SOME DEGENERACY THEOREMS FOR ENTIRE FUNCTIONS WITH VALUES IN AN ALGEBRAIC VARIETY[1]

BY

JAMES A. CARLSON

Abstract. In the first part of this paper we prove the following extension theorem. Let P_q^* be a q-dimensional punctured polycylinder, i.e. a product of disks and punctured disks. Let W_n be a compact complex manifold such that the bundle of holomorphic q-forms is positive in the sense of Grauert. Let $f: P_q^* \to W_n$ be a holomorphic map whose Jacobian determinant does not vanish identically. Then f extends as a rational map to the full polycylinder P_q. In the second half of the paper we prove the following generalization of the little Picard theorem to several complex variables: Let $V \subset P_n$ be a hypersurface of degree $d \geq n+3$ whose singularities are locally normal crossings. Then any holomorphic map $f: C^n \to P_n - V$ has identically vanishing Jacobian determinant.

0. **Introduction.** The purpose of this paper is twofold. First, we prove an extension theorem for holomorphic maps into a compact complex manifold W_n such that $\bigwedge^q T_{W_n}^*$ is positive in the sense of Grauert. Second, we discuss a generalization of the little Picard theorem in several complex variables. Thus, in the first section we apply the Ahlfors-Chern-Kobayashi version of the Schwarz lemma to show

THEOREM A. *Let $P_q^* = (\Delta^*)^k \times \Delta^{q-k}$ be a punctured polycylinder, and let W_n be a compact complex manifold such that $\bigwedge^q T_{W_n}^*$ is positive in the sense of Grauert. Then any nondegenerate map $f: P_q^* \to W_n$ extends to a rational map $f': P_q \to W_n$. If $\bigwedge^q T_{W_n}^*$ is very ample, then f' is actually holomorphic.*

We say that a map is rational on P_q if it has no essential singularities along any of the branches $z_1 = 0, \ldots, z_q = 0$. From this we obtain

COROLLARY. *Let $\bigwedge^q T_{W_n}^*$ be positive in the sense of Grauert. Then any holomorphic map $f: C^q \to W_n$ is totally degenerate, i.e., the Jacobian determinant $|J(f)|$ vanishes identically.*

The above theorem was first proved by Griffiths [3] in the equidimensional case when $\bigwedge^n T_{W_n}^* = K_W$ is very ample[2]. The more general case, for $q \neq n$, is based on

[2] A proof for K_W positive has recently been given by S. Kobayashi.

the proof in [3] plus simple curvature arguments. It should be remarked that Griffiths' equidimensional theorem is the most important for the applications, for example, to Picard-type problems in several complex variables.

In the second section we give a simplified proof of a fundamental Nevanlinna estimate of Kodaira. Kodaira's estimate deals with holomorphic mappings of the n-ball of radius R in C^n into W_n, where W_n is a complex manifold of general type (see §2 for definitions). A consequence of this estimate is that if $f: B_n(R) \to W_n$ is normalized so that $f(0) = p_0$ and $|Jf(0)| = 1$, then there is an absolute constant $R_{\max}(n, p_0)$ such that $R \leq R_{\max}$. This is a generalization of the classical Schottky-Landau theorem.

In the third section we apply the theorems of Griffiths and Kodaira to prove the following generalization of the little Picard theorem:

THEOREM B. *Let $f: C^n \to P_n - V$ be holomorphic, and suppose that V is a hyper-surface of degree $d \geq n+3$ whose singularities are locally normal crossings. Then f is totally degenerate.*

Thus, if $n = 1$, this says that $f: C \to P_1 - \{0, 1, a, \infty\}$ is constant, a crude version of Picard's theorem. Two interesting special cases of the above theorem occur when V is smooth and when V is the union of smooth hypersurfaces $V_1, \ldots, V_k$ meeting transversely and such that no more than n components pass through any point. For example, if the V_i are hyperplanes, we conclude that any holomorphic map of C^n into $P_n - (V_1 \cup \cdots \cup V_{n+3})$ must be totally degenerate. This gives a partial answer to a problem of Chern, who asked the same question for $n+2$ hyperplanes[3]. We should remark that the more obvious generalization of Picard's theorem, namely, that $f: C^n \to P_n - V$ be constant, is false in the generality of the above theorem. In fact, we have the following instructive example of Peter Kiernan.

EXAMPLE. Let $F_d \subset P_n$ be the Fermat hypersurface of degree d defined by the homogenous equation $z_0^d + \cdots + z_n^d = 0$. Set $p = $ greatest integer in $n/2$. Then there is a nondegenerate holomorphic map $f: C^p \to P_n - F_d$ for any d. To see this, consider the affine equation for F_d : $1 + w_1^d + \cdots + w_n^d = 0$. We define f for n even by $f(\mu_1, \ldots, \mu_p) = (\mu_1, \varepsilon_1 \mu_1, \mu_2, \varepsilon_2 \mu_2, \ldots, \mu_p, \varepsilon_p \mu_p)$ where the ε_i are dth roots of -1.

What Kiernan's example shows here is that for more refined versions of Picard's theorem one must do more than require V to have high degree.

In the fourth section we prove a version of the big Picard theorem in several complex variables under the assumption that V be smooth and have high degree. This is applied to give the following rigidity theorem:

PROPOSITION. *Let V_1 and V_2 be nonsingular hypersurfaces of degree $d \geq n+3$ in P_n. Then $P_n - V_1$ and $P_n - V_2$ are biholomorphic if and only if there is a projective linear automorphism β which carries V_1 onto V_2. Moreover, the complex automorphism group of $P_n - V_1$ is finite.*

([3]) Chern's conjecture has been answered by recent work of Mark Green.

It is my pleasure to acknowledge the many helpful conversations with Phillip Griffiths which made this work possible.

I. **Intermediate volume forms and the Schwarz lemma.**   The aim of this section is to formulate and prove an extension theorem for holomorphic mappings of a q-dimensional punctured polycylinder into an n-dimensional complex manifold, where $q \leq n$. The first step is to prove a generalization of the Ahlfors comparison theorem for volume elements on the unit disk. The proof is a rephrasing of the one found in [3]. Recall that if $P_q(R) = P(R)$ is the q-dimensional polycylinder $\{z \mid |z_i| < R, \ i = 1, \ldots, q\}$, then there is a volume element invariant under all complex automorphisms of $P(R)$ given by

$$(1) \qquad \mu_R = i^{q^2+q} a^q R^{2q} \frac{dz_1 \wedge \cdots \wedge dz_q \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q}{\prod_{j=1}^{q} (R^2 - |z_j|^2)^2}.$$

Let $v = i^{q^2+q} g\, dz_1 \wedge \cdots \wedge dz_q \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$ be a pseudovolume element on $P(R)$, i.e., $g > 0$ except on an analytic subset $\mathcal{D}$. Now the Ricci form of a volume element v given locally by $i^{q^2+q} h\, dz_1 \wedge \cdots \wedge dz_q \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$ on an arbitrary complex manifold W_q is given by

$$(2) \qquad \text{Ric}\,(v) = dd^c \log h = 2i\partial\bar{\partial} \log h.$$

Thus Ric (v) is a real nonnegative $(1, 1)$ form. If v is a pseudovolume element, we will define the Ricci form on $P(R) - \mathcal{D}$. We want to prove the following lemma:

LEMMA 1 (AHLFORS, CHERN, KOBAYASHI). *Let v be a pseudovolume element on $P_q(R)$ such that Ric (v) is positive definite and Ric $(v)^q \geq c_0 v$ on $P(R) - \mathcal{D}$. Then the normalization constant a in (1) can be chosen such that $v \leq \mu_R$ on $P(R)$.*

Proof. (i) We first recall that with respect to μ_R, P is *Einstein-Kähler*, i.e., there is a constant c_1 such that

$$(3) \qquad \text{Ric}\,(\mu)^q = c_1 \mu.$$

To see this, note that the group G of complex automorphisms generated by Möbius transformations in each of the variables z_j together with the maps $z_i \rightarrow z_j$ leaves μ invariant. Moreover, G is transitive on P, so that if (3) holds at one point $x_0 \in P$, then it holds for all $x \in P$.

(ii) Write $v = h_\rho \mu_\rho$ on $P(\rho)$. Thus we have

$$h_\rho = \frac{g \prod_{j=1}^{q} (\rho^2 - |z_j|^2)^2}{a^q \rho^{2q}}.$$

We must prove $h_R \leq c$ for some constant c. Note that it is sufficient to prove $h_\rho \leq c$ for $\rho < R$, since $h_\rho \rightarrow h_R$ as $\rho \rightarrow R$. For $\rho < R$, h_ρ is either identically zero or has a positive interior maximum. This follows from the fact that g is continuous and bounded on $P(\rho)$, so that $h_\rho(w) \rightarrow 0$ as $w \rightarrow \partial P(\rho)$. There is nothing to prove in

the first case, so we assume h_ρ has a positive maximum at x_ρ. Note that $g(x_\rho) > 0$ if $h_\rho(x_\rho) > 0$, so that $x_\rho \in P(\rho) - \mathcal{D}$. Thus we have

$$0 \geqq dd^c \log h_\rho(x_\rho) = \operatorname{Ric} \nu(x_\rho) - \operatorname{Ric} \mu_\rho(x_\rho),$$

or

$$\operatorname{Ric} \mu_\rho(x_\rho) \geqq \operatorname{Ric} \nu(x_\rho).$$

Using the fact that P is Einstein-Kähler together with $\operatorname{Ric} \nu^q \geqq c_0\nu$, we obtain

$$c_1\mu_\rho(x_\rho) \geqq c_0\nu(x_\rho).$$

This shows that $h_\rho(x) \leqq h_\rho(x_\rho) \leqq c_1/c_0 = c$, as desired.

