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MAPPINGS FROM 3-MANIFOLDS ONTO 3-MANIFOLDS(1)

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Abstract. Let f be a compact, boundary preserving mapping from the 3-manifold M^3 onto the 3-manifold N^3. Let Z_p denote the integers mod a prime p, or, if $p=0$, the integers. (1) If each point inverse of f is connected and strongly 1-acyclic over Z_p, and if M^3 is orientable for $p>2$, then all but a locally finite collection of point inverses of f are cellular. (2) If the image of the singular set of f is contained in a compact set each component of which is strongly acyclic over Z_p, and if M^3 is orientable for $p\neq 2$, then N^3 can be obtained from M^3 by cutting out of Int M^3 a compact 3-manifold with 2-sphere boundary, and replacing it by a Z_p-homology 3-cell. (3) If the singular set of f is contained in a 0-dimensional set, then all but a locally finite collection of point inverses of f are cellular.

I. **Introduction.** We suppose throughout the introduction that $f\colon M^3 \twoheadrightarrow N^3$ is a compact, boundary preserving mapping from the 3-manifold M^3 onto the 3-manifold N^3 (where M^3 and N^3 may or may not have boundary). Let Z_p denote the integers modulo a prime p, or, if $p=0$, the integers.

If $f^{-1}(x)$ is connected and strongly 1-acyclic over Z_p for all $x \in N^3$, and if M^3 is orientable for $p>2$, then in Corollary 1 it is shown that all but a locally finite collection of point inverses are cellular. This implies that N^3 can be obtained from M^3 by cutting out of Int M^3 a locally finite collection of compact 2-manifolds, each bounded by a 2-sphere, and replacing them by a 3-cell (see Corollary 3). Thus, if M^3 is compact, N^3 is a factor in a connected sum decomposition of M^3.

Now suppose that the image of the singular set of f is contained in a compact set X each component of which is strongly acyclic over Z_p. If M^3 is orientable for $p\neq 2$, then N^3 can be obtained from M^3 by cutting out of M^3 a finite number of compact 3-manifolds, each bounded by a 2-sphere, and replacing each by a Z_p-homology 3-cell. In particular, if X has a neighborhood which is an irreducible 3-manifold with boundary (or if N^3 is irreducible), then N^3 is a factor in a connected sum decomposition of M^3. This extends Theorem 1 of Lambert in [9]. In the special case where the image of the singular set is contained in a Cantor set,

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we can say in addition that all but a finite number of point inverses are cellular. This was previously proved by the author using other techniques.

Lemma 5 restates one of Armentrout's results on approximating cellular maps with homeomorphisms. Using this lemma, we combine the results of Theorems 1 and 3 in Theorem 5. Thus if M^3 is compact and orientable for $p \neq 2$, and if the image of the point inverses of f which are not connected and strongly 1-acyclic over Z_p is contained in a compact set X each component of which is strongly acyclic over Z_p, then N^3 can be obtained from M^3 by cutting out of Int M^3 a finite number of 3-manifolds each bounded by a 2-sphere, and replacing each by a Z_p-homology 3-cell. Theorem 6 combines Theorems 1 and 4 in a similar fashion.

In Theorem 7, we extend a result of McMillan [13] to show that if the image of the singular set of f is contained in a (nonclosed) 0-dimensional set, then all but a locally finite collection of point inverses are cellular.

Let G be a nontrivial abelian group. A compact set $X \subset M$ is *strongly k-acyclic over G* if for each open set $U \subset M$ containing X, there is an open set V such that $X \subset V \subset U$ and such that the inclusion induced homomorphism $i_*: H_k(V; G) \to H_k(U; G)$ is zero. (If X is connected and strongly k-acyclic over G for $1 \leqq k \leqq n$, then $X \subset M$ has property $uv^n(G)$ in the sense of [8].) The compact set $X \subset M$ is *strongly acyclic over* G if it is connected and strongly k-acyclic over G for all $k \geqq 1$. We refer the reader to [13 (especially Lemma 1)] for further facts about strong acyclicity. In particular, for any positive integer k, a compact set X in the interior of a 3-manifold M^3 is strongly k-acyclic over G if and only if each component of X is strongly k-acyclic over G. Also X is strongly acyclic over Z if and only if X is connected and $H^*(X; Z) = 0$ (see [7]).

The compact set $X \subset M$ has property UV^∞ if for each open set $U \subset M$ containing X, there is an open set V such that $X \subset V \subset U$ and such that V is contractable in U. A set X in a 3-manifold M^3 is *cellular* in M^3 if $X = \bigcap_{i=1}^\infty F_i$ where each F_i is a 3-cell, and $F_{i+1} \subset \operatorname{Int} F_i$ for all i.

If σ is a loop in a space M, we will denote its homology class in $H_1(M; G)$ by $[\sigma]$. The symbol Z_p for $p > 0$ will denote the finite cyclic group of order p. The symbol Z_0 will denote the integers.

A manifold will be assumed to be connected and to have no boundary unless otherwise specified. We assume that all manifolds have a piecewise-linear structure. A 3-manifold is *irreducible* if every polyhedral 2-sphere in it bounds a polyhedral 3-cell. If M^3 and N^3 are 3-manifolds, possibly with boundary, the *connected sum* $M^3 \# N^3$ of M^3 and N^3 is obtained by removing the interior of a 3-cell from the interior of each, and then sewing the two manifolds together along the resulting boundary components, using an orientation reversing homeomorphism if M^3 and N^3 are oriented.

A *map* or *mapping* is a continuous function. A *monotone map* is a map all of whose point inverses are connected. A map $f: M \twoheadrightarrow N$ is *compact* (proper) if, for any compact set K in N, $f^{-1}(K)$ is compact. If $f: M \twoheadrightarrow N$ is a compact monotone map, then the point inverses of M form a monotone upper semicontinuous decomposition of M whose associated decomposition space is homeomorphic to N. Conversely, if G is a monotone upper semicontinuous decomposition of M, the projection map $p: M \twoheadrightarrow M/G$ is a compact monotone map.

Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of compact subsets of a space M. Then $\{X_\alpha\}_{\alpha\in A}$ is a *locally finite collection* if for $y \in M$, y has a neighborhood U which intersects only a finite number of elements of the collection.

II. Maps all of whose point inverses are strongly acyclic.

LEMMA 1. *If X is a compact connected subset of a space M and if X is strongly k-acyclic over Z in M for $1 \leqq k \leqq n$, then X is strongly k-acyclic over Z_p in M for $1 \leqq k \leqq n$ and for any prime $p > 1$.*

Proof. Let W and V be chosen so that $X \subset W \subset V \subset U$ and so that the inclusion induced homomorphisms $i_*: H_k(V; Z) \to H_k(U; Z)$ and $j_*: H_k(W; Z) \to H_k(V; Z)$ are zero for $1 \leqq k \leqq n$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_k(W;Z)\otimes Z_p \arrow[r] \arrow[d, "i_*\otimes \mathrm{id}"] & H_k(W;Z_p) \arrow[r] \arrow[d, "i'_*"] & \mathrm{Tor}_1\,(H_{k-1}(W;Z), Z_p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_k(V;Z)\otimes Z_p \arrow[r] \arrow[d, "j_*\otimes \mathrm{id}"] & H_k(V;Z_p) \arrow[r] \arrow[d, "j'_*"] & \mathrm{Tor}_1\,(H_{k-1}(V;Z), Z_p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_k(U;Z)\otimes Z_p \arrow[r] & H_k(U;Z_p) \arrow[r] & \mathrm{Tor}_1\,(H_{k-1}(U;Z), Z_p) \arrow[r] & 0
\end{tikzcd}$$

The horizontal rows, which are exact, are from the universal coefficient theorem. By our choice of W and V, the outer vertical maps are zero. Using a diagram chasing argument, we see that $j'_*i'_*$ is the zero homomorphism.

