq 2$. The function $g_1(\lambda) = f(k+\lambda(\pi(x)-\pi(x_i)))$ is defined for all $|\lambda| \leqq 2$. The Cauchy inequalities for this function at 0 imply that

$$|\hat{d}^m f(k)(\pi(x)-\pi(x_i))| \leqq m!2^{-m} \sup_{t \in W_i} |f(t)|$$

for $m=0, 1, 2, \ldots$. These inequalities and (1) imply

$$|f'(x)| \leqq \sup_{t \in W_i} |f(t)| \cdot \sum_{m=0}^{\infty} 2^{-m} \leqq 2 \sup_{t \in W} |f(t)|.$$

Therefore

$$\sup_{x \in B(W)} |f'(x)| \leqq 2 \sup_{t \in W} |f(t)|$$

for every open subset W of D such that K is contained in W. It follows that for every open subset W of D containing K there is $c(B(W))>0$ such that

$$p(f') \leqq 2c(B(W)) \sup_{t \in W} |f(t)|$$

for all f in $\mathscr{H}(D)$. The supremum of the family of all algebra seminorms s in $\mathscr{H}(D)$, ported by K and such that

$$s(f) \leqq 2c(B(W)) \sup_{t \in W} |f(t)|$$

for all open sets W containing K and all f in $\mathscr{H}(D)$, is a member of this family. This supremum is the seminorm q we need to complete the proof of this theorem.

Now it is easy to show the following result:

Proposition 2. *If (U, φ) is a Riemann domain over E such that $\mathscr{H}(U)$ separates the points of U, then $(E(U), \pi)$ and the biholomorphism i from U onto the open subset U_S of $E(U)$ are such that* (a) *$\mathscr{H}(E(U))$ separates the points of $E(U)$*; (b) *$(E(U), U)$ is a normal extension pair. Moreover, $(E(U), \pi)$ is maximum relative to* (a) *and* (b) *in the following sense: if (M, ψ) is a Riemann domain over E and j is a biholomorphism from U onto an open subset U_M of M and* (a) *and* (b) *are satisfied when we replace $E(U)$ by M, then M may be identified to an open subset of $E(U)$ by a biholomorphism preserving the points of U.*

Throughout the remaining part of this paper $E=C^N$ and the points and the subsets of C^n are identified to the points and subsets of $C^n \times (0, 0, \ldots) \subset C^N$, $n=1, 2, \ldots$.

A Riemann domain (U, φ) over C^N is of order n at a point u of U if n is the smallest positive integer such that there is an open polydisc B in $\pi_n^{-1}(C^N)$ with center 0 for which $u+v \in U$ for every v in $\pi_n^{-1}(B)$. Here π_n denotes the projection mapping from C^N onto the space of the first n variables. (U, φ) is of order (at most) n in a subset A of U if (U, φ) is of order (at most) n at each point of A. (U, φ) is locally pseudoconvex if (U_V, φ_V) is pseudoconvex for each affine subspace V of C^N of dimension two. U_V denotes the topological subspace $\varphi^{-1}[\varphi(U) \cap V]$ of U and

φ_V denotes the restriction of φ to U_V. In this case, for each v in V, the mapping $z \in U_V \mapsto -\log \delta_{U_V}(z, v) \in C$ is plurisubharmonic. Recall that

$$\delta_{U_V}(z, v) = \inf \{|\lambda| ; u+\lambda v \in U_V\}.$$

PROPOSITION 3. *If (U, φ) is a locally pseudoconvex Riemann domain over C^N, then there is a positive integer n such that (U, φ) is of order n in U and*

$$\varphi(U) = \pi_n \circ \varphi(U) \times C^{N-[0,n-1]}.$$

LEMMA 1. *Let (U, φ) be locally pseudoconvex and of order n at a point u of U. Let r be the largest real positive number such that $u+b \in U$ for each b in the open polydisc B_r in $\pi_n(C^N)$ with center 0 and radius r. Then $u+v \in U$ for each v in $\pi_n^{-1}(B_r)$. (U, φ) is of order at most n at each one of these $u+v$.*