To state the next result, we recall a few standard facts about curvature. Let h be a hermitian metric in a complex vector bundle E on V, and let $\theta(h)$ be the curvature form. Thus θ is a section of $\operatorname{Hom}(E, E) \otimes T_V^* \otimes \bar{T}_V^*$, where T_V is the holomorphic tangent bundle of V. Now choose an orthonormal frame field $\{e_\sigma\}$ of E near $x \in V$ and let θ_σ^ρ be the curvature form of E relative to $\{e_\sigma\}$. Thus θ is given locally by

$$\theta = \sum \theta_{\sigma ij}^\rho dz_i \wedge d\bar{z}_j$$

and we have $\theta_{\sigma ij}^\rho = \bar{\theta}_{\rho ij}^\sigma$. Hence

$$\theta(\xi \otimes \eta) = \sum \theta_{\sigma ij}^\rho \xi^\rho \bar{\xi}^\sigma \eta^i \bar{\eta}^j$$

defines a quadratic form, where $\xi \in E_x$ and $\eta \in T_{V,x}$ (see [2, p. 201]). We say that E is *negative* if $\theta(\xi \otimes \eta)$ is everywhere negative definite. Now let V_n be n-dimensional, and suppose henceforth that there is a metric ν on $\bigwedge^q T_V$ such that $\theta(\nu)$ is negative. Note that a volume form on V_q, given locally by $\nu = i^{q^2+q} g dz_1 \wedge \cdots \wedge dz_q \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$, is naturally a metric on $\bigwedge^q T_V = K_V^*$. Thus we call such a ν on V_n an *intermediate volume form*, or a *q-volume*.

In order to prove the Schwarz lemma for a q-volume, we need a generalization of the Ricci form. To give this, let Σ_V be the sphere bundle in $\bigwedge^q T_V$ defined by the metric ν, and let $\pi: \Sigma_V \to V$ be the projection. Let ξ be the canonical section of length one of the bundle $\pi^* \bigwedge^q T_V$ on Σ_V. The *generalized Ricci form* of ν is a real, positive definite section of $\pi^*(T_V \otimes \bar{T}_V)$ which is defined locally by

$$(4) \qquad \operatorname{Ric}^\#(\nu) = -i \sum \theta_{\sigma ij}^\rho \xi^\rho \bar{\xi}^\sigma dz_i \wedge d\bar{z}_j,$$

where θ_σ^ρ is the curvature form of $\bigwedge^q T_V$ relative to an orthonormal frame $\{e_\sigma\}$ near $p \in V$.

LEMMA 2. *Let $W \subseteq V$ be a complex submanifold. Hence ν_V restricts to give a metric ν_W in $\bigwedge^q T_W$. Then $\Sigma_W \subseteq \Sigma_V$ and $\operatorname{Ric}^\#(\nu_W) \geqq \operatorname{Ric}^\#(\nu_V)$ on Σ_W.*

Proof. $\bigwedge^q T_W$ is a subbundle of $\bigwedge^q T_V|_W$. The usual argument (see [2, pp. 196–201]) with the second fundamental form shows that curvature decreases in the

subbundle: $\theta(\nu_W)(\xi \otimes \eta) \leq \theta(\nu_V)(\xi \otimes \eta)$ where $\xi \in \bigwedge^q T_{W,x}$ and $\eta \in T_{W,x}$. This means that

$$-i \sum \theta(\nu_W)^\rho_{\sigma ij} \xi^\rho \bar{\xi}^\sigma dz_i \wedge d\bar{z}_j \geq -i \sum \theta(\nu_V)^\rho_{\sigma ij} \xi^\rho \bar{\xi}^\sigma dz_i \wedge d\bar{z}_j$$

as desired.

REMARKS. (i) Note that if V is n-dimensional and $\bigwedge^n T_V$ is negative relative to a volume element ν, then both Ric (ν) and Ric$^\#$ (ν) are positive. For example, the unit disk with Poincaré volume

$$\mu = (idz \wedge d\bar{z})/(1-|z|^2)^2$$

has constant negative Gauss curvature -4, whereas Ric (μ) is positive. Thus negative curvature corresponds to positive Ricci form. Ric$^\#$ should be thought of as the generalization of Gauss curvature needed to generalize the Ahlfors lemma.

(ii) For $q=1$ the above lemma reduces to the statement that if $W \subseteq V$ is a submanifold, then the holomorphic sectional curvatures of W are no greater than the sectional curvatures of V. The proof is also the same, except that we have done things with a globally defined form which is positive.

The next lemma is a generalization of the statement that if V is compact and has negative sectional curvatures, then the sectional curvatures are bounded away from zero.

LEMMA 3. *Let V be compact, and suppose that $\bigwedge^q T_V$ is negative relative to ν. If ξ is the canonical section of length one on $\pi^* \bigwedge^q T_V$, then* Ric$^\#$ $(\nu)^q(\xi) \geq c > 0$.

Proof. Note that Ric$^\#$ $(\nu)^q$ is a positive section of $\pi^*(\bigwedge^q T_V^* \otimes \bigwedge^q \bar{T}_V^*)$. Evaluating Ric$^\#$ $(\nu)^q$ on ξ gives a positive function on a compact manifold, from which the result follows.

We can now reinterpret the condition Ric $(\nu)^q \geq c_0 \nu$ in Lemma 1. Note that both Ric $(\nu)^q$ and ν give metrics in $K_{P'}^*$ $(P' = P - \mathscr{D})$, so that to verify the above inequality, it is enough to verify it on sections ξ of K_P^* which have length one relative to ν. Thus the Ricci condition is locally Ric $(\nu)^q(\xi) \geq c_0$ for $\nu(\xi)=1$, or equivalently, Ric$^\#$ $(\nu)^q(\xi) \geq c_0$ where ξ is the canonical section of length one. This leads to the following result.

PROPOSITION 1. *Let V_n be a compact complex manifold with a q-volume ν_V such that $\bigwedge^q T_V$ is negative. Let $f: P_q(R) \to V_n$ be a nondegenerate holomorphic map. Then by a suitable choice of the normalization constant in* (1), *we obtain $f^*\nu_V \leq \mu_R$. Moreover, the choice of constant is independent of f.*

Proof. Let $\mathscr{D}$ be the locus of $J(f)=0$ on $P(R)$. Then $f^*\nu_V$ is a pseudometric on $K_{P(R)}^*$ which is a metric on $K_{P'}^*$ $(P' = P - \mathscr{D})$. By Lemma 1, we must show Ric $(f^*\nu_V)^q \geq c_0 f^*\nu_V$ on P'. By the above remarks, this is equivalent to showing Ric$^\#$ $(f^*\nu_V)^q(\xi) \geq c_0$ on $\Sigma_{P'}$, where Σ_P is the circle bundle of K_P^* defined by the metric $f^*\nu_V$. Now f is locally a holomorphic diffeomorphism of P' onto $f(P')$ which gives an isomorphism $f_*: \bigwedge^q T_{P'} \to \bigwedge^q T_{f(P')}$ preserving metrics. Here $\bigwedge^q T_{f(P')}$ is equipped

with the metric ν_f gotten by restriction of ν_V. Therefore it is enough to prove $\mathrm{Ric}^{\#}(\nu_f)^q(\xi) \geqq c_0$ on $\Sigma_{f(P')}$. From Lemma 2 we get $\mathrm{Ric}^{\#}(\nu_f)(\xi) \geqq \mathrm{Ric}^{\#}(\nu_V)(\xi)$ in $\pi^*(T^*_{f(P')} \otimes \overline{T}^*_{f(P')})$. These are positive sections, hence $\mathrm{Ric}^{\#}(\nu_f)^q \geqq \mathrm{Ric}^{\#}(\nu_V)^q(\xi)$ in $\bigwedge^q T^*_{f(P')} \otimes \bigwedge^q \overline{T}^*_{f(P')}$. Now Lemma 3 gives $\mathrm{Ric}^{\#}(\nu_V)^q(\xi) \geqq c_0$ everywhere, hence $\mathrm{Ric}^{\#}(\nu_f)^q(\xi) \geqq c_0$, as desired.

REMARKS. This argument specializes to give the usual result on distance-decreasing for negative sectional curvatures when $q = 1$. When $q = n$, it specializes to Griffiths' Schwarz lemma in [3]. The proof in the general case is by interpolation.

To state the next result, suppose that V is an n-dimensional complex manifold with a metric ν_V in $S^l \bigwedge^q T_V$, where $S^l E = S^l(E)$ denotes the lth symmetric power of the bundle E. Let $f: W \to V$ be a holomorphic map, where W is q-dimensional. Then $f^* \nu_V$ is a metric on $S^l \bigwedge^q T_W$ which has the local form

$$g(i^{q^2+q} dz_1 \wedge \cdots \wedge dz_q \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q)^l.$$

This defines a volume form $(f^* \nu_V)^{1/l}$ which has the local form

$$i^{q^2+q} g^{1/l} dz_1 \wedge \cdots \wedge dz_q \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q.$$

PROPOSITION 2. *Let V_n be a compact complex manifold such that $S^l \bigwedge^q T_V$ is negative relative to a metric ν_V. Let $f: P_q(R) \to V$ be nondegenerate and holomorphic. Then the normalization constant in* (1) *can be chosen independently of f such that $(f^* \nu_V)^{1/l} \leqq \mu_R$.*

Proof. We want to mimic the proof of Proposition 1. First note that ν_V defines a Ricci form by the same formula (4) as before, except that the Ricci form is now defined on $\pi^*(T^*_V \otimes \overline{T}^*_V)$, where $\pi: \Sigma_V \to V$ is the projection from the sphere bundle of $S^l \bigwedge^q T_V$. By the same arguments which establish Lemmas 2 and 3, we get

LEMMA 2′. *Let $W \subseteq V$ be a complex submanifold. Hence ν_V restricts to give a metric ν_W in $S^l \bigwedge^q T_W$. Then $\Sigma_W \subseteq \Sigma_V$ and $\mathrm{Ric}^{\#}(\nu_W) \geqq \mathrm{Ric}^{\#}(\nu_V)$ on Σ_W.*