LEMMA 2. *Let M^3 and N^3 be 3-manifolds, and let $f: M^3 \twoheadrightarrow N^3$ be a compact, monotone, onto map. Let p be 0 or a prime, and suppose M^3 is orientable if $p \neq 2$. If $f^{-1}(y)$ is strongly 1-acyclic over Z_p for every $y \in N^3$, then each $f^{-1}(y)$ is strongly acyclic over Z_p in M^3.*

Proof. By Alexander duality and Theorem 3 of [8] we see that $H^k(f^{-1}(y); Z_p) = 0$ for $k \geqq 2$. Then the continuity of H^* and the universal coefficient theorem for cohomology show that $f^{-1}(y)$ is strongly acyclic over Z_p for all $y \in N^3$. (For more details, see Theorems 4.4 and 3.2 of [7].)

LEMMA 3. *Let M^3 and N^3 be 3-manifolds, and $f: M^3 \twoheadrightarrow N^3$ be a compact, monotone, onto map such that $f^{-1}(y)$ is strongly 1-acyclic over G for each $y \in N^3$. If $H_1(N^3; G) = 0$, then $H_1(M^3; G) = 0$.*

The proof of Lemma 3 is similar to the proof of Theorem 2.1 of [15].

If M^n and N^n are n-manifolds with boundary, a map f: $M^n \to N^n$ is said to be *boundary preserving* if $f|$ Bd M^n is a homeomorphism of Bd M^n onto Bd N^n, and if $f^{-1}(\text{Bd } N^n) = \text{Bd } M^n$. A 2-manifold with boundary S is *properly embedded* in a 3-manifold with boundary M^3 if $S \cap \text{Bd } M^3 = \text{Bd } S$.

A Z_p*-homology* (*homotopy*) 3-*cell* is a compact Z_p-acyclic (contractible) 3-manifold with boundary. A cube-with-handles is obtained by adding orientable 1-handles to a 3-cell. We define a Z_p*-homology* (*homotopy*) *cube-with-handles* similarly. We will say that a set X is the intersection of a decreasing sequence of (Z_p-homology, homotopy) cubes-with-handles if $X = \bigcap_{i=1}^{\infty} K_j^3$ where each K_j^3 is a (Z_p-homology, homotopy) cube-with-handles and $K_{j+1}^3 \subset \text{Int } K_j^3$.

THEOREM 1. *Let p denote* 0 *or a prime, and let* M^3 *and* N^3 *be compact* 3-*manifolds, possibly with boundary, where* M^3 *is orientable if* $p>2$. *Let* f: $M^3 \twoheadrightarrow N^3$ *be a monotone, onto, boundary preserving map. Let* U *be an open subset of* N^3. *If* $f^{-1}(x)$ *is strongly* 1-*acyclic over* Z_p *for all* $x \in U$, *then* $\{x \in U : f^{-1}(x)$ *is not cellular*$\}$ *is a finite set.*

REMARK. This theorem was first proved for $p=0, 2$ in [16]. It has since been generalized by D. R. McMillan in [13].

Proof. The case where $p=0$ reduces to the case where $p=2$ by Lemma 1. By the proofs of Theorems 1 and 2 of [11] and by Kneser's Theorem [6] it is sufficient to prove that $\{x \in U : f^{-1}(x)$ is not $UV^{\infty}\}$ is finite.

We can apply Lemma 2 to see that $f^{-1}(x)$ is strongly acyclic over Z_p for each $x \in U$. By Theorem 2 of [12], $f^{-1}(x)$ is the intersection of a decreasing sequence of Z_p-homology cubes-with-handles.

Let q be the rank (i.e. the minimum number of generators) of $\pi_1(M^3)$. By a corollary to the Grushko-Neumann Theorem (p. 192 of [10]), there are at most q disjoint Z_p-homology 3-cells in M^3 which are not homotopy 3-cells. Thus there are at most q points in U whose inverse images are not the intersection of a decreasing sequence of homotopy cubes-with-handles.

Let $x \in U$, where $f^{-1}(x)$ is the intersection of a decreasing sequence of homotopy cubes-with-handles. We will complete the proof by showing that $f^{-1}(x)$ is UV^{∞}. Let U' be an open set in M^3 containing $f^{-1}(x)$. There is a homotopy cube-with-handles H^3 such that

$$f^{-1}(x) \subset \text{Int } H^3 \subset H^3 \subset U' \cap f^{-1}(U).$$

Let W be an open 3-cell in U such that $x \in W$ and $f^{-1}(W) \subset \text{Int } H^3$. Define inductively $G_0, G_1, G_2, \dots$ by letting $G_0 = \pi_1(f^{-1}(W))$, and by letting

$$G_i = G_{i-1}(X_1 X_2 X_1^{-1} X_2^{-1} X_3^p).$$

(See p. 74 of [10] for notation.) In other words, G_i is the subgroup of G_{i-1} generated by all elements of the form $uvu^{-1}v^{-1}\tau^p$ where $u, v, \tau \in G_{i-1}$. Let $F_0, F_1, F_2, \dots$ be the corresponding subgroups of $\pi_1(H^3)$.

The subgroup G_1 certainly contains the commutator subgroup of G_0. The image of G_1 in $H_1(f^{-1}(W); Z)$ is $p \cdot H_1(f^{-1}(W); Z)$. Thus

$$\pi_1(f^{-1}(W))/G_1 \cong H_1(f^{-1}(W); Z)/p \cdot H_1(f^{-1}(W); Z) \cong H_1(f^{-1}(W); Z_p).$$

Let $\delta \in \pi_1(f^{-1}(W))$. Since $H_1(f^{-1}(W); Z_p)=0$ (by Lemma 3), $\delta \in G_1$. Thus δ is a product of elements of the form $uvu^{-1}v^{-1}\tau^p$ where $u, v, \tau \in G_0$. By applying the same argument to u, v, and τ, we see that $u, v, \tau \in G_1$. Thus $\delta \in G_2$. By repeating this argument, $\delta \in \bigcap_{i=0}^{\infty} G_i$. By Corollary 2.12 on p. 109 of [10], $\bigcap_{i=1}^{\infty} F_i = 1$. Thus $\delta = 1$ in $\pi_1(H^3)$, and $f^{-1}(x)$ is UV^{∞}.

COROLLARY 1. *Let M^3 and N^3 be 3-manifolds, possibly with boundary, and let $f: M^3 \twoheadrightarrow N^3$ be a compact, monotone, boundary preserving, onto map. Let p denote 0 or a prime, and suppose that M^3 is orientable if $p>2$. If $f^{-1}(x)$ is strongly 1-acyclic over Z_p in M^3 for all $x \in U$, then $\{x \in U : f^{-1}(x)$ is not cellular$\}$ is a locally finite set in N^3.*

III. **Maps where the image of the singular set lies in a strongly acyclic set.** We state below a slightly strengthened version of Theorem 2 of [13]: here we assume that M^3 is orientable only if $p>2$, and thus the 1-handles which are attached to Bd Q_i to obtain H_i may be attached in a nonorientable fashion. (See the statement of Theorem 2 for the definition of Q_i and H_i.) The only additional difficulty in the proof is when we have $S_i \subset$ Bd Z_i^* and $S_k \subset$ Bd Z_k^* topologically parallel. (See p. 133 of [12].) As before, each loop in S_i Z_p-bounds in Z_i^*, and the same argument shows that S_i is a 2-sphere if S_i is not homeomorphic to a projective plane. But if S_i is a projective plane, it must contain an orientation-reversing simple closed curve since S_i is two-sided. This contradicts the fact that every simple closed curve in S_i Z_p-bounds in Z_i^*, since $p=0, 2$.