Proof. $\varphi(u)$ may be considered equal to zero with no loss of generality. Let $w=(w_j)_{j\in N} \in \pi_n^{-1}(B_r)$. If $\pi_n(w)=0$, then $u+w \in U$ because (U, φ) is of order n at u. If $\pi_n(w)\neq 0$ and $w_j=0$ for each $j \geqq n$, $u+w \in U$ because $w \in B_r$. If $\pi_n(w)\neq 0$ and $w_j\neq 0$ for some $j\geqq n$, consider $z=(z_j)_{j\in N}=(0, \ldots, 0, w_n, w_{n+1}, \ldots) \in C^N$. The subspace V of C^N generated by z and w has dimension two. Since $(U_V, \varphi| U_V)$ is pseudoconvex, $-\log \delta_{U_V}(t, w-z)$ is a plurisubharmonic function of t in U_V. (U, φ) of order n at u implies that there are positive real numbers $\varepsilon_0, \ldots, \varepsilon_{n-1}$ such that $u+v \in U$ for each v in C^N such that $|v_i| < \varepsilon_i$, $i=0, 1, \ldots, n-1$. Hence $u+\lambda z \in U_V$ for each $\lambda \in C$ because $\lambda z_i=0$, $i=0, 1, \ldots, n-1$, and $\varphi(u+\lambda z)=\lambda z \in V$. Consequently $-\log \delta_{U_V}(u+\lambda z, w-z)$ is a subharmonic function of λ in C. If ε is the minimum of the ε_i, $i=0, 1, \ldots, n-1$ and δ is the product of ε by the inverse of the supremum of the $|w_i|$, $i=0, 1, \ldots, n$, then $|\alpha(w_i-z_i)| < \varepsilon$ for each $|\alpha| < \delta$ and $i=0, 1, \ldots, n-1$. It follows that $u+\lambda z+\alpha(w-z)$ is in U_V for all λ in C and $|\alpha| < \delta$. Thus $-\log \delta_{U_V}(u+\lambda z, w-z)$ is a bounded above subharmonic function of λ in C, hence constant. Since $\delta_{U_V}(u+0z, w-z)>1$, $\delta_{U_V}(u+z, w-z)>1$ and $u+w \in U$.

LEMMA 2. *Let (U, φ) and u be as in Lemma 1. Let W be an open connected neighborhood of u such that $\varphi|W$ is a homeomorphism from W onto $\varphi(u)+A_0\times \cdots \times A_s \times C^{N-[0,s]}$, where each A_i is an open ball in C with center 0 and $s \geqq n-1$. Then (U, φ) is of order at most n in W and $\varphi(U) \supset \varphi(u)+A_0 \times \cdots \times A_{n-1} \times C^{N-[0,n-1]}$.*

Proof. $\varphi(u)$ may be considered equal to 0 without any loss of generality. Let $W_n=A_0\times \cdots \times A_{n-1}$ and W_n' be the set of all points w of W_n such that (U, φ) is of order at most n at $[\varphi|W]^{-1}(w)$. By Lemma 1, if $w \in W_n'$, W_n' contains the largest polydisc of radius $r>0$ with center w which is contained in W_n. Since $0 \in W_n'$, it follows that $W_n=W_n'$. If $w \in W$, $\varphi(w) \in W_n' \times C^{N-[0,n-1]}$. Hence, applying Lemma 1 for the case $u=\varphi^{-1}[\pi_n\varphi(w)]$, it is easy to see that (U, φ) is of order at most n at $w=u+(\varphi(w)-\pi_n\varphi(w))$.

Proof of Proposition 1. Let V be the set of all points of U where (U, φ) is of order at most n. By Lemma 2, V is open. Let $(x_k)_{k=0}^{\infty}$ be a sequence of points of V converging to x in U. Let W be an open connected neighborhood of x in U such that $\varphi|W$ is a homeomorphism from W onto $\varphi(x)+A_0\times\cdots\times A_s\times C^{N-[0,s]}$, where each A_i is an open ball in C with center 0. Thus $x_k\in W$ for k large enough and, by Lemma 2, (U,φ) is of order at most n in W. Hence $x\in V$ and V is closed in U. Since U is connected, V is equal to U. Now the remaining part of the proof follows easily.