LEMMA 3′. *Let V be compact, and suppose that $S^l \bigwedge^q T_V$ is negative relative to ν. If ξ is the canonical section of length one of $\pi^* S^l \bigwedge^q T_V$, then $[\mathrm{Ric}\,(\nu)^q]^l(\xi) \geqq c > 0$.*

Now let $f^* \nu_V = g(i^{q^2+q} dz_1 \wedge \cdots \wedge d\bar{z}_q)^l$ so that $(f^* \nu_V)^{1/l} = i^{q^2+q} g^{1/l} dz_1 \wedge \cdots \wedge d\bar{z}_q$. To apply Lemma 1 to $(f^* \nu_V)^{1/l}$ and thus to complete the proof, we must show
  (a) $\mathrm{Ric}\,(f^* \nu_V)^{1/l}$ is positive definite,
  (b) $(\mathrm{Ric}\,(f^* \nu_V)^{1/l})^q \geqq c_0 (f^* \nu_V)^{1/l}$.
Proof of (a). Note that (a) follows if $\mathrm{Ric}\,(\nu_f)^{1/l}$ is positive definite on $f(P')$, where $\nu_f = \nu_V|_{S^l \bigwedge^q T_{f(P')}}$. Let $w_1, \ldots, w_q$ be local coordinates around $x \in f(P')$, and set $\nu_f = h(i^{q^2+q} dz_1 \wedge \cdots \wedge d\bar{z}_q)^l$. If ξ and ξ' are the canonical sections of the sphere bundles $\Sigma(S^l \bigwedge^q T_{f(P')})$ and $\Sigma(\bigwedge^q T_{f(P')})$, then we have

$$\mathrm{Ric}^{\#}(\nu_f)(\xi) = dd^c \log h = l \cdot dd^c \log h^{1/l} = l \cdot \mathrm{Ric}^{\#}(\nu_f^{1/l})(\xi').$$

By Lemma 2′, $\text{Ric}^{\#}(\nu_f)$ is positive definite, from which we conclude that $\text{Ric}^{\#}(f^{*}\nu_V)^{1/l}$ is positive definite.

Proof of (b). We claim that (b) follows from

(c) $[(\text{Ric}\, f^{*}\nu_V)^q]^l \geqq c_1 f^{*}\nu_V$.

Note that (c) holds because of Lemma 3′. Expressing (b) and (c) in local coordinates we have

(b′)
$$(dd^c \log g^{1/l})^q \geqq i^{q^2+q} c_0 g^{1/l} dz_1 \wedge \cdots \wedge d\bar{z}_q,$$

(c′)
$$[(dd^c \log g)^q]^l \geqq c_1 g [i^{q^2+q} dz_1 \wedge \cdots \wedge d\bar{z}_q]^l.$$

Let $dd^c \log g = i \sum g_{ij} dz_i \wedge d\bar{z}_j$, so that $(dd^c \log g)^q = \det(g_{ij})(i^{q^2+q} dz_1 \wedge \cdots \wedge d\bar{z}_q)$. Hence (b′) and (c′) can be written as

(b″)
$$(1/l)^q \det(g_{ij}) \geqq c_0 g^{1/l},$$

(c″)
$$(\det(g_{ij}))^l \geqq c_1 g.$$

Since $\text{Ric}^{\#}(\nu_f)$ is positive, we know that (g_{ij}) is a positive definite matrix. It is now clear that (b″) follows from (c″) with $c_0 = (1/l)^q c_1$.

To state the next result, we recall that a complex vector bundle E is *negative in the sense of Grauert* if the zero section can be blown down to a point (see [5]). This is equivalent to the existence of a plurisubharmonic function on E which is strictly plurisubharmonic on the complement of the zero section. A complex vector bundle E is *positive in the sense of Grauert* if its dual is negative in the sense of Grauert. We recall that if E is Grauert-positive then $S^l E$ has positive curvature for l sufficiently large. We can now state the following:

PROPOSITION 3. *Let V_n be a compact complex manifold such that $\bigwedge^q T_V^{*}$ is positive in the sense of Grauert. Let $f: P_q^{*} \to V$ be a nondegenerate holomorphic map defined on the punctured polycylinder $(\Delta^{*})^k \times \Delta^{q-k}$. Then f extends to a rational[4] map $\bar{f}: P_q \to V$.*

Proof. (i) Because $\bigwedge^q T_V^{*}$ is Grauert-positive, $S^l \bigwedge^q T_V^{*}$ has positive curvature for some $l \gg 0$ (see [5], [2]). By choosing l even larger, we may assume that $S^l \bigwedge^q T_V^{*}$ is very ample (see [2]). This means that V is immersed into a Grassmannian $\text{Gr}(m, N)$ by the global sections of $S^l \bigwedge^q T_V^{*}$. To be explicit, we choose a basis $\omega_1, \ldots, \omega_N$ for the vector space Γ of global sections. Evaluation of a section on the fiber gives the exact sequence

$$0 \to F \to V \times \Gamma \to S^l \bigwedge^q T_V^{*} \to 0,$$

which defines a map $\gamma: V \to \text{Gr}(m, N)$ by sending x to the subspace F_x of Γ. We now fix a metric in $S^l \bigwedge^q T_V$ by $\nu_V = \omega_1 \wedge \bar{\omega}_1 + \cdots + \omega_N \wedge \bar{\omega}_N$.

(ii) To extend $f: P_q^{*} \to V_n$ to a rational map $\bar{f}: P_q \to V_n$, it is sufficient to extend each of the holomorphic sections $f^{*}\omega_i$ of $S^l \bigwedge^q T_{P^{*}}^{*}$ to a rational section of

---

[4] Rational means that f has no essential singularities along $z_1 = 0, \ldots, z_k = 0$.

$S^l \wedge^q T_P^*$. This is because the ω_i give local coordinates on V. We will show that $f^*\omega_i$ has at most a pole of order $l-1$ along $z_i = 0$, $i = 1, \ldots, k$. Write $f^*\omega_i = g_i(i^{q^2+q}dz_1 \wedge \cdots \wedge d\bar{z}_q)^l$. If g_i has an essential singularity or a pole of order $> N$ along some $z_j = 0$, then there is a $c > 0$ such that, in the punctured polycylinder $P^*(\frac{1}{2})$ of radius $\frac{1}{2}$, we have $|g_i| > c/(z_1 \cdots z_k)^N$. For if not, the Riemann extension theorem yields $g_i = h/(z_1 \cdots z_k)^N$, where h_i is holomorphic on $P(\frac{1}{2})$, a contradiction. Suppose that we can prove

$$(*) \qquad\qquad \int (f^*(\omega_i \wedge \bar{\omega}_i))^{1/l} < \infty.$$

Then we have

$$\infty > \int_{P^*} (f^*(\omega_i \wedge \bar{\omega}_i))^{1/l}$$
$$= \int_{P^*} |g_i|^{2/l} \, dz_1 \wedge \cdots \wedge d\bar{z}_q > \lim_{\varepsilon \to 0} c' \int_\varepsilon^{1/2} \frac{r_1 \, dr_1}{r_1^{2N/l}} \cdots \int_\varepsilon^{1/2} \frac{r_k \, dr_k}{r_k^{2N/l}}.$$

Hence

$$\infty > \text{const} + \frac{\text{const}}{(2 - 2N/l)^k} \cdot \lim_{\varepsilon \to 0} (\varepsilon^{2 - 2N/l})^k,$$

from which we conclude $2 - 2N/l > 0$, or $N < l$.

(iii) It remains to show $(*)$. First, we show that

$$\int_{P^*} (f^*(\omega_i \wedge \bar{\omega}_i))^{1/l} \leq N\left(\frac{1}{N}\right)^{1/l} \int_{P^*} (f^*\nu_V)^{1/l}.$$

Now $f^*\nu_V = \sum_{i=1}^N f^*(\omega_i \wedge \bar{\omega}_i)$ so that

$$f^*\nu_V = \left(\sum_{i=1}^N |g_i|^2\right)(i^{q^2+q}dz_1 \wedge \cdots \wedge d\bar{z}_q)^l.$$

Using the fact that the function $\chi \to \chi^{1/l}$ is concave[5], we get

$$\frac{1}{N}|g_i|^{2/l} \leq \frac{1}{N}\sum_{i=1}^N |g_i|^{2/l} \leq \left(\frac{1}{N}\sum |g_i|^2\right)^{1/l},$$

which gives $(1/N)(f^*(\omega_i \wedge \bar{\omega}_i))^{1/l} \leq (1/N)^{1/l}(f^*\nu_V)^{1/l}$, and hence the claim.

Now let $\kappa: P \to P^*$ be the universal covering map. The Poincaré volume μ on P is invariant under covering transformations, so it descends to give a volume form μ_{P^*} on P^*. Therefore, $(f^*\nu_V)^{1/l} \leq \mu_{P^*}$ on P^* is equivalent to $(\kappa^*f^*\nu_V)^{1/l} \leq \mu_P$ on P. The latter statement follows from the Schwarz lemma applied to the function $f \circ \kappa$.

_____

[5] If E is a measure space such that $\mu(E) = 1$, and ψ is concave, then Jensen's inequality says $\int_E \psi \circ f \, d\mu = \psi(\int_E f \, d\mu)$ (see [10, p. 61]).

Now the volume of the concentric polycylinder $P(\tfrac{1}{2})$ is finite in the Poincaré volume, which implies that $\mu_{P^*}(P^*(\tfrac{1}{2})) < \infty$. Therefore we conclude that

$$\int_{P^*(1/2)} (f^*(\omega_i \wedge \bar{\omega}_i))^{1/l} \leq N\left(\frac{1}{N}\right)^{1/l} \int_{P^*(1/2)} (f^*\nu_V)^{1/l}$$

$$\leq N\left(\frac{1}{N}\right)^{1/l} \int_{P^*(1/2)} \mu_{P^*} < \infty,$$

as desired.