THEOREM 2. *Let p denote 0 or a prime. Let X be a compact, proper subset of* Int M^3, *where M^3 is a 3-manifold, possibly with boundary. Suppose M^3 is orientable if $p>2$, and suppose that X has the following property relative to M^3 and p. For each open set $U \subset M^3$ with $X \subset U$, there is an open set V, $X \subset V \subset U$, such that, under inclusion, $H_1(V-X; Z_p) \to H_1(U; Z_p)$ is zero. Then $X = \bigcap_{i=1}^{\infty} H_i$, where H_i is a compact polyhedron in M^3, each component of H_i is a 3-manifold with nonempty boundary, $H_{i+1} \subset$* Int *H_i and each H_i has the following structure: it is obtained from a compact polyhedron Q_i, each component of which is a 3-manifold whose boundary consists entirely of 2-spheres, by adding to* Bd *Q_i a finite number of (solid, possibly nonorientable) 1-handles.*

Let $f: M \to N$ be a map. Then let $S_f = \{x \in M : f^{-1}f(x)$ is nondegenerate$\}$.

THEOREM 3. *Let p denote 0 or a prime. Let M^3 and N^3 be piecewise-linear 3-manifolds, possibly with boundary, where M^3 is orientable if $p \neq 2$. Let X be a compact subset of* Int *N^3 such that each component of X is strongly acyclic over Z_p. Let $f: M^3 \twoheadrightarrow N^3$ be a compact, boundary preserving map with $f(S_f) \subset X$. Then N^3 can* *be obtained from* M^3 *by cutting out of* Int M^3 *a finite number of polyhedral* 3-*manifolds which are each bounded by a* 2-*sphere, and replacing each by a polyhedral* Z_p-*homology* 3-*cell.*

Proof. By Theorem 2 of [12], X is the intersection of a decreasing sequence of Z_p-homology cubes-with-handles. Thus we can assume that N^3 is a Z_p-homology cube-with-handles, and that each two-sided surface in Int N^3 separates N^3.

The first half of the proof will be to show that $f^{-1}(X)$ has the following property in Int M^3: for each open set $U \subset \text{Int } M^3$ with $f^{-1}(X) \subset U$, there is an open set V, with $f^{-1}(X) \subset V \subset U$, such that, under inclusion, $H_1(V - f^{-1}(X); Z_p) \to H_1(U; Z_p)$ is zero.

Let U be an open set in Int M^3 with $f^{-1}(X) \subset U$. Since $\text{Cl}(S_f) \subset U$, $f(U)$ is open. Let Z^3 be a compact polyhedron in $f(U)$ such that each component of Z^3 is a 3-manifold with boundary, and such that $X \subset \text{Int } Z^3$. Since X is strongly 1-acyclic over Z_p, there is an open set W containing X such that, under inclusion

$$H_1(W - X; Z_p) \to H_1(Z^3; Z_p)$$

is zero.

Let $V = f^{-1}(W)$, and let $[\sigma] \in H_1(V - f^{-1}(X); Z_p)$ where we can assume that σ is a finite, pairwise disjoint collection of (oriented, if $p \neq 2$) simple closed curves such that $f(\sigma)$ is polyhedral in Z^3. Let F^3 be a regular neighborhood of $f(\sigma)$ in $(\text{Int } Z^3) - X$. We can triangulate Z^3 so that F^3 and $f(\sigma)$ are subcomplexes of the triangulation. Then the homeomorphism $f^{-1} | (\text{Bd } Z^3 \cup F^3)$ induces a triangulation of $f^{-1}(\text{Bd } Z^3 \cup F^3)$. Since each of the finite number of components of $f^{-1}(Z^3)$ is a 3-manifold with boundary, by Theorem 5 of [2] there is a triangulation of $f^{-1}(Z^3)$ which is compatible with the above triangulation of $f^{-1}(\text{Bd } Z^3 \cup F^3)$. Using the relative simplicial approximation theorem, there is a piecewise-linear, nondegenerate map g from $f^{-1}(Z^3)$ onto Z^3 such that

$$\begin{aligned} g | f^{-1}(\text{Bd } Z^3 \cup F^3) &= f | f^{-1}(\text{Bd } Z^3 \cup F^3), \\ g^{-1}(\text{Bd } Z^3 \cup F^3) &= f^{-1}(\text{Bd } Z^3 \cup F^3). \end{aligned}$$

By subdividing we can assume that g is simplicial.

At this point we divide the remainder of the first half of the proof into three cases: Case 1 ($p=0$), Case 2 ($p=2$), and Case 3 ($p>2$).

Case 1 ($p=0$). Since $f(\sigma) \subset W - X$, $[f(\sigma)] = 0$ in $H_1(Z^3; Z)$. Thus $f(\sigma)$ must bound a 2-complex L^2 in Z^3 where each component of L^2 is an orientable, two-sided 2-manifold with boundary. We can adjust L^2 slightly so that it is in general position mod $f(\sigma)$ with respect to our last triangulation of Z^3. Then $g^{-1}(L^2)$ will be a 2-complex in $f^{-1}(Z^3) \subset U$, where each component of $g^{-1}(L^2)$ is a two-sided 2-manifold with boundary. Thus, since M^3 is orientable, each component of $g^{-1}(L^2)$ is orientable. Since σ bounds $g^{-1}(L^2)$, $[\sigma] = 0$ in $H_1(U; Z)$, and the inclusion-induced homomorphism $H_1(V - f^{-1}(X); Z) \to H_1(U; Z)$ is trivial.

Case 2 ($p=2$). The proof is essentially the same as Case 1, except that L^2 and $g^{-1}(L^2)$ may not be orientable.

Case 3 ($p>2$). Note that

$$H_1(Z^3; Z)/G \cong H_1(Z^3; Z) \otimes Z_p \cong H_1(Z^3; Z_p)$$

where G is the subgroup of $H_1(Z^3; Z)$ generated by elements of the form $p[\gamma]$ where $[\gamma] \in H_1(Z^3; Z)$. Since $[f(\sigma)]=0$ in $H_1(Z^3; Z_p)$, there is a 1-cycle $[\tau] \in H_1(Z^3; Z)$ so that $[f(\sigma)]=p[\tau]$ in $H_1(Z^3; Z)$. We can assume that τ is a finite, pairwise disjoint collection of polyhedral, oriented, simple closed curves which are in general position with respect to our last triangulation of Z^3. Then $g^{-1}(\tau)$ is a finite, pairwise disjoint collection of simple closed curves in $f^{-1}(Z^3)$. We can find a regular neighborhood T^3 of τ so close to τ that $g^{-1}(T^3)$ is a regular neighborhood of $g^{-1}(\tau)$. We can find a 1-cycle $[\delta] \in H_1(\mathrm{Bd}\, T^3; Z)$ so that $[f(\sigma)]=[\delta]$ in $H_1(Z^3 - \mathrm{Int}\, T^3; Z)$. We can assume that δ is a finite collection of mutually exclusive, oriented, simple closed curves on Bd T^3. Then there is a 2-complex $L^2 \subset Z^3 - \mathrm{Int}\, T^3$ where each component of L^2 is a two-sided, orientable, 2-manifold, and where $\mathrm{Bd}\, L^2 = f(\sigma) \cup \delta$ (homologically $f(\sigma)-\delta$). We can assume that L^2 is in general position mod $f(\sigma)$ with respect to our last triangulation of Z^3. Then $g^{-1}(L^2)$ will be a 2-complex where each component of $g^{-1}(L^2)$ is a two-sided 2-manifold with boundary. Thus $g^{-1}(L^2)$ is orientable.