PROPOSITION 4. *Let (U,φ) be a locally pseudoconvex Riemann domain over C^N. There is $n>0$ in N such that (U,φ) is of order n in U and (U_n,φ_n) is a manifold of holomorphy spread over C^n if $U_n=\varphi^{-1}[\pi_n\circ\varphi(U)]$ and $\varphi_n=\varphi|U_n$.*

Proof. Proposition 1 implies that there is a positive n in N such that (U,φ) is of order n in U. Thus $\varphi(U)=\pi_n\circ\varphi(U)\times C^{N-[0,n-1]}$. If (U_n,φ_n) is defined as above, it is a manifold spread over C^n. Since (U,φ) is locally pseudoconvex, (U_n,φ_n) is locally pseudoconvex, hence a manifold of holomorphy spread over C^n.

PROPOSITION 5. *Let (U,φ) be a Riemann domain over C^N of order n in U and let v be a point of C^N. Consider the manifolds spread over C^n (U_n,φ_n) and (V_n,ψ_n) given by $U_n=\varphi^{-1}[\pi_n\circ\varphi(U)]$, $V_n=\varphi^{-1}[v-\pi_n(v)+\pi_n\circ\varphi(U)]$, $\varphi_n=\varphi|U_n$, $\psi_n=\varphi|V_n$. Then there is a biholomorphism between them. In particular, if (U_n,φ_n) is of holomorphy, (V_n,ψ_n) is also of holomorphy.*

Proof. Consider the mappings

$$b_1\colon x\in V_n\mapsto x+[\pi_n\circ\varphi(x)-\varphi(x)]\in U_n,$$

$$b_2\colon z\in U_n\mapsto z+[v-\pi_n(v)]\in V_n.$$

It is easy to see that $b_1\circ b_2$ and $b_2\circ b_1$ are the identity mappings in U_n and in V_n respectively. They are also local homeomorphisms. In fact consider x in V_n and an open neighborhood W' of x in U such that $\varphi|W'$ is a homeomorphism and $\varphi(W')=\varphi(x)+A_0\times\cdots\times A_t\times C^{N-[0,t]}$, where each A_i is an open ball in C with center 0. Let W be an open neighborhood of $x+[\pi_n\varphi(x)-\varphi(x)]$ in U such that $\varphi|W$ is a homeomorphism and $\varphi(W)=\pi_n\varphi(x)+B_0\times\cdots\times B_s\times C^{N-[0,s]}$ is contained in $\pi_n\varphi(x)=A_0\times\cdots\times A_t\times C^{N-[0,t]}$, each B_i being an open ball in C with center 0. Let W'' be the open neighborhood of $x\in U$ given by

$$[\varphi|W']^{-1}[\varphi(x)+B_0\times\cdots\times B_s\times C^{N-[0,s]}].$$

Now it is easy to see that $b_1|W''\cap V_n$ is a homeomorphism from $W''\cap V_n$ onto $W\cap U_n$. It is also easy to verify that $f\circ b_1\in\mathscr{H}(V_n)$ and $g\circ b_2\in\mathscr{H}(U_n)$ for every $f\in\mathscr{H}(U_n)$ and $g\in\mathscr{H}(V_n)$.

A Riemann domain (U,φ) over C^N is a domain of holomorphy if there is $f\in\mathscr{H}(U)$ with no extension $f'\in\mathscr{H}(U')$ for every Riemann domain (U',φ') over C^N extending (U,φ) properly. (U',φ') extends (U,φ) properly if there is a biholomorphism j from U onto a proper open subset U_0 of U'. In this case, f has an extension $f'\in\mathscr{H}(U')$ if $f'\circ j=f$.

PROPOSITION 6. *Let (U,φ) be a Riemann domain over C^N of order n in U and such that (U_n,φ_n), defined as above, is a manifold of holomorphy spread over C^n. Then (U,φ) is a domain of holomorphy.*

Proof. There is $f_n \in \mathscr{H}(\boldsymbol{U}_n)$ with no extension $f'_n \in \mathscr{H}(\boldsymbol{U}'_n)$ for each manifold $(\boldsymbol{U}'_n, \varphi'_n)$ spread over $\boldsymbol{C}^n$ extending $(\boldsymbol{U}_n, \varphi_n)$ properly.