REMARK. Note that if $l=1$, i.e. if $\bigwedge^q T_V^*$ is very ample, then f extends as a *holomorphic* mapping. For $q=n$ this is Griffiths' extension theorem for canonical algebraic varieties [3]. The proof also establishes Griffiths' theorem when K_V is positive.

Before giving the applications of this theorem, we must recall the notion of smooth compactifications of algebraic varieties (see [2]). Let U be a smooth projective Zariski open set, i.e. $U=V-W$, where V and W are projective varieties. A smooth compactification of U is a nonsingular projective variety U' such that $U \subseteq U'$ is a dense open set and $U'-U=D$ is a union of nonsingular divisors D_i with normal crossings. Thus a local analytic equation for D in U' is $z_1, \ldots, z_k=0$. This has the useful consequence that if P is a polycylinder in U', then $U \cap P = (U'-D) \cap P$ is just a punctured polycylinder $(\Delta^*)^k \times \Delta^{n-k}$. Recall that U' is obtained from V by quadratic transformations. Thus there is a rational holomorphic map $U' \xrightarrow{\pi} V$ which is biholomorphic on U. Smooth compactifications are not unique. Given two compactifications U', U'', there is always a third one U''' with rational holomorphic maps $\pi_1: U''' \to U'$ and $\pi_2: U''' \to U''$. If $f: X \to Y$ is a rational map between algebraic varieties, then there is a subvariety $B \subseteq X$ along which f is indeterminate. However, we can always find a smooth compactification X' of $X-B$ such that $\pi: X' \to X$ pulls f back to an everywhere defined function, i.e. a rational holomorphic function.

As an example, we note that P_n is a smooth compactification of C_n, where $D=P_{n-1}$ is the hyperplane at infinity. Let z_0, z_1, z_2 be homogeneous coordinates on P_2 such that $D=\{z_0=0\}$. Then z_1/z_0 is holomorphic on $C^2=P_2-D$ and has a pole along D. However, z_1/z_0 is indeterminate at $[0, 0, 1]$ (homogeneous coordinates). To remedy this, we blow up $[0, 0, 1]$ to obtain a rational variety P_2' and a holomorphic map $\pi: P_2' \to P_2$. If $\mathscr{E} = \pi^{-1}[0, 0, 1]$ is the exceptional set, then π is biholomorphic on $P_2'-\mathscr{E}$. Now z_1/z_0 extends from $P_2'-\mathscr{E}$ to P_2 as an everywhere defined rational function $f: P_2' \to P_1$.

An immediate corollary of Proposition 3 and the above remarks is the following extension theorem.

COROLLARY 1. *Let U_q be a q-dimensional Zariski open set as above, and $f: U_q \to V_n$ a nondegenerate holomorphic map. If $\bigwedge^q T_V^*$ is positive in the sense of*

Grauert, then f extends as a holomorphic map f': $U'_q \to V_n$ *for a suitable smooth compactification* U'_q.

This, in turn, gives the following degeneracy theorem:

COROLLARY 2. *Let* V_n *be an n-dimensional complex manifold such that* $\bigwedge^q T^*_V$ *is positive in the sense of Grauert. Then any holomorphic map* f: $C^q \to V_n$ *is totally degenerate.*

Proof. By Corollary 1, f extends to a holomorphic map f': $P'_q \to V_n$. We know that for $l \gg 0$, $S^l \bigwedge^q T^*_V$ is very ample. Therefore we can find a global section $\omega \in \Gamma(S^l \bigwedge^q T^*_V)$ such that $f^*\omega \in \Gamma(K^l_{P'_q})$ is nonzero. But $\Gamma(K^l_{P'_q}) \simeq \Gamma(K^l_{P_q}) = \{0\}$, a contradiction.

REMARKS. (1) The extension theorem is also valid when $\bigwedge^q T^*_V$ is very ample and f: $U_m \to V_n$ has rank $\geq p$ (dim $U_m = m$).

(2) If $\bigwedge^q T^*_V$ is positive in the sense of Grauert, then the *Schottky-Landau theorem* holds. This says that if f: $P_q(R) \to V_n$ is normalized so that $f(0) = p$ and $|J(f)(0)| = 1$ ([6]), then there is a constant R_{max} independent of f such that $R \leq R_{max}$. The proof follows from the Schwarz lemma in exactly the same way as for the equidimensional Schottky-Landau theorem in [3].

II. **The basic Kodaira estimate.** We will give a modified proof of a Schottky-Landau theorem due to Kodaira [8]. To state the theorem, we recall that an algebraic manifold is of *general type* if

$$\liminf_{m \to \infty} m^{-n} \dim H^0(V, K^m) > 0$$

where $n = \dim V$ and K is the canonical bundle.

THEOREM (KODAIRA). *Let* $B(R) = \{z \in C^n \mid \sum |z_i|^2 < R\}$, *and let* f: $B(R) \to V$ *be holomorphic and nondegenerate. Suppose f is normalized such that* $f(0) = p$ *and* $|J(f)(0)| = 1$ ([7]). *If V is of general type, then there is a constant* R_{max} *independent of f such that* $R \leq R_{max}$.

In particular, this implies that there is no nondegenerate map f: $C^n \to V$, where V is of general type. The usefulness of Kodaira's theorem is that the numerical criterion of general type is usually much easier to compute than curvature conditions. The main point in the proof given below is that balls are used instead of polycylinders.

Conventions. We begin by establishing some conditions regarding curvature and volume forms. This will be done to avoid complicated and irrelevant constants

---

([6]) $|J(f)(0)|$ is defined by measuring $f_*((\partial/\partial z_1) \wedge \cdots \wedge (\partial/\partial z_q))$ with the metric ν in $\bigwedge^q T^*_V$, for example.

([7]) $|J(f)(0)|$ can be defined relative to coordinates on a fixed polycylindrical neighborhood U of $f(0)$.

in the final theorems. First, we let $d^c = (-1)^{1/2}(\bar{\partial} - \partial)/4\pi$ ([8]). This is defined so that $\int_{|z|=1} d^c \log |z|^2 = 1$. Let $\|z\|^2 = |z_1|^2 + \cdots + |z_m|^2$, and define $\eta = d^c \log \|z\|$, $\psi = d\eta = dd^c \log \|z\|$. If we let $\sigma = \eta \wedge \psi^{n-1}$, then the sphere of radius r in C^n has volume 1 relative to σ: $\int_{S[r]} \sigma = 1$. If L is a line bundle given by transition functions $f_{\alpha\beta}$, then sections of L are locally defined functions s_α such that $s_\alpha = f_{\alpha\beta} s_\beta$. If $\{a_\alpha\}$ is a collection of positive C^∞ functions satisfying $a_\alpha = |f_{\alpha\beta}|^2 a_\beta$, then the length of a section is measured by $|s|^2 = |s_\alpha|^2 / a_\alpha$. The first Chern class of L is defined by $c_1(L) = dd^c \log a_\alpha$. With these definitions one verifies that an ample bundle has positive definite first Chern class. By this, we mean that if $dd^c \log a_\alpha = ((-1)^{1/2}/4\pi) \sum g_{ij}^\alpha dz_i \wedge d\bar{z}_j$, then the hermitian matrix (g_{ij}^α) is positive definite. For example, one easily checks that the hyperplane bundle on P_n is positive according to this definition([9]).

Proof of the basic estimates. We begin by defining a volume form on V. Let L be the hyperplane line bundle on V determined by some projective imbedding. If V is of general type, then there is a section $s \in H^0(V, K^m \otimes L^*)$ for some $m > 0$([10]). Let a be the metric on L defined above. Thus a is a positive C^∞ section of $L \otimes \bar{L}$. Now $s \otimes \bar{s}$ is a real nonnegative C^∞ section of $K_V^m \otimes \bar{K}_V^m \otimes L^* \otimes \bar{L}^*$, so that $as \otimes \bar{s}$ is a nonnegative section of $K_V^m \otimes \bar{K}_V^m$. Therefore it has a globally defined mth root

$$\Omega = (as \otimes \bar{s})^{1/m} \in \Gamma(K_V \otimes \bar{K}_V),$$

which is the volume element we want. If s is defined locally by s_α, then the (n, n) form Ω is defined locally by $\Omega_\alpha = a_\alpha^{1/m} |s_\alpha|^{2/m}$.

Now suppose that $f: B(R) \to V$ is holomorphic and nondegenerate. We will also suppose that $f(0) = p \notin (s)$, where $(s) = \{x \in V \mid s(x) = 0\}$. Let ξ be the comparison function defined by $f^* \Omega = (-1)^{n/2} \xi dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$. Thus

$$\xi = (a_\alpha^{1/m} \circ f) |s_\alpha \circ f|^{2/m} |J(f)|^2.$$

Since f is normalized so that $f(0) = p$ and $|J(f)(0)| = 1$, we can multiply Ω by a constant independent of f so that $\xi(0) = 1$. The idea of Kodaira's result is to apply the proof of Jensen's theorem to ξ. We define two order functions by

$$T(r) = \int_{B[r]} \xi \beta^n, \qquad T^\#(r) = \int_{B[r]} \xi^{1/n} \beta^n,$$

where $\beta = ((-1)^{1/2}/4\pi) \sum_{j=1}^n dz_j \wedge d\bar{z}_j$, $B[r] = \{z \in C^n \mid \|z\| < r\}$ and $r < R$. The order function $T(r)$ has the natural interpretation as the volume of the image $f(B[r])$.

([8]) This is different from the d^c used in §1.

([9]) If L is very ample, take a basis $s^0, \ldots, s^N$ for the global sections, and define $a_\alpha = |s_\alpha^0|^2 + \cdots + |s_\alpha^N|^2$. Since L is very ample, a has the form $1 + |z_1|^2 + |z_n|^2 +$ higher order terms. Here the z_i are coordinates centered at an arbitrary point x. Then $dd^c \log a_\alpha$ is positive definite at x (see [2]).