Since L^2 is two-sided in Z^3, δ is two-sided in Bd T^3. Thus $g^{-1}(\delta)$ is two-sided in $g^{-1}(\mathrm{Bd}\, T^3)$, and using this two-sidedness, we can induce an orientation of $g^{-1}(\delta)$ which is consistent with that on $g^{-1}(L^2)$. Thus $[g^{-1}(\delta)]=[\sigma]$ in $H_1(f^{-1}(Z^3); Z)$.

Let α be a meridional curve on Bd T^3 which is in general position with respect to δ. Then α will intersect δ algebraically $\pm p$ times. Since the two-sidedness of δ is preserved by g^{-1}, each component of $g^{-1}(\alpha)$ which is a meridional curve must intersect $g^{-1}(\delta)$ algebraically $\pm p$ times. Thus, $[g^{-1}(\delta)]=p[g^{-1}(\tau)]$ in $H_1(T^3; Z)$.

Therefore, $[\sigma]=p[g^{-1}(\tau)]$ in $H_1(Z^3; Z)$, and the inclusion-induced homomorphism $H_1(V-X; Z_p) \to H_1(U; Z_p)$ is trivial. This completes Case 3.

By Theorem 2, we can find a compact polyhedron H_0^3, where each component of H_0^3 is a 3-manifold with nonempty boundary, and where H_0^3 has the following structure: it is obtained from a compact polyhedron Q_0^3, each component of which is a 3-manifold whose boundary consists entirely of 2-spheres, by adding to Bd Q_0^3 a finite number of (solid, possibly nonorientable) 1-handles.

We can also assume that each 1-handle is attached to only one boundary component of Bd Q_0^3 since we can add 1-handles to Bd Q_0^3 which join different components of Bd Q_0^3 without destroying the property that Bd Q_0^3 consists entirely of 2-spheres.

We claim that each component of Bd Q_0^3 separates M^3. For suppose that S_0 is a component of Bd Q_0^3 that does not separate M^3. Then there is a polyhedral simple closed curve J which intersects S_0 at exactly one point which is a piercing point. It is easy to see that we can choose J so that it does not intersect any of the 1-handles which are added to Q_0^3 to obtain H_0^3. Let S_1 be the component of Bd H_0^3 which is obtained from S_0 by adding handles. Then J intersects S_1 only in the same piercing point. Since $f^{-1}|f(\text{Bd } H_0^3)$ is a homeomorphism, $f(J)$ is a loop in N^3 which intersects $f(S_1)$ in exactly one piercing point. Thus $f(S_1)$ does not separate N^3. But $f(S_1)$ is a 2-sided surface in N^3, so $f(S_1)$ must separate N^3. This is a contradiction, so S_0 does separate M^3.

Let Q^3 be the closure of the "inside" complementary domains of the "outermost" boundary components of Q_0^3. (Here, "inside" and "outermost" are relative to Bd M^3, which is connected.) Thus we have "filled in the holes" in Q_0^3 to obtain Q^3, and each component of Q^3 has connected boundary. We define H^3 to be Q^3 union the 1-handles of $H_0^3-Q_0^3$ which are not already contained in Q^3.

There are properly embedded polyhedral disks $B_1^2, \ldots, B_r^2$ in H^3 such that the 1-handles which are added to Q^3 to obtain H^3 are regular neighborhoods of $B_1^2, \ldots, B_r^2$ in H^3. Let these 1-handles be $N(B_1^2), \ldots, N(B_r^2)$. Since $S_f \subset f^{-1}(X) \subset \text{Int } H^3$, each component of $f(H^3)$ is a 3-manifold with boundary in Int N^3. Each B_i^2 is mapped properly into $f(H^3)$ by f, and furthermore, $f|B_i^2$ has no singularities near Bd B_i^2. So by Dehn's Lemma, there exist nonsingular properly embedded polyhedral disks $D_1^2, \ldots, D_r^2$ in $f(H^3)$ with Bd $D_i^2 = f(\text{Bd } B_i^2)$. By a cutting and pasting argument, we can choose $D_1^2, \ldots, D_r^2$ to be disjoint. We can also find disjoint regular neighborhoods $N(D_1^2), \ldots, N(D_r^2)$ of $D_1^2, \ldots, D_r^2$ in $f(H^3)$ so that

$$f(N(B_i^2) \cap \text{Bd } H^3) = N(D_i^2) \cap \text{Bd} f(H^3).$$

For each i, there is a homeomorphism $h_i\colon N(B_i^2) \twoheadrightarrow N(D_i^2)$ such that

$$h_i|(\text{Bd } H^3 \cap N(B_i^2)) = f|(\text{Bd } H^3 \cap N(B_i^2)).$$

We define a homeomorphism

$$h\colon M^3 - \text{Int } Q^3 \twoheadrightarrow (N^3 - \text{Int} f(H^3)) \cup \left(\bigcup_{i=1}^{r} N(D_i^2)\right)$$

by $h|(M^3 - \text{Int } H^3) = f|(M^3 - \text{Int } H^3)$, and by $h|N(B_i^2) = h_i$ for each $i = 1, \ldots, r$.

Then $h(\text{Bd } Q^3)$ is a finite disjoint collection of 2-spheres in N^3 each of which bounds a Z_p-homology 3-cell. Furthermore, these homology 3-cells are disjoint since each component of $h(\text{Bd } Q^3)$ is outermost in the sense that it can be joined to Bd N^3 with an arc which misses $h(\text{Bd } Q^3)$ except at one end point.

Let $K_1^3, \ldots, K_m^3$ be these homology 3-cells, and let $Q_1^3, \ldots, Q_m^3$ be the corresponding components of Q^3 so that $h^{-1}(\text{Bd } K_i^3) = \text{Bd } Q_i^3$. Each Q_i^3 is a 3-manifold with 2-sphere boundary. Then h is a homeomorphism from $M^3 - (\bigcup_{i=1}^m Q_i^3)$ onto $N^3 - (\bigcup_{i=1}^m K_i^3)$. Thus we obtain N^3 from M^3 by cutting out the Q_i^3's and replacing each with the corresponding K_i^3.

Remark. If we define $*Q_i^3$ to be the closed 3-manifold obtained from Q_i^3 by sewing a 3-cell onto Bd Q_i^3, and if we define $*K_i^3$ to be the closed 3-manifold obtained from K_i^3 in the same way, then

$$M^3 \# *K_1^3 \# \cdots \# *K_m^3 \cong N^3 \# *Q_1^3 \# \cdots \# *Q_m^3.$$

We should also note that we have shown that for any open set U in M^3 which contains X, then $f^{-1}(X)$ has a polyhedral neighborhood $H^3 \subset U$ where each component of H^3 is formed by adding 1-handles to a 3-manifold with 2-sphere boundary. Furthermore, we have shown that these 1-handles are attached in an orientable fashion to the 2-sphere boundary.

COROLLARY 2. *Let M^3 and N^3 be compact 3-manifolds, possibly with boundary. Let X be a compact proper set in* Int N^3 *with the following property: For each open set $U \subset \text{Int } N^3$ with $X \subset U$, there is an open set V, $X \subset V \subset U$, such that under inclusion $H_1(V-X; Z_p) \to H_1(U; Z_p)$ is zero. Suppose also that X has a polyhedral neighborhood each component of which is an orientable, irreducible 3-manifold with boundary. If there is a boundary preserving map f from M^3 onto N^3 such that $f(S_f) \subset X$, then M^3 can be obtained from N^3 by removing the interiors of a finite number of 3-manifolds each of which is bounded by a 2-sphere, and by replacing each by a 3-cell.*

Proof. By using Theorem 2 and the fact that X has a polyhedral neighborhood each component of which is an irreducible 3-manifold with boundary, we see that X has a polyhedral neighborhood each component of which is a cube-with-handles. Thus we can assume that N^3 is a cube-with-handles. The remainder of the proof of Theorem 3 now goes through with the weaker hypothesis on X.