$$f: \boldsymbol{x} \in \boldsymbol{U} \mapsto f(\boldsymbol{x}) = f_n[\boldsymbol{x} + (\pi_n \varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}))] \in \boldsymbol{C}$$

is holomorphic in $\boldsymbol{U}$. In fact: if $\boldsymbol{x}$ is in $\boldsymbol{U}$, let $\boldsymbol{W}''$ and $\boldsymbol{W}$ be considered as in the proof of Proposition 5. It is quite clear that

$$(*) \qquad f \circ [\varphi | \boldsymbol{W}'']^{-1}(\varphi(\boldsymbol{x}) + b) = f \circ [\varphi | \boldsymbol{W}]^{-1}[\pi_n \varphi(\boldsymbol{x}) + b]$$

for each b in $B_0 \times \cdots \times B_s \times \boldsymbol{C}^{N-[0,s]}$. But, in $\varphi(\boldsymbol{W})$, $f \circ [\varphi | \boldsymbol{W}]^{-1}$ depends only on the first n variables and it is holomorphic in $\pi_n \varphi(\boldsymbol{W})$ since it is equal to

$$f_n \circ [\varphi | \boldsymbol{W} \cap \boldsymbol{U}_n]^{-1}$$

there. Hence it is holomorphic in $\varphi(\boldsymbol{W})$ ([3], [4], [5]). Since $\varphi(\boldsymbol{W})$ is a translation of $\varphi(\boldsymbol{W}'')$ and $(*)$ holds, $f \circ [\varphi | \boldsymbol{W}'']$ is holomorphic in $\varphi(\boldsymbol{W}'')$. Thus f is an element of $\mathscr{H}(\boldsymbol{U})$ and the restriction of it to $\boldsymbol{U}_n$ is equal to f_n. If f has an extension f' in $\mathscr{H}(\boldsymbol{U}')$ for some Riemann domain $(\boldsymbol{U}', \varphi')$ over $\boldsymbol{C}^N$ extending $(\boldsymbol{U}, \varphi)$ properly, there is some manifold $(\boldsymbol{V}_n, \psi_n)$ spread over $\boldsymbol{C}^n$ (of the type used in the proof of Proposition 5) which is not of holomorphy. Proposition 5 implies that $(\boldsymbol{U}_n, \varphi_n)$ is not a manifold of holomorphy spread over $\boldsymbol{C}^n$, a contradiction to the hypothesis of this proposition. Therefore $(\boldsymbol{U}, \varphi)$ is a domain of holomorphy.

PROPOSITION 7. *Let $(\boldsymbol{U}, \varphi)$ be a Riemann domain of holomorphy over $\boldsymbol{C}^N$ such that $\mathscr{H}(\boldsymbol{U})$ separates the points of $\boldsymbol{U}$. Then $(E(\boldsymbol{U}), \pi)$ is canonically identified to $(\boldsymbol{U}, \varphi)$.*

The proof of this proposition is an immediate consequence of Proposition 2.

A Riemann domain $(\boldsymbol{U}, \varphi)$ over $\boldsymbol{C}^N$ is holomorphically convex if, for each compact subset $\boldsymbol{K}$ of $\boldsymbol{U}$ and each balanced convex open neighborhood U of 0 in $\boldsymbol{C}^N$ such that $\boldsymbol{K} + U \subset \boldsymbol{U}$ and $\boldsymbol{K} + L$ is compact for every compact subset L of U, $\hat{\boldsymbol{K}}_U + U \subset \boldsymbol{U}$, where

$$\hat{\boldsymbol{K}}_U = \left\{ \boldsymbol{u} \in \boldsymbol{U};\ |f(\boldsymbol{u})| \leqq \sup_{\boldsymbol{t} \in \boldsymbol{K}} |f(\boldsymbol{t})|,\ \forall f \in \mathscr{H}(\boldsymbol{U}) \right\}.$$

PROPOSITION 8. *Let $(\boldsymbol{U}, \varphi)$ be a Riemann domain over $\boldsymbol{C}^N$ such that $\mathscr{H}(\boldsymbol{U})$ separates the points of $\boldsymbol{U}$ and $(E(\boldsymbol{U}), \pi)$ is canonically identified to $(\boldsymbol{U}, \varphi)$. Then $(\boldsymbol{U}, \varphi)$ is holomorphically convex.*