([10]) We recall Kodaira's proof: Since $L = [X]$, where X is a nonsingular hyperplane section of V, we have the exact sequence $0 \to H^0(V, K^m \otimes L^*) \to H^0(V, K^m) \to H^0(X, K_X^m) \cdots$. Now $\dim H^0(V, K^m) = O(m^n)$ whereas $\dim H^0(X, K_X^m) = O(m^{n-1})$, so that $\dim H^0(V, K^m \otimes L^*) > 0$ for sufficiently large m.

Unfortunately, $T^\#(r)$ is required for technical reasons. Counting functions are defined by

$$n(t) = \int_{Z[t]} \psi^{n-1}, \qquad N(t) = \int_0^t n(s) \frac{ds}{s},$$

where $Z[t] = f^{-1}((s)) \cap B[t]$,

$$n_1(t) = \int_{R[t]} \psi^{n-1}, \qquad N_1(t) = \int_0^t n_1(s) \frac{ds}{s},$$

where $R[t] = (J) \cap B[t]$ and (J) is the divisor of $J(f) = 0$. The normalization conditions are simply that neither $Z = f^{-1}((s))$ nor (J) pass through the origin so that both $n(t)$ and $n_1(t)$ converge. The functions $n(t)$ and $n_1(t)$ should be thought of as the area of $Z[t]$, $R[t]$, respectively. Hence $n(t)$ and $n_1(t)$ are zero for small values of t, so the improper integrals $N(t)$ and $N_1(t)$ converge. We can now state

PROPOSITION 1. *Under the assumptions of Theorem 1 we have* $T(r) \geqq r^{2n}$ *and* $T^\#(r) \geqq l_0 r^{2n}$, *where* l_0 *is an absolute constant depending only on* $n = \dim V$.

PROPOSITION 2. *Under the assumptions of Theorem 1 we have*

$$\frac{nk^{1/n}}{m} \int_0^r T^\#(r) \frac{dt}{t^{2n-1}} + \frac{1}{m} N(r) + N_1(r) \leqq n \log \left[\frac{1}{r^{2n-1}} \frac{dT^\#}{dr} \right] + l_1.$$

Here k *is a positive constant such that* $c_1(L)^n \geqq k(n!/(4\pi)^n)\Omega$, *and* l_1 *is an absolute constant depending only on* n.

The Schottky-Landau theorem will follow easily from Propositions 1 and 2.

Preliminaries to the proof of both propositions. Consider $I(t) = \int_{B[t]} dd^c \log \xi \wedge \psi^{n-1}$. Let $T_\varepsilon(Z[t])$ and $T_\varepsilon(R[t])$ be ε-tubes around $Z[t]$, $R[t]$, respectively. Since $dd^c \log \xi$ is C^∞ on the region $B_\varepsilon^0[t] = B[t] - T_\varepsilon(Z[t]) - T_\varepsilon(R[t])$, Stokes' theorem gives

$$\int_{B_\varepsilon^0[t]} dd^c \log \xi \wedge \psi^{n-1} = \int_{S[t]} d^c \log \xi \wedge \psi^{n-1} - \int_{\partial T_\varepsilon(Z[t])} d^c \log \xi \wedge \psi^{n-1}$$

(1)

$$- \int_{\partial T_\varepsilon(R[t])} d^c \log \xi \wedge \psi^{n-1}.$$

Recall the residue formula for an analytic function f:

$$\lim_{\varepsilon \to 0} \int_{\partial T_\varepsilon(f)} d^c \log |f|^2 \wedge \psi^{n-1} = \int_{(f)} \psi^{n-1}$$

where $T_\varepsilon(f)$ is an ε-tube around the divisor (f). Moreover, if g is a function which does not vanish on (f), we easily find that

(2) $$\lim_{\varepsilon \to 0} \int_{\partial T_\varepsilon(f)} d^c \log g \wedge \psi^{n-1} = 0.$$

Now

$$d^c \log \xi = (1/m)d^c \log (a_\alpha \circ f) + (1/m)\, d^c \log |s_\alpha \circ f|^2 + d^c \log |J(f)|^2.$$

Using this, together with the residue formula, we find that

$$(3) \quad \lim_{\varepsilon \to 0} \int_{B^0_\varepsilon[t]} dd^c \log \xi \wedge \psi^{n-1} = \int_{S[t]} d^c \log \xi \wedge \psi^{n-1} - \frac{1}{m}\int_{Z[t]} \psi^{n-1} - \int_{R[t]} \psi^{n-1}.$$

On the other hand, since $dd^c \log |h|^2 = 0$ if h is a nonvanishing holomorphic function, we find that

$$
(4) \quad
\begin{aligned}
\int_{B^0_\varepsilon[t]} dd^c \log \xi \wedge \psi^{n-1} &= \frac{1}{m}\int_{B^0_\varepsilon[t]} dd^c \log (a_\alpha \circ f) \wedge \psi^{n-1} \\
&= \frac{1}{m}\int_{B^0_\varepsilon[t]} f^*c_1(L) \wedge \psi^{n-1}.
\end{aligned}
$$

Taking the limit in (4) and combining the result with (3), we obtain

$$(5) \quad \frac{1}{m}\int_{B[t]} f^*c_1(L) \wedge \psi^{n-1} + \frac{1}{m}\int_{Z[t]} \psi^{n-1} + \int_{R[t]} \psi^{n-1} = \int_{S[t]} d^c \log \xi \wedge \psi^{n-1}.$$

For convenience, we introduce $K(t) = \int_{B[t]} f^*c_1(L) \wedge \psi^{n-1}$. Since ψ has a singularity at the origin, we should show that $K(t)$ converges. This will be answered by the following lemma of Lelong (see [9, pp. 72–73]).

LEMMA. *Let θ be a closed positive current of type $(1, 1)$. Then*

$$(i) \quad \int_{B[r_1] - B[r_2]} \theta \wedge \psi^{n-1} = \frac{1}{r_1^{2n-2}}\int_{B[r_1]} \theta \wedge \beta^{n-1} - \frac{1}{r_2^{2n-2}}\int_{B[r_2]} \theta \wedge \beta^{n-1}.$$

(ii) *If θ is C^∞ at the origin, then the integral $\int_{B[r]} \theta \wedge \psi^{n-1}$ converges and* $\int_{B[r]} \theta \wedge \psi^{n-1} = (r^{2n-2})^{-1} \int_{B[r]} \theta \wedge \beta^{n-1}$.

(iii) *If θ is C^∞ at the origin, then $\int_{B[r]} \theta \wedge \psi^{n-1} \leq \mathrm{const} \cdot r^2$ for $r \leq r_0$.*

Since our notation is somewhat different from Lelong's we recall the proof. Since $B[r]$ is contractible, there is a current α such that $d\alpha = \theta$. An easy computation shows that $\psi^{n-1}|S[r] = (r^{2n-2})^{-1}\beta^{n-1}$. Then Stokes' theorem gives

$$
\begin{aligned}
\int_{B[r_1] - B[r_2]} \theta \wedge \psi^{n-1} &= \int_{S[r_1]} \alpha \wedge \psi^{n-1} - \int_{S[r_2]} \alpha \wedge \psi^{n-1} \\
&= \frac{1}{r_1^{2n-2}}\int_{S[r_1]} \alpha \wedge \beta^{n-1} - \frac{1}{r_2^{2n-2}}\int_{S[r_2]} \alpha \wedge \beta^{n-1}.
\end{aligned}
$$

Another application of Stokes' theorem gives (i). If θ is C^∞ at the origin, then for $r \leq r_0$ sufficiently small, we have

$$
\begin{aligned}
(*) \quad \frac{1}{r^{2n-2}}\int_{S[r]} \alpha \wedge \beta^{n-1} &= \frac{1}{r^{2n-2}}\int_{B[r]} \theta \wedge \beta^{n-1} \\
&\leq \frac{\mathrm{const}}{r^{2n-2}}\int_{B[r]} \beta^n \leq \frac{\mathrm{const}}{r^{2n-2}} \cdot \mathrm{vol}\,(B[r]) \leq \mathrm{const} \cdot r^2.
\end{aligned}
$$

Using (∗) in (i) to compute the limit as $r_2 \to 0$, we get (ii). The estimate (∗) applied to (ii) gives (iii).   Q.E.D.

Returning to the proof, we write (5) as

$$\frac{1}{m} K(t) + \frac{1}{m} n(t) + n_1(t) = \int_{S[t]} d^c \log \xi \wedge \psi^{n-1}.$$

We now integrate this from ε to r with respect to dt/t to get

(6) $$\frac{1}{m} \int_{\varepsilon}^{r} K(t) \frac{dt}{t} + \frac{1}{m} \int_{\varepsilon}^{r} n(t) \frac{dt}{t} + \int_{\varepsilon}^{r} n_1(t) \frac{dt}{t} = \int_{\varepsilon}^{r} \frac{dt}{t} \int_{S[t]} d^c \log \xi \wedge \psi^{n-1}.$$

Since $dt/t = d \log \|z\|$, Fubini's theorem gives

(7) $$\int_{\varepsilon}^{r} \frac{dt}{t} \int_{S[t]} d^c \log \xi \wedge \psi^{n-1} = \int_{B[t] - B[\varepsilon]} d \log \|z\| \wedge d^c \log \xi \wedge \psi^{n-1}.$$

If α and β are 1-forms and ϕ is an $(n-1, n-1)$ form, then $d\alpha \wedge d^c\beta \wedge \phi = -d^c\alpha \wedge d\beta \wedge \phi$. Therefore

$$d \log \|z\| \wedge d^c \log \xi \wedge \psi^{n-1} = d \log \xi \wedge d^c \log \|z\| \wedge \psi^{n-1}$$
$$= d(\log \xi \wedge d^c \log \|z\| \wedge \psi^{n-1}).$$

In the last step we have used the fact that $\psi^n = 0$. Hence another application of Stokes' theorem to (7) reduces (6) to

$$\frac{1}{m} \int_{\varepsilon}^{r} K(t) \frac{dt}{t} + \frac{1}{m} \int_{\varepsilon}^{r} n(t) \frac{dt}{t} + \int_{\varepsilon}^{r} n_1(t) \frac{dt}{t} = \int_{S[r]} \log \xi \sigma - \int_{S[\varepsilon]} \log \xi \sigma.$$

Now $K(t) = \int_{B[t]} f^* c_1(L) \wedge \psi^{n-1} \leqq \text{const} \cdot t^2$ by (iii) in the lemma above, so that the improper integral $\int_{0}^{r} K(t)(dt/t)$ converges. Moreover, $\int_{S[\varepsilon]} \log \xi \sigma$ is just the mean value of $\log \xi$ on $S[\varepsilon]$. Because of our normalization, $\log \xi(0) = 0$, so that $\lim_{\varepsilon \to 0} \int_{S[\varepsilon]} \log \xi \sigma = 0$.