THEOREM 4. *Let M^3 and N^3 be 3-manifolds, possibly with boundary, and let $f: M^3 \twoheadrightarrow N^3$ be an onto, compact, boundary preserving mapping from M^3 onto N^3 such that $f(S_f) \subset X$ where X is a closed 0-dimensional set in N^3. Then f is monotone, and $\{x \in N^3 : f^{-1}(x)$ is not cellular in $M^3\}$ is a locally finite subset of N^3.*

Proof. Let $x \in X$, and let U be an arbitrarily small open 3-cell containing x. Then there is a polyhedral 3-manifold with boundary K^3 so that $x \in \text{Int } K^3 \subset K^3 \subset U$ and so that Bd $K^3 \cap X = \varnothing$. In fact, using Theorem 2 of [12] and the fact that U is irreducible, we can see that K^3 can be chosen to be a cube-with-handles. Then $f^{-1}(K^3)$ is a connected neighborhood of $f^{-1}(x)$ which can be chosen "arbitrarily close" to $f^{-1}(x)$. Thus f is monotone.

We can cover X with the interiors of a locally finite collection of mutually exclusive collection of cubes-with-handles. Thus, in order to prove the theorem, it suffices to consider the case where N^3 is a cube-with-handles, and where M^3 is a compact 3-manifold with connected boundary. In this case, we will prove that all but a finite number of point inverses of f are cellular.

The set X is strongly 1-acyclic over Z_2 in N^3, and thus by the remark following the proof of Theorem 3, we have $f^{-1}(X) = \bigcap_{i=1}^{\infty} H_i^3$, where H_i^3 is a 3-manifold with connected boundary, and where $H_i^3 \subset \text{Int } H_{i-1}^3$. We can assume that H_i^3 is obtained from a compact polyhedron Q_i^3 where each component of Q_i^3 is a 3-manifold with 2-sphere boundary, by adding to Bd Q_i^3 a finite number of (orientable, solid) 1-handles. We also have that each 1-cycle in Bd H_i^3 bounds in Int H_{i-1}^3. We have assumed that M^3 is compact and that $H_1(M^3; Z_2)$ is finitely generated;

so it is easy to show that there is an integer N so that there are not more than N disjoint 3-manifolds with 2-sphere boundary and nontrivial Z_2-homology in Int M^3. Therefore, all but at most N components of $f^{-1}(X)$ are the intersection of a decreasing sequence of Z_2-homology cubes-with-handles.

If Z_i^3 is a Z_2-homology cube-with-handles, the inclusion-induced homomorphism $H_1(\text{Bd } Z_i^3; Z_2) \to H_1(Z_i^3; Z_2)$ is onto. Thus, if $Z_i^3 \subset \text{Int } Z_{i-1}^3$ where Z_{i-1}^3 is another Z_2-homology cube-with-handles, and if each 1-cycle in Bd Z_i^3 Z_2-bounds in Int Z_{i-1}^3, then the inclusion-induced homomorphism $H_1(Z_i^3; Z_2) \to H_1(Z_{i-1}^3; Z_2)$ is trivial. Therefore, each component of $f^{-1}(X)$ which is the intersection of Z_2-homology cubes-with-handles must be strongly 1-acyclic over Z_2. This shows that at most a finite number of point inverses of f are not strongly 1-acyclic over Z_2.

We can now apply Theorem 1 which implies that only a finite number of the strongly 1-acyclic over Z_2 point inverses of f are not cellular.

IV. Maps almost all of whose point inverses are strongly 1-acyclic over Z_p.

LEMMA 4. *Let $f\colon M \twoheadrightarrow N$ be a compact map from a metric space M onto a metric space N. Let X be a closed set in N. Let G be a decomposition of M defined by*

$$G = \{f^{-1}(y) : y \in X\} \cup \{x \in M : f(x) \notin X\}.$$

Let $Q = M/G$ and let $\pi\colon M \to Q = M/G$ be the projection map for the decomposition G. Let $p\colon Q \to N$ be defined so as to make the following diagram commute:

$$\begin{tikzcd} M \arrow[rr, two heads, "\pi"] \arrow[dr, "f"'] & & Q \arrow[dl, "p"] \\ & N & \end{tikzcd}$$

Then

(1) *G is upper semicontinuous and hence π is continuous and compact.*

(2) *The decomposition $\{p^{-1}(y) : y \in N\}$ is upper semicontinuous and hence p is continuous and compact.*

Proof. Lemma 4 follows from the fact that $\{f^{-1}(y) : y \in N\}$ is an upper semicontinuous decomposition of M.

LEMMA 5. *Let $p\colon Q \twoheadrightarrow N^3$ be a compact, monotone map from a metric space Q onto a 3-manifold N^3, possibly with boundary. Let X be a closed set in N^3 containing* Bd N^3. *Suppose that $p|p^{-1}(X)$ is a homeomorphism, and that $W = Q - p^{-1}(X)$ is an open 3-manifold. If $p^{-1}(x)$ is cellular for all $x \in N^3 - X$, then there is a homeomorphism $h\colon N^3 \twoheadrightarrow Q$ such that $h|X = p^{-1}|X$.*

The proof of Lemma 9 is the same as the proof of Theorem 1 of [1].

Suppose $f\colon M^3 \twoheadrightarrow N^3$ is a mapping. We let $A_f^p = \{x \in M^3 : f^{-1}f(x)$ is either not connected or is not strongly 1-acyclic over $Z_p\}$.

THEOREM 5. *Let p denote* 0 *or a prime, and let M^3 and N^3 be compact* 3*-manifolds, possibly with boundary, where M^3 is orientable if $p \neq 2$. Let Y be a compact set in* Int N^3 *each component of which is strongly acyclic over Z_p. Let $f\colon M^3 \twoheadrightarrow N^3$ be an onto, boundary preserving map such that $f(A_f^p) \subset Y$. Then N^3 can be obtained from M^3 by cutting out of M^3 a finite number of polyhedral* 3*-manifolds, each bounded by a* 2*-sphere, and replacing each by a Z_p-homology* 3*-cell.*

Proof. By Theorem 1 there are only a finite number of points $x_1, x_2, \ldots, x_n$ in $N^3 - Y$ whose inverses under f are not cellular in M^3. Let

$$X = Y \cup \{x_1, x_2, \ldots, x_n\} \cup \operatorname{Bd} N^3.$$

We use this X to define Q, $\pi\colon M^3 \twoheadrightarrow Q$, and $p\colon Q \twoheadrightarrow N^3$ as in Lemma 4. Since $\pi|(M^3 - f^{-1}(X))$ is a homeomorphism from $M^3 - f^{-1}(X)$ onto $W = Q - p^{-1}(X)$, W is an open 3-manifold. And since $p|p^{-1}(X)$ is one-to-one and continuous, $p|p^{-1}(X)$ is a homeomorphism. Therefore, by Lemma 5, there is a homeomorphism $h\colon N^3 \twoheadrightarrow Q$. In particular, Q is a 3-manifold Q^3. Let

$$X' = Y \cup \{x_1, \ldots, x_n\}.$$

Then $\pi(S_\pi) \subset p^{-1}(X') = h(X')$, and X' is strongly acyclic over Z_p, so the map π satisfies the hypotheses of Theorem 3.

THEOREM 6. *Let p denote* 0 *or a prime, and let M^3 and N^3 be* 3*-manifolds, possibly with boundary, where M^3 is orientable if $p > 2$. Let Y be a closed* 0*-dimensional set in* Int N^3, *and let $f\colon M^3 \twoheadrightarrow N^3$ be an onto, compact, boundary preserving map such that $f(A_f^p) \subset Y$. Then $\{x \in N^3 : f^{-1}(x)$ is not cellular in $M^3\}$ is a locally finite subset of N^3.*

Proof. By Corollary 1, the set $\{x \in N^3 - Y : f^{-1}(x)$ is not cellular in $M^3\}$ is a locally finite subset of N^3.