Proof. Let $\boldsymbol{K}$ be a compact subset of $\boldsymbol{U}$ and let U be an open balanced convex neighborhood of 0 in $\boldsymbol{C}^N$ such that $\boldsymbol{K} + U \subset \boldsymbol{U}$ and $\boldsymbol{K} + L$ is compact for every compact subset L of U. Then, by the remarks we have done after Proposition 1, $\varphi(\hat{\boldsymbol{K}}_U) + U \subset \varphi(\boldsymbol{U})$ and $N_{i(\boldsymbol{x}), U} \subset S(\boldsymbol{U})$ for each $\boldsymbol{x}$ in $\hat{\boldsymbol{K}}_U$. Since $N_{i(\boldsymbol{x}), U}$ is open connected and $\boldsymbol{x}$ is in $E(\boldsymbol{U})$, it follows that $N_{i(\boldsymbol{x}), U}$ is contained in $E(\boldsymbol{U})$ for every $\boldsymbol{x}$ in $\hat{\boldsymbol{K}}_U$. But $\boldsymbol{U}$ is identified to $E(\boldsymbol{U})$ and $N_{i(\boldsymbol{x}), U}$ is the same set as $\boldsymbol{x} + U$ for each $\boldsymbol{x}$ in $\hat{\boldsymbol{K}}_U$. Hence $\hat{\boldsymbol{K}}_U + U \subset \boldsymbol{U}$.

PROPOSITION 9. *If (U, φ) is a holomorphically convex Riemann domain over C^N, it is locally pseudoconvex.*

Proof. If V is an affine subspace in C^N of dimension two, (U_V, φ_V) is a manifold spread over C^2, where $U_V=\varphi^{-1}[\varphi(U) \cap V]$ and $\varphi_V=\varphi|U_V$. To show that U_V is pseudoconvex, it is enough to prove that $d(\hat{K}_{U_V})>0$ for every compact subset K of U_V. Let K be a compact subset of U_V and let U be an open balanced convex neighborhood of 0 in C^N such that $K+U\subset U$ and $K+L$ is compact for each compact subset L of U. Then $\hat{K}_U+U\subset U$ and $\varphi(\hat{K}_U)+U\subset\varphi(U)$. Since $\varphi(\hat{K}_U)$ is contained in the closed convex hull of $\varphi(K)$ and $\varphi(K)=\varphi_V(K)\subset V$, it follows that

$$\varphi(\hat{K}_U)+U \cap V \subset \varphi(U) \cap V = \varphi_V(U_V).$$

Now

$$\hat{K}_{U_V}+U \cap V \subset \hat{K}_U \cap U_V+U \cap V \subset (\hat{K}_U+U) \cap U_V \subset U \cap U_V \subset U_V.$$

It follows that there is a polydisc B in V with center 0 and radius $r>0$ such that $\hat{K}_{U_V}+B\subset U_V$. This means that $d(\hat{K}_{U_V})>0$.

Now it is possible to enunciate the following theorem whose proof we have just finished.

THEOREM 2. *Let (U, φ) be a Riemann domain over C^N such that $\mathscr{H}(U)$ separates the points of U. The following conditions are equivalent:*

(1) *(U, φ) is a domain of holomorphy.*

(2) *$(E(U), \pi)$ is canonically identified to (U, φ).*

(3) *(U, φ) is holomorphically convex.*

(4) *(U, φ) is locally pseudoconvex.*

(5) *There is $n>0$ in N such that (U, φ) is of order n in U and (U_n, φ_n) is a manifold of holomorphy spread over C^n, if $\varphi_n=\varphi|U_n$ and $U_n=\varphi^{-1}[\pi_n \circ \varphi(U)]$.*

REMARKS. (a) Theorem 2 was proved by Hirschowitz in [5] for the case in which (U, φ) is an open subset of C^N.

(b) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are true for a Riemann domain (U, φ) over a locally convex space E such that the closed convex hull of each compact subset is compact.

Let (U, φ) be a Riemann domain over C^N such that $\mathscr{H}(U)$ separates the points of U. The envelope of holomorphy of (U, φ) is a Riemann domain (U_0, φ_0) over C^N which is maximum in the sense stated in Proposition 2 with the word "normal" erased in condition (b).

Theorem 2 and Proposition 2 imply that the following result is true:

PROPOSITION 10. *If (U, φ) is a Riemann domain over C^N and $\mathscr{H}(U)$ separates the points of U, then $(E(U), \pi)$ is the envelope of holomorphy of (U, φ).*

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