Taking the limit as $\varepsilon \to 0$, the expression above becomes

(8) $$\frac{1}{m} \int_{0}^{r} K(t) \frac{dt}{t} + \frac{1}{m} N(t) + N_1(t) = \int_{S[r]} \log \xi \sigma.$$

Proof of Proposition 1. The idea is to reinterpret $\int_{S[r]} \log \xi \sigma$ using concavity of the logarithm and polar coordinates. We give the proof for $T^\#$, the proof for T being essentially the same. First, $\log \xi = n \log \xi^{1/n}$. Then concavity of the logarithm gives

$$\int_{S[r]} \log \xi \sigma \leqq n \log \int_{S[r]} \xi^{1/n} \sigma.$$

The formula for polar coordinates tells us that

$$\left(\frac{2\pi^n}{(n-1)!} \right) \int_{0}^{r} t^{2n-1} dt \int_{S[t]} \xi^{1/n} \sigma = \int_{B[r]} \xi^{1/n} \Phi = \left(\frac{(2\pi)^m}{n!} \right) \int_{B[r]} \xi^{1/n} \beta^n,$$

where $2\pi^n/(n-1)!$ is the volume of the unit sphere in C^n and $\Phi=((-1)^{1/2}/2)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$ is the standard euclidean volume form on $C^n \cong R^{2n}$. Differentiating the above identity, we get

$$\left(\frac{2^{n-1}}{n}\right)\frac{1}{r^{2n-1}}\frac{d}{dr}\int_{B[r]}\xi^{1/n}\beta^n = \int_{S[r]}\xi^{1/n}\sigma.$$

Applying these remarks to (8), we get

(9) $$\frac{1}{m}\int_0^r K(t)\frac{dt}{t} + \frac{1}{m}N(t) + N_1(t) \leqq n\log\left[\left(\frac{2^{n-1}}{n}\right)\frac{1}{r^{2n-1}}\frac{dT^\#}{dr}\right].$$

Now the left-hand side of (9) is nonnegative, which implies that

$$\left(\frac{2^{n-1}}{n}\right)\frac{1}{r^{2n-1}}\frac{dT^\#}{dr} \geqq 1.$$

Integrating the inequality $dT^\#/dr \geqq (n/2^{n-1})r^{2n-1}$ from 0 to r gives $T^\#(r) \geqq (n/(2n-1)2^{n-1})r^{2n}$, which we write as $T^\#(r) \geqq l_0 r^{2n}$.

Proof of Proposition 2. The idea is to reinterpret the term $\int_0^r K(t)\,dt/t$ in (9) using the order function $T^\#$. It is here that the assumption that L be positive is critical. We first remark that if G is a positive semidefinite hermitian matrix, then $\det G^{1/n} \leqq \text{trace } G/n$. This follows from the theorem on the geometric and arithmetic means, $(\lambda_1, \ldots, \lambda_n)^{1/n} = (\lambda_1 + \cdots + \lambda_n)/n$ applied to the eigenvalues of G. Now recall that if θ is a positive C^∞ $(1, 1)$ form, we have the relation between the projective and euclidean trace (see [9, pp. 72–73])

(10) $$\int_{B[r]}\theta \wedge \psi^{n-1} = \frac{1}{r^{2n-2}}\int_{B[r]}\theta \wedge \beta^{n-1}.$$

Now let $\theta = f^*c_1(L) = ((-1)^{1/2}/4\pi)\sum g_{kl}dz_k \wedge d\bar{z}_l$, where the hermitian matrix $G = (g_{kl})$ is positive definite. An easy computation gives $\theta \wedge \beta^{n-1} = (\text{trace } G)\beta^n$. Substituting into (10), we get

(11)
$$\int_{B[t]}f^*c_1(L) \wedge \psi^{n-1} = \frac{1}{t^{2n-2}}\int_{B[t]}(\text{trace } G)\beta^n$$
$$\geqq \frac{n}{t^{2n-2}}\int_{B[t]}(\det G)^{1/n}\beta^n.$$

Now V is compact and $c_1(L)^n$ is an everywhere positive (n, n) form, so there is a constant $k > 0$ such that $c_1(L)^n \geqq (n!k/(4\pi)^n)\Omega$. Pulling this back to $B[t]$, we get $f^*c_1(L)^n \geqq (n!k/(4\pi)^n)f^*\Omega$, or $(\det G)\beta^n \geqq k\xi\beta^n$. Therefore $\det G \geqq k\xi$ and

$$(\det G)^{1/n} \geqq k^{1/n}\xi^{1/n}.$$

Substituting this into (11) gives

$$\int_{B[t]}f^*c_1(L) \wedge \psi^{n-1} \geqq \frac{nk^{1/n}}{t^{2n-2}}\int_{B[t]}\xi^{1/n}\beta^n.$$

Recalling the definitions of K and $T^{\#}$, the estimate (9) becomes

$$(12) \qquad \left(\frac{nk^{1/n}}{m}\right)\int_0^r T^{\#}(t)\,\frac{dt}{t^{2n-1}} \quad \frac{1}{m}\,N(r)+N_1(r) \;\leqq\; n\log\left[\frac{1}{r^{2n-1}}\,\frac{dT^{\#}}{dr}\right]+l_1$$

where $l_1 = \log\,(2^{n-1}/n)\geqq 0$. Note that for $t\leqq t_0$,

$$\int_{B[t]} \xi^{1/n}\beta^n \;\leqq\; \mathrm{const}\cdot\mathrm{vol}\,(B[t]) \;\leqq\; \mathrm{const}\cdot t^{2n},$$

so that the integral $\int_0^r T^{\#}(t)\,dt/t^{2n-1}$ converges. This ends the proof of Proposition 2.

Proof of the Schottky-Landau theorem. Before giving the formal proof, we will give a heuristic argument. First, the basic estimate gives

$$(13) \qquad \frac{k^{1/n}}{m}\int_0^r T^{\#}(t)\,\frac{dt}{t^{2n-1}} \;\leqq\; n\log\left[\frac{1}{r^{2n-1}}\,\frac{dT^{\#}}{dr}\right]+l_1.$$

Suppose that $T^{\#}$ is extremely nice, let us say $T^{\#}(r)=r^{\alpha+2n}$. By Proposition 1, we know $\alpha\geqq 0$. Substituting into (8) we obtain

$$\frac{k^{1/n}}{m}\int_0^r t^{\alpha+1}\,dt \;\leqq\; \log\,[(\alpha+2n)t^{\alpha}]+l_1,$$

hence

$$(14) \qquad \mathrm{const}\cdot t^{\alpha+2} \;\leqq\; \mathrm{const}\cdot\log t+\mathrm{const}.$$

The left-hand side of this equation grows much faster than the right-hand side, so it is valid only for $r\leqq R_{\max}$ for some $R_{\max}<\infty$. A crude version of the Schottky-Landau theorem can be obtained by a slight refinement of this argument.

PROPOSITION. *If V is of general type and $f\colon B(R)\to W$ is nondegenerate, then $R<\infty$.*

Proof. We recall a standard calculus lemma of value distribution theory (see [11, p. 54]).

LEMMA. *Let ψ be a continuously differentiable, positive, increasing function on $[r_0,\infty]$, where $r_0\geqq 1$. Then $\psi'\leqq\psi^{\beta}$ for any $\beta>1$ on $J=[r_0,\infty]-I$, where I is an open set such that $\int_I d\log x<\infty$.*

One easily checks that $T^{\#}(r)$ satisfies the hypotheses of the lemma[11]. The inequality (13) yields

$$A\int_0^r T^{\#}(t)\,\frac{dt}{t^{2n-1}} \;\leqq\; \log\left[\frac{T^{\#2}(r)}{r^{2n-1}}\right]+l_1 \quad \text{on } J_1,$$

---

[11] For example, $dT^{\#}/dr=\mathrm{const}\cdot r^{2n-1}\int_{S[r]}\xi^{1/n}\sigma$. Since $\xi^{1/n}$ is continuous, Stokes' theorem shows $\int_{S[r]}\xi^{1/n}\sigma$ is a continuous function of r, so that $dT^{\#}/dr$ is continuous.

where $A = k^{1/n}/m$. Introduce $S(r) = \int_0^r T^\#(r)\, dt/t^{2n-1}$ and write the above equation as

$$AS(r) \leqq \log T^\#(r) + \log (T^\#(r)/r^{2n-1}) + l_1 \quad \text{on } J_1.$$

Now $dS/dr = T^\#(r)/r^{2n-1}$, and the lemma gives $dS/dr \leqq S^2$ on J_2. Therefore we get

$$AS(r) \leqq \log T^\#(r) + 2 \log S(r) + l_1 \quad \text{on } J_1 \cap J_2.$$

We rewrite this as

$$AS(r) \leqq (2n-1) \log r + \log (dS/dr) + 2 \log S(r) + l_1$$

and apply $dS/dr \leqq S^2$ on J_2 once more to get

$$AS(r) \leqq (2n-1) \log r + 4 \log S(r) + l_1 \quad \text{on } J_1 \cap J_2.$$

Dividing by $S(r)$ we get

$$A \leqq (2n-1) \frac{\log r}{S(r)} + \frac{4 \log S(r)}{S(r)} + \frac{l_1}{S(r)} \quad \text{on } J_1 \cap J_2.$$

Now $T^\#(r) \geqq l_0 r^{2n}$, so $S(r) \geqq (l_0/2) r^2$. Moreover, the complement of any set of finite logarithmic measure contains a sequence $\{r_j\}$ such that $r_j \to \infty$. Therefore we find that

$$(15) \qquad 0 < A \leqq \frac{2(2n-1)}{l_0} \frac{\log r_j}{r_j^2} + \frac{4 \log S(r_j)}{S(r_j)} + \frac{2l_1}{l_0 r_j^2}.$$

Using l'Hôpital's rule to compute $\lim (\log S(r_j)/S(r_j))$, we see that the right-hand side of (15) tends to zero as $r_j \to \infty$, thus the inequality is valid only for $r \leqq R_{\max} < \infty$.   Q.E.D.