Let

$$X = Y \cup \operatorname{Bd} N^3 \cup \{x \in N^3 - Y : f^{-1}(x) \text{ is not cellular}\}.$$

Let Q, $\pi\colon M^3 \twoheadrightarrow Q$, $p\colon Q \twoheadrightarrow N^3$, and $h\colon N^3 \twoheadrightarrow Q$ be defined as in Lemmas 4 and 5. Let

$$X' = Y \cup \{x \in N^3 - Y : f^{-1}(x) \text{ is not cellular}\}.$$

Then $\pi(S_\pi) \subset p^{-1}(X') = h(X')$, and thus $\pi(S_\pi)$ is contained in a closed 0-dimensional set in Q. Theorem 4 can be applied to the map $\pi\colon M^3 \twoheadrightarrow Q^3$ to say that

$$\{y \in Q^3 : \pi^{-1}(y) \text{ is not cellular in } M^3\}$$

is a locally finite subset of Q^3. The image under p (or h^{-1}) of this set is

$$\{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3\}$$

which must then be a locally finite subset of N^3.

V. **Further applications.** The following lemma is a slight generalization of Lemma 5 of [13]. While the proof of Lemma 5 of [13] suffices to prove our Lemma 6, a proof is included here for completeness and since part of the proof will be needed to prove Theorem 7.

LEMMA 6. *Let M^3 and N^3 be 3-manifolds. Let $f\colon M^3 \twoheadrightarrow N^3$ be a compact, monotone mapping so that $f(S_f)$ is 0-dimensional. Let $x \in N^3$. If there is an open set U containing $f^{-1}(x)$ so that the inclusion-induced homomorphism from $H_1(U; Z)$ into $H_1(M^3; Z)$ is trivial, then $f^{-1}(x)$ is strongly 1-acyclic over Z.*

Proof. Let B^3 be an open 3-cell in N^3 with compact closure so that $x \in B^3$ and $W = f^{-1}(B^3)$ is contained in U. Let $K_1, K_2, K_3, \ldots$ be a locally finite collection of compact sets in W so that $\bigcup_{i=1}^{\infty} K_i = W$ and each K_i is contained in an open 3-cell $B_i^3 \subset W$. Let

$$\varepsilon_i = \inf\{\rho(x, y) : x \in K_i \text{ and } y \in W - B_i^3\}$$

where ρ is a metric on M^3. Let

$$C_i = \{x \in N^3 : \operatorname{diam}(f^{-1}(x)) \geqq \varepsilon_i \text{ and } f^{-1}(x) \cap K_i \neq \varnothing\}.$$

It is easy to see that each C_i is a closed set. Let $C = \bigcup_{i=1}^{\infty} C_i$.

We will show that $\{f(K_i)\}$ is a locally finite collection in B^3. Let $x_0 \in B^3$ and let V be a neighborhood of x_0 in B^3 with compact closure. Since f is a compact map, $f^{-1}(V)$ has compact closure. Since $\{K_i\}$ is a locally finite collection in W, $f^{-1}(V)$ intersects only a finite number of the K_i's, and thus V intersects only a finite number of the $f(K_i)$'s. Using the fact the $\{f(K_i)\}$ is a locally finite collection, we see that C is a closed 0-dimensional subset of B^3.

Consider the following commutative diagram where the horizontal maps are induced by inclusion, and the vertical maps are induced by f.

$$\begin{array}{ccc} H_1(W - f^{-1}(C); Z) & \xrightarrow{\alpha} & H_1(W; Z) \\ \downarrow & & \downarrow \\ H_1(B^3 - C; Z) & \longrightarrow & H_1(B^3; Z) \end{array}$$

First, we claim that α is an epimorphism. Let $[\delta] \in H_1(W; Z)$ where δ is a simple closed curve. Let O be an open set in B^3 so that $f(\delta) \subset O$ and $(\operatorname{Bd} O) \cap C = \varnothing$. By applying Lemma 2 of [13], we see that δ is homologous in $f^{-1}(O)$ to a 1-cycle in $f^{-1}(O) - f^{-1}(O \cap C) \subset W - f^{-1}(C)$.

Finally, we claim that α is the zero homomorphism. Let $[\tau] \in H_1(W - f^{-1}(C); Z)$ where τ is a simple closed curve. We can also suppose that $f(\tau)$ is a simple closed curve, and that $f(\tau)$ bounds an orientable surface S in $B^3 - C$. By our choice of the ε_i's, for each $y \in B^3 - C$, there is an open set V_y so that $f^{-1}(V_y)$ is contractible in W. Let $\mathscr{V} = \{V_y : y \in B^3 - C\}$. We can find a triangulation T of S which is so fine that for each 2-simplex $\sigma \in T$, there is a $V_\sigma \in \mathscr{V}$ so that $\sigma \subset V_\sigma$. Using the fact that f is monotone, we can find a map h from the 1-skeleton of T into $W - f^{-1}(C)$ so that, if σ is a 2-simplex of T, $h(\partial\sigma) \subset f^{-1}(V_\sigma)$. (See the proof of Theorem 2.1 of [15] for details.) We can also suppose that $hf|\tau$ is the identity. Since each V_σ is contractible in W, h can be extended to a map H which takes the surface S into W and which takes ∂S onto τ. Thus, $\alpha[\tau]=0$ in $H_1(W; Z)$.

THEOREM 7. *Let M^3 and N^3 be 3-manifolds, possibly with boundary. Let f be a compact, monotone, boundary preserving mapping from M^3 onto N^3 such that $f(S_f)$ is 0-dimensional. Then $\{x \in N^3 : f^{-1}(x)$ is not cellular$\}$ is a locally finite subset of N^3.*

Proof. By a procedure similar to the first part of the proof of Lemma 6, we can find a closed set $C \subset f(S_f) \subset N^3$ so that, if $x \notin C$, then there is an open set U_x where $f^{-1}(x) \subset U_x$ and U_x is contractible in M^3. By Lemma 6, if $x \in N^3 - C$, then $f^{-1}(x)$ is strongly 1-acyclic over Z. Thus $f(A_f^0) \subset C$, and C is a closed 0-dimensional set. Theorem 7 now follows from Theorem 6.

Let $f: M^3 \twoheadrightarrow N^3$ be an onto, compact, boundary preserving map as before. Many of our earlier results have shown that $\{x \in N^3 : f^{-1}(x)$ is not cellular in $M^3\}$ is a locally finite subset of N^3. The following three corollaries concern mappings of this type.

COROLLARY 3. *Let M^3 and N^3 be 3-manifolds, possibly with boundary. Let $f: M^3 \twoheadrightarrow N^3$ be a compact, monotone, boundary preserving map such that $\{x \in N^3 : f^{-1}(x)$ is not cellular$\}$ is a locally finite subset of N^3. Then*

(i) *For each $x \in N^3$ and each open set U containing $f^{-1}(x)$, there is an open set V with $f^{-1}(x) \subset V \subset U$, such that $V - f^{-1}(x)$ is homeomorphic to $S^2 \times (0, 1)$.*

(ii) *N^3 can be obtained from M^3 by cutting out of M^3 a locally finite collection of mutually exclusive, polyhedral 3-manifolds, each with 2-sphere boundary, and replacing each by a 3-cell.*

Proof. (i) If $f^{-1}(x)$ is cellular, this follows from Theorem 1 of [3].

Let $x_1, x_2, x_3, \ldots$ be the points in N^3 such that $f^{-1}(x_i)$ is not cellular for $i=1, 2, 3, \ldots$. Let $X=\{x_1, x_2, x_3, \ldots\} \cup \mathrm{Bd}\, N^3$. Let the 3-manifold Q^3, the maps $\pi: M^3 \twoheadrightarrow Q^3$, $p: Q^3 \twoheadrightarrow N^3$, and the homeomorphism $h: N^3 \twoheadrightarrow Q^3$ be defined as in Lemmas 4 and 5. It will be sufficient to show that $f^{-1}(x_1)$ has the required neighborhood. We are given an open set $U \supset f^{-1}(x_1)$. Let U' be an open set in M^3 so that $f^{-1}(x_1) \subset U' \subset U$ and $U' \cap f^{-1}(x_i) = \varnothing$ for $i \geqq 2$. Then $h^{-1}\pi(U')$ is an open set containing x_1 in N^3. Let W be an open 3-cell so that $x_1 \subset W \subset h^{-1}\pi(U')$. Let $V = \pi^{-1}h(W)$. Then $V - f^{-1}(x_1)$ is homeomorphic by $\pi^{-1}h$ to $W - \{x_1\}$ which is homeomorphic to $S^2 \times (0, 1)$.

(ii) As in part (i) let $x_1, x_2, x_3, \ldots$ be the points of N^3 whose inverses are not cellular. We can find pairwise disjoint closed neighborhoods $K_1, K_2, K_3, \ldots$ of $f^{-1}(x_1), f^{-1}(x_2), f^{-1}(x_3), \ldots$ respectively so that $K_i - f^{-1}(x_i)$ is homeomorphic to $S^2 \times (0, 1]$. Then each K_i is a 3-manifold with 2-sphere boundary, and $\pi|K_i$ is a boundary preserving map of K_i onto a 3-cell. Furthermore, $\pi | M^3 - \bigcup_{i=1}^{\infty} K_i$ is a homeomorphism. Thus Q^3 can be obtained by cutting $K_1, K_2, K_3, \ldots$ out of M^3, and replacing each by a 3-cell.

COROLLARY 4. *Let M^3 and N^3 be compact 3-manifolds, possibly with boundary. Let $f: M^3 \twoheadrightarrow N^3$ be a boundary preserving, onto map such that $\{x \in N^3 : f^{-1}(x)$ is not cellular in $M^3\}$ is a finite set. If M^3 is homeomorphic to N^3, then $f^{-1}(x)$ is cellular for every $x \in N^3$.*

Proof. By Corollary 3, part (ii), there are closed 3-manifolds $*K_0^3, \ldots, *K_n^3$ such that

$$M^3 = N^3 \# *K_0^3 \# \cdots \# *K_n^3.$$

By a corollary to the Grushko-Neumann Theorem (see p. 192 of [10]), the rank of $\pi_1(M^3)$ is equal to the sum of the ranks of $\pi_1(N^3), \pi_1(K_0^3), \ldots, \pi_1(K_n^3)$. Therefore

$$\pi_1(*K_0^3) = \cdots = \pi_1(*K_n^3) = 1,$$

and each $*K_i^3$ $(i=0, \ldots, n)$ is a homotopy 3-sphere.

If M^3 is closed and orientable, we use the unique decomposition theorem of Milnor [14] to show that $*K_0^3, \ldots, *K_n^3$ are all 3-spheres. This shows that $f^{-1}(x)$ is cellular for every $x \in N^3$.

If M^3 is orientable with boundary, we can sew a cube-with-handles onto each boundary component of M^3 to obtain a closed manifold M_0^3. The homeomorphism from M^3 to N^3 induces a similar sewing of cubes-with-handles onto Bd N^3 to give a closed 3-manifold N_0^3 which is homeomorphic to M_0^3. We have

$$M_0^3 = N_0^3 \# *K_0^3 \# \cdots \# *K_n^3$$

and the argument for the closed orientable case applies.

If M^3 is nonorientable, we apply the previous argument to the orientable double covering of M^3.

COROLLARY 5. *Let M^3 and N^3 be compact (i.e., closed) 3-manifolds. Let $f: M^3 \twoheadrightarrow N^3$ be an onto map such that $\{x \in N^3 : f^{-1}(x)$ is not cellular in $M^3\}$ is finite, and let $g: N^3 \twoheadrightarrow M^3$ be an onto map such that $\{x \in M^3 : g^{-1}(x)$ is not cellular in $N^3\}$ is finite. Then M^3 is homeomorphic to N^3.*

Proof. By Corollary 2, we have $M^3 = N^3 \# *K_0^3 \# \cdots \# *K_n^3$ and $N^3 = M^3 \# *Q_0^3 \# \cdots \# *Q_m^3$. By a corollary to the Grushko-Neumann Theorem (p. 192 of [10]) we see that $*K_0^3, \ldots, *K_n^3, *Q_0^3, \ldots, *Q_m^3$ are all homotopy 3-spheres. This implies that all of the point inverses of f and g have property UV^∞. Then Corollary 5 follows from Corollary 2.3 of [11].

VI. **On Haken's finiteness theorem.** In [5], Wolfgang Haken stated a finiteness theorem for incompressible surfaces in a compact 3-manifold M^3. We are interested here only in the special case of the theorem where the surfaces are closed: this case is stated as Theorem C. Some difficulties arise with Haken's proof in the case where M^3 is not irreducible. Haken's proof is correct and can be simplified considerably in the case where M^3 is irreducible. We give here an argument due to John Hempel to show that the finiteness theorem holds in the case where M^3 may not be irreducible. Haken intended to prove Kneser's Theorem [7] as a special case of the finiteness theorem; our argument uses Kneser's Theorem. The previous results of this paper depend on the finiteness theorem directly through Theorem 2 of [12].

In this section we will be working in the piecewise-linear category. A *surface* is a 2-manifold. If F^2 is a surface in a 3-manifold M^3, and if F^2 is not a 2-sphere, then F^2 is *incompressible* in M^3 if every simple closed curve in F^2 that bounds an (open) disk in M^3-F^2 also bounds a disk in F^2. A 2-sphere is *incompressible* in M^3 if it does not bound a 3-cell in M^3. A 3-manifold M^3 is irreducible if every 2-sphere in M^3 bounds a 3-cell in M^3.

Two surfaces F_0^2 and F_1^2 in a 3-manifold M^3 are *parallel* in M^3 if there is an embedding $\alpha\colon F_0^2\times[0,1]\to M^3$ such that $\alpha_0\colon F_0^2\to M^3$ is the inclusion map, and $\alpha_1\colon F_0^2\to M^3$ takes F_0^2 homeomorphically onto F_1^2. If $F_1^2,\dots,F_n^2$ are disjoint surfaces in a 3-manifold M^3, and if L^3 is the closure of a complementary domain of $M^3-\bigcup_{i=1}^n F_i^2$, then L^3 is a *parallelity component* if, for some $i=1,\dots,n$, there is a homeomorphism $h\colon F_i^2\times[0,1]\twoheadrightarrow L^3$ such that $h_0\colon F_i^2\to L^3$ is the inclusion map, and $h_1\colon F_i^2\to L^3$ takes F_i^2 homeomorphically onto F_j^2 for some $j=1,\dots,n$, $j\neq i$.

If C^3 is a 3-manifold, possibly with boundary, we define $\hat{C}^3$ to be the 3-manifold, possibly with boundary, obtained from C^3 by capping off each 2-sphere boundary component of C^3 with a 3-cell.

If B^3 is a 3-cell, and if $B_1^3,\dots,B_k^3$ are disjoint polyhedral 3-cells in Int B^3, then we call the manifold-with-boundary $B^3-(\bigcup_{i=1}^k \operatorname{Int} B_i^3)$ a *punctured 3-cell.*

LEMMA A. *If F^2 is an incompressible surface in the product $M^2\times[0,1]$, where M^2 is a compact 2-manifold, then F^2 is parallel to $M^2\times\{0\}$ and $M^2\times\{1\}$.*

This lemma is stated and proved by Haken on pp. 91–96 of [5].