   Proof of the precise Schottky-Landau theorem. The problem with the above "proof" is that the choice of J_1 and J_2 is not *a priori* independent of the order function $T^\#$, which certainly depends on f. The precise Schottky-Landau theorem is proved by a more precise analysis of the integrated order function $S(r) = \int_0^r T^\#(t)\, dt/t^{2n-1}$ (see [8]). For convenience, we write $\Omega(r) = (r^{2n-1})^{-1} dT^\#/dr$. Then (13) becomes

$$(16) \qquad AS(r) \leqq n \log B\Omega(r)$$

where $B = e^{l_1} = 2^{n-1}/n$.

   The first step in the proof is to obtain a tentative bound for $R_{\max}$ by assuming

$$(*) \qquad \Omega(r) \leqq r^{2n-1} S(r)^4.$$

Assuming (*), inequality (16) becomes

$$S(r) \leqq (n/A)(\log B + (2n-1) \log r + 4 \log S(r)).$$

Dividing by $S(r)$ and simplifying constants we get

$$1 \leq \frac{n}{A}\left(\frac{l_1}{S(r)} + (2n-1)\frac{\log r}{S(r)} + \frac{4\log S(r)}{S(r)}\right).$$

Using $S(r) \geq (l_0/2)r^2$ and $\log x \leq e^{-1}x$ for $x > 0$, the first two terms are dominated by $(2l_1/l_0)r^{-2}$, $(2(2n-1)e^{-1}/l_0)r^{-1}$, respectively. Writing $(4\log S)/S = (8\log S^{1/2})/S$ $\leq 8e^{-1}/S^{1/2} \leq 8(2)^{1/2}e^{-1}/l_0^{1/2}r$, and assuming $r \geq 1$, the above estimate becomes

$$1 \leq \frac{n}{A}\left\{\frac{2l_1}{l_0} + \frac{2(2n-1)e^{-1}}{l_0} + \frac{8(2)^{1/2}e^{-1}}{l_0^{1/2}}\right\}\frac{1}{r}.$$

We write this as $1 \leq r_0(k, n, p_0)r^{-1}$ where r_0 is a constant depending only on n, k, and the point p_0. Note that we have chosen $r_0 \geq 1$. We conclude that *if* $\Omega(r)$ $\leq r^{2n-1}S(r)^4$, *then* $r \leq r_0$.

Now assume that $r > r_0$. By what we have just shown, we know that one of the following must hold:

    (a) $T^{\#2}(r) < r^{2n-1}\Omega(r)$,

    (b) $r^{2n-1}S^2(r) < T^{\#}(r)$.

Suppose that (a) holds on an interval I. Using $\Omega(r) = (r^{2n-1})^{-1}dT^{\#}/dr$ we find that $dr < dT^{\#}/T^{\#2}$. Suppose (b) holds on an interval J. Using $dS/dr = T^{\#}(r)/r^{2n-1}$ we find $dr < dS/S^2$. Since both T and S are nondecreasing, this implies that on the interval $(r_0, R_{\max})$ we have $dr < dT^{\#}/T^{\#2} + dS/S^2$. Integrating this inequality we obtain

$$r - r_0 = \int_{r_0}^{r} dt < \int_{r_0}^{r} d\left(-\frac{1}{T^{\#}}\right) + \int_{r_0}^{r} d\left(-\frac{1}{S}\right),$$

$$r - r_0 < \frac{1}{T^{\#}(r_0)} + \frac{1}{S(r_0)} \leq \frac{l_0^{-1}}{r_0^{2n}} + \frac{2l_0^{-1}}{r_0^2}.$$

Since $r_0 \geq 1$, this yields $R_{\max} \leq r_0 + 3/l_0$. Thus we have obtained a bound on $R_{\max}$ depending only on n, k, and p_0.

III. A degeneracy theorem for holomorphic maps $f: C^n \to P_n - D$. In this section we discuss an n-dimensional generalization of Picard's theorem. The first result is the following:

THEOREM 1. *Let* $D \subseteq P_n$ *be a nonsingular divisor of degree* $d \geq n+3$. *Then any holomorphic mapping* $f: C^n \to P_n - D$ *is totally degenerate, i.e.* $|J(f)| \equiv 0$.

In dimension one this says that any entire function $f: C \to P_1 - \{\text{four points}\}$ is constant, a crude version of Picard's theorem. Another way of stating Theorem 1 is that any nondegenerate holomorphic map $f: C^n \to P_n$ must intersect D if D is smooth and has high enough degree.

Proof of the theorem. We construct a cyclic branched cover W_n of P_n satisfying

    (a) $W_n \xrightarrow{\pi} P_n$ is branched exactly along D.

    (b) W_n is a nonsingular hypersurface of degree d in P_{n+1}.

If $f_d(z_0, \ldots, z_n) = 0$ is a homogeneous equation for D, one easily verifies that such a W_n is given by $F(z_0, \ldots, z_{n+1}) = z_{n+1}^d - f_d(z_0, \ldots, z_n)$. Note that $W_n - D = W_n - \pi^{-1}(D)$ because the cover is cyclic. Because of condition (a), $\pi: W_n - D \to P_n - D$ is a covering map[12], so f lifts to a map $f': C^n \to W_n - D \subseteq W_n$.

$$
\begin{array}{ccc}
W_n - D & \longrightarrow & W_n \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
C^n \xrightarrow{\ f\ } P_n - D & \longrightarrow & P_n
\end{array}
$$
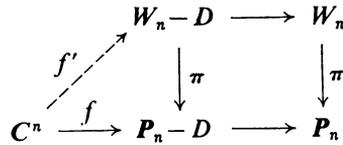
The idea is now to apply Griffiths' theorem on canonical algebraic varieties (see [3]), which implies that any holomorphic map $f: C^n \to W_n$ is totally degenerate if K_{W_n} is very ample. Thus it remains to show that the cyclic cover constructed above has very ample canonical bundle.

To see this recall that a line bundle L on a variety V is very ample if its global sections give an immersion of V into P_N. The immersion g_L is constructed by taking a basis $s^0, \ldots, s^N$ for the global sections of L and sending $x \to [s^0, \ldots, s^N]$, where $[w_0, \ldots, w_N]$ are homogeneous coordinates on P_N. Moreover, if $W \subseteq V$ is a submanifold of V, then the global sections of L on V restrict to global sections of L on W, so that if L is very ample on V, it is very ample on any subvariety W. By explicit computation, we see that any positive multiple of the hyperplane bundle $[H]$ on P_{n+1} is very ample, and hence very ample when restricted to any subvariety of P_{n+1}. Now we can explicitly compute K_W from the adjunction formula $K_W = K_{P_{n+1}} \otimes [W]|_W$, where $[W]$ is the line bundle determined by the divisor W. Recall that $K_{P_{n+1}} = [H]^{-(n+2)}$. Moreover, W is linearly equivalent to $[H]^d$. To see this, just observe that $g = f_d/z_0^d$ exhibits the linear equivalence. We conclude that $K_W = [H]^{d-(n+2)}|_W$, so that K_W is very ample if $d > n+2$.   Q.E.D.

To state the next theorem, we let $D \subseteq V_n$ be a possibly singular divisor. D is said to have *good singularities* if for each point $x \in D$ there are local analytic coordinates $z_1, \ldots, z_n$ on V_n centered at x such that a local equation for D is $z_1 \ldots z_k = 0$. Thus the singularities of D are locally normal crossings. For example, a union of nonsingular hypersurfaces meeting transversely with no more than n components passing through any point is such a divisor. An interesting special case is the union of hyperplanes in general position. Other examples are surfaces in P_2 with a nonsingular double curve, etc.

THEOREM 1'. *Let $D \subseteq P_n$ be a divisor of degree $d \geq n+3$ having good singularities. Let $f: B(R) \to P_n - D$ be a nondegenerate holomorphic map normalized so that $f(0) = p_0 \in P_n - D$ and $|J(f)(0)| = 1$. Then there is an absolute constant $R_{\max}(n, p_0)$ such that $R \leq R_{\max}(n, p_0)$.*

---

[12] In fact, $\pi_1(P_n - D) \cong Z_d$. The covering transformations are given by $[z_0, \ldots, z_{n+1}] \to [z_0, \ldots, pz_{n+1}]$, where $p = \exp(2\pi i/d)$. Hence $W_n - D$ is the universal cover of $P_n - D$.