LEMMA B. *If C^3 is a 3-manifold, possibly with boundary, and $\hat{C}^3$ is irreducible, then the finiteness theorem holds for C^3. In other words, there is an integer $n=n(C^3)$ such that if $F_1^2,\dots,F_{n+1}^2$ are $n+1$ disjoint incompressible polyhedral surfaces in C^3, then two of these surfaces are parallel.*

Proof. We have assumed the finiteness theorem for irreducible 3-manifolds, so there is an integer $n(\hat{C}^3)$ such that if there are more than $n(\hat{C}^3)$ disjoint incompressible surfaces in $\hat{C}^3$, then two of them are parallel. There are disjoint 3-cells $B_1^3,\dots,B_k^3$ such that $C^3=\hat{C}^3-[\bigcup_{i=1}^k \operatorname{Int} B_i^3]$. Let $n=n(C^3)=n(\hat{C}^3)+2k$. Let $F_1^2,\dots,F_{n+1}^2$ be $n+1$ disjoint incompressible surfaces in C^3. Then $n-k+1$ of these surfaces are incompressible in $\hat{C}^3$. There are $k+1$ distinct pairs from $F_1^2,\dots,F_{n+1}^2$ which are parallel in $\hat{C}^3$. (We say that the pair (F_i^2, F_j^2) is distinct from the pair

(F_r^2, F_s^2) if either $i \neq r, s$ or $j \neq r, s$.) Then, using Lemma A, we see that there are $k+1$ parallelity components in $\hat{C}^3$ whose interiors are disjoint. Thus, one of these parallelity components does not contain any B_i^3, and is a parallelity component in C^3.

THEOREM C. *Let M^3 be a compact 3-manifold, possibly with boundary. Then there is an integer $n_0 = n(M^3)$ such that if $F_1^2, \ldots, F_{n_0+1}^2$ are n_0+1 disjoint polyhedral-incompressible surfaces in M^3, then two of these surfaces are parallel.*

Proof. Let $\Sigma = \{S_1^2, \ldots, S_l^2\}$ be a disjoint collection of 2-spheres in M^3. Let $N_1^3, \ldots, N_l^3$ be disjoint regular neighborhoods of $S_1^2, \ldots, S_l^2$ respectively. Let $C_1^3, \ldots, C_k^3$ be the components of $\mathrm{Cl}\,(M^3 - \bigcup_{i=1}^l N_i^3)$. (The C_i^3's are determined up to homeomorphism by the S_i^2's and do not depend on the choice of the N_i^3's. Note that k may not equal l since some of the S_i^2's may not separate M^3.) We will call Σ a complete system of 2-spheres in M^3 if $\hat{C}_1^3, \ldots, \hat{C}_k^3$ are each irreducible. We will let $n(M^3, \Sigma) = \sum_{i=1}^k n(C_i^3)$ where $n(C_i^3)$ $(i = 1, \ldots, k)$ is defined in Lemma B.

Kneser's Theorem [7] shows that there is a complete system Σ_0 of 2-spheres in M^3. We will assume Σ_0 is a fixed complete system and we will let $n_0 = n(M^3, \Sigma_0)$.

Let $F_1^2, \ldots, F_{n_0+1}^2$ be disjoint incompressible surfaces in M^3. Let $F^2 = \bigcup_{i=1}^{n_0+1} F_i^2$. Suppose $\Sigma = \{S_1^2, \ldots, S_l^2\}$ is a complete system of 2-spheres in M^3, each of which is in general position with respect to F^2, and suppose that $n(M^3, \Sigma) = n_0$. Let $m(M^3, \Sigma, F^2)$ be the number of components of $(\bigcup_{i=1}^l S_i^2) \cap F^2$. (Each of these components is a simple closed curve.) We can suppose $m(M^3, \Sigma, F^2)$ is minimal over all such complete systems of 2-spheres in M^3. Theorem C will be proved if $m(M^3, \Sigma, F^2)$ is zero. For then there will be more than $n(C_j^3)$ of the surfaces $F_1^2, \ldots, F_{n_0+1}^2$ in one of the components C_j^3, and two of these surfaces must be parallel in C_j^3 by Lemma B. (Let $N_1^3, \ldots, N_l^3$ and $C_1^3, \ldots, C_k^3$ be defined as before.)

So we suppose that $m(M^3, \Sigma, F^2) > 0$. Any simple closed curve of $(\bigcup_{i=1}^l S_i^2) \cap F^2$ must bound a disk in F^2, since F^2 is incompressible. Therefore, we can choose an "innermost" (on F^2) simple closed curve J of $(\bigcup_{i=1}^l S_i^2) \cap F^2$; suppose $J \subset S_r^2 \cap F_s^2$ for some $r = 1, \ldots, l$ and $s = 1, \ldots, n_0+1$. Let D^2 be the disk that J bounds in F_s^2. Then D^2 is contained in some C_q^3 (where $q = 1, \ldots, k$) except for a regular neighborhood of Bd D^2.

Let E_1^2 and E_2^2 be the two disks bounded by J in S_r^2. We can push each of the 2-spheres $E_1^2 \cup D^2$ and $E_2^2 \cup D^2$ to one side so that they each miss D^2, and so that they are each contained in C_q^3. Then one of these 2-spheres must be in the boundary of a punctured cube P^3 in C_q^3 since $\hat{C}_q^3$ is irreducible. Let $S_r'^2$ be the 2-sphere that is not in the boundary of P^3, and let $\Sigma' = \{S_1^2, \ldots, S_{r-1}^2, S_r'^2, S_{r+1}^2, \ldots, S_l^2\}$. We will show that Σ' is a complete system of 2-spheres in M^3, that $n(M^3, \Sigma') = n_0$, and that $m(M^3, \Sigma', F^2) < m(M^3, \Sigma, F^2)$.

Let C_t^3 $(t = 1, \ldots, k)$ be the component of $\mathrm{Cl}\,(M^3 - \bigcup_{i=1}^l N_i^3)$ on the "other side" of S_r^2. (If S_r^2 does not separate M^3, then C_q^3 may equal C_t^3.) If we choose a small regular neighborhood $N_r'^3$ of $S_r'^2$ (so that $N_r'^3 \cap D^2 = \varnothing$) and let $N_i'^3 = N_i^3$ for

$i \neq r$, we can define $C_q'^3$ and $C_t'^3$ to be components of Cl $(M^3 - \bigcup_{i=1}^{l} N_i'^3)$. A subdisk D_0^2 of D^2 is a spanning disk of C_q^3 and if we remove the interior of a regular neighborhood of D_0^2, this separates C_q^3 into two components, one homeomorphic to $C_q'^3$, and the other homeomorphic to the punctured cube P^3. Thus $\hat{C}_q^3$ is homeomorphic to $\hat{C}_q'^3$. Furthermore, $C_t'^3$ is homeomorphic to the manifold obtained by sewing P^3 to C_t^3 along a disk on the boundary of each. Thus $\hat{C}_t^3$ is homeomorphic to $\hat{C}_t'^3$. We also have $n(C_q^3)+n(C_t^3)=n(C_q'^3)+n(C_t'^3)$ since the 2-sphere boundary components of $C_q^3 \cap P^3$ which were removed from C_q^3 to obtain $C_q'^3$ were added to C_t^3 to obtain $C_t'^3$. Thus $n(M^3, \Sigma')=n_0$.

Since $S_r'^2 \cap D^2=\varnothing$, $m(M^3, \Sigma', F^2)<m(M^3, \Sigma, F^2)$, and this contradicts our assumption that $m(M^3, \Sigma, F^2)$ was minimal.

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