As an immediate consequence we have

COROLLARY 1. *Let $D \subseteq P_n$ be a divisor of degree $d \geq n+3$ with good singularities. Then any holomorphic map $f: C^n \to P_n - D$ is totally degenerate.*

The proof is somewhat more complicated than in the previous case, although the idea is the same. We begin by defining a cyclic branched cover $W_n \xrightarrow{\pi} P_n$ which is branched exactly along D. Just as before, W_n is defined by $z_{n+1}^d - f_d(z_0, \ldots, z_n) = 0$. The technical problem is that W_n has singularities lying over the singularities of D. By means of explicit quadratic transformations we desingularize W_n to obtain an algebraic manifold $W_n' \xrightarrow{\pi'} P_n$. Because the singularities of W_n lie in the singular locus of D, $W_n' - \pi'^{-1}(D) \to W_n - \pi^{-1}(D)$. Moreover, $\pi': W_n' - \pi'^{-1}(D) \to P_n - D$ is a covering map, so we get a lifting $f': C^n \to W_n' - \pi'^{-1}(D) \subseteq W_n'$. If we can show that W_n' is of general type, Kodaira's theorem will apply to give Theorem 1'.

$$
\begin{array}{ccccc}
W_n - \pi^{-1}(D) & \simeq & W_n' - \pi'^{-1}(D) & \longrightarrow & W_n' \\
\nearrow f' & & \downarrow & \downarrow & \downarrow \\
C^n \xrightarrow{\quad f \quad} & P_n - D & = & P_n - D & \longrightarrow P_n
\end{array}
$$

The theorem is therefore proved modulo the lemma:

LEMMA 1. *Let $D \subseteq P_n$ be a divisor with good singularities such that $\deg(D) \geq n+3$. Let W_n' be a desingularization of the standard d-fold cyclic cover W_n of P_n branched exactly along D. Then W_n' is of general type.*

To give the proof of the lemma, we review the notion of rational differential forms and the residue operator [4]. Next we discuss the resolution of singularities. Then we will be able to calculate the so-called adjoint conditions which the singularities of W_n impose upon the rational differential forms in the ambient projective space, from which the lemma will follow.

(A) *The residue operator.* Let V_{n+1} be an $(n+1)$-dimensional complex manifold, and $W \subseteq V_{n+1}$ a *nonsingular* divisor. Given a rational $(n+1)$-form w with a first order pole along W, we want to define a holomorphic n-form $R(\omega)$ on W. We first solve the problem locally. The global result will follow by taking sections of the appropriate sheaves. Thus we set $V_{n+1} = P_{n+1}$, the $(n+1)$-dimensional poly-cylinder. On P_{n+1}, $\omega = (h\, dz_1 \wedge \cdots \wedge dz_{n+1})/f$, where f is a local defining equation for W. Since W is nonsingular, we may assume by shrinking P_{n+1} that one of the partial derivatives, say $\partial f/\partial z_{n+1}$, is nowhere zero on W. Then we have

$$
\left(\frac{\partial f}{\partial z_{n+1}}\right)^{-1} df = \left(\frac{\partial f}{\partial z_{n+1}}\right)^{-1}\left(\frac{\partial f}{\partial z_1} dz_1 + \cdots + \frac{\partial f}{\partial z_n} dz_n\right) + dz_{n+1},
$$

hence

$$
\omega = \frac{h\, dz_1 \wedge \cdots \wedge dz_n}{\partial f/\partial z_{n+1}} \wedge \frac{df}{f}.
$$

Define

$$R(\omega) = \frac{h \, dz_1 \wedge \cdots \wedge dz_n}{\partial f/\partial z_{n+1}},$$

i.e. we integrate out the logarithmic term df/f. One easily checks (see [4]) that the definition is independent of the choice of local defining equation, etc. The result of this is the Poincaré residue sequence

$$0 \longrightarrow K_V \longrightarrow K_V \otimes [W] \xrightarrow{R} K_W \longrightarrow 0.$$

The exact cohomology sequence globalizes this to give

$$H^0(K_V) \longrightarrow H^0(K_V \otimes [W]) \xrightarrow{R} H^0(K_W).$$

(B) *Rational differential forms.* A rational differential form on P_{n+1} with a first order pole along D is just a section of $K_{P_{n+1}} \otimes [D] = K_{P_{n+1}}(D)$. If $C^{n+1} \subseteq P_{n+1}$ is an affine piece of P_{n+1} given by $z_0 \neq 0$, then a section $\omega \in \Gamma(K_{P_{n+1}}(D))$ is given by

$$\omega = (P dz_1 \wedge \cdots \wedge dz_{n+1})/Q$$

where Q is the affine equation for D and P is a polynomial of degree $d-(n+2)$.

(C) *Resolution of singularities.* Suppose that $Z \subseteq V$ is a nonsingular subvariety of codimension $\geqq 2$. Then there are a variety V' and a holomorphic map $p: V' \to V$ such that

(a) $p: V' - p^{-1}(Z) \to V - Z$ is biholomorphic,

(b) $p^{-1}(Z) = P(N(Z))$.

Here $N(Z)$ is the normal bundle of Z and $P(N(Z))$ is the variety in V obtained by replacing each fiber N_z by the associated projective space $P(N_z)$. Note that V' is nonsingular and $P(N(Z))$ is a divisor. If $W \subseteq V$ is a singular subvariety of V, we set $W' = $ closure of $p^{-1}(W - Z)$. Notice that $p^{-1}(W) = W' \cup \mathscr{E}$, where $\mathscr{E} = P(N(Z))$ is the so-called exceptional set. We call $p^{-1}(W)$ the total transform of W and W' the proper transform. We say that V', W' are obtained by blowing up V, W, respectively, along Z. The aim is to resolve the singularities of V by a sequence of such transformations.

In order to perform the necessary computations, we need an explicit local description of the map p. Therefore, let P_n be a polycylindrical neighborhood of a point $x \in V$ such that $P_n \cap Z$ is defined by $z_1 = 0, \ldots, z_k = 0$ where $k = \operatorname{codim} Z$. Then $p^{-1}(P_n) = P_n'$ is defined as a subvariety of $P_n \times P_{k-1}$ as follows. Let $w_1, \ldots, w_k$ be homogeneous coordinates on P_{k-1}. Then $P_n' = \{(z_1, \ldots, z_n, w_1, \ldots, w_k) \mid w_i z_j = w_j z_i, 1 \leq i, j \leq k\}$. Now $P_n \times P_{k-1}$ is covered by open sets $P_n \times U_\alpha$, where $U_\alpha = P_{k-1} - \{w_\alpha = 0\}$. Coordinate functions on U_α are $\xi_j^\alpha = w_j/w_\alpha$, $1 \leq j \leq k$, $j \neq \alpha$. Thus coordinates on P_α are given by $\{z_\alpha, \xi_1^\alpha, \ldots, \hat{\xi}_\alpha^\alpha, \ldots, \xi_k^\alpha, z_{k+1}, \ldots, z_n\}$[13]. The map

_____

[13] Here $\frown$ denotes omission.

$p: P'_n \to P_n$ is induced by the projection $P_n \times P_{k-1} \to P_n$. Therefore, p is given locally by

$$p_\alpha(z_\alpha, \xi_1^\alpha, \ldots, \xi_k^\alpha, z_{k+1}, \ldots, z_n) = (z_\alpha \xi_1^\alpha, \ldots, z_\alpha, \ldots, z_\alpha \xi_k^\alpha, z_{k+1}, \ldots, z_n).$$

Since this notation is somewhat disgusting and cumbersome, we usually write

$$p_\alpha(\eta_1, \ldots, \eta_n) = (\eta_\alpha \eta_1, \ldots, \eta_\alpha, \ldots, \eta_\alpha \eta_k, \eta_{k+1}, \ldots, \eta_k).$$

EXAMPLE. Consider the variety defined by $f = z_1^2 + z_2^2 = 0$ in P_3. We perform a quadratic transformation (blowing up) centered along $\{z_1 = 0\} \cap \{z_2 = 0\}$. One of the projection maps is given by $p(\eta_1, \eta_2, \eta_3) = (\eta_1, \eta_1 \eta_2, \eta_3)$. The total transform is $f \circ p = \eta_1^2 + \eta_1^2 \eta_2^2 = \eta_1^2(1 + \eta_2^2)$. The smooth variety $1 + \eta_2^2 = 0$ is the proper transform, and $\eta_1^2 = 0$ is the exceptional set.

(D) *Calculation of the adjoint conditions.* The problem now is to find a $W' \subseteq P'_{n+1}$ which desingularizes W, and from this to calculate $K_{W'}$. The residue operator gives sections of $K_{W'}$ from sections of $K_{P'_{n+1}}(W')$, so what we really need is a description of the global sections of $K_{P'_{n+1}}(W')$. If ω is a global section of $K_{P_{n+1}}(W)$, then $p^*\omega$ has a pole along the total transform $W' \cup \mathscr{E}$. Therefore we must impose restrictions on sections of $K_{P_{n+1}}(W)$ such that $p^*\omega$ has a simple pole on W' but is holomorphic elsewhere. These are the so-called adjoint conditions on W. If we look at the problem locally, what we really want is a description of the subsheaf $p_*K_{P'_{n+1}}(W') \subseteq K_{P_{n+1}}(W)$. Then every global section of $p_*K_{P'_{n+1}}(W')$ pulls back to a global section of $K_{P_{n+1}}(W)$, and this will give us what we need. The calculation of the adjoint conditions is thus reduced to an essentially local problem. Our immediate goal is to prove

LEMMA 2. *For W as in Theorem $1'$, $p_*K_{P'_{n+1}}(W') = K_{P_{n+1}}(W)$, i.e. no adjoint conditions are imposed.*

The proof of the lemma is somewhat tedious, so we give an example to illustrate how the computations go.

EXAMPLE. Let $W \subseteq P_{n+1}$ be a hypersurface of degree d which is nonsingular except at the point x. Suppose that a local analytic equation for W at x is $f = z_1^k + \cdots + z_{n+1}^k = 0$. A rational $(n+1)$-form is given locally by

$$\omega = (h \, dz_1 \wedge \cdots \wedge dz_{n+1})/(z_1^k + \cdots + z_{n+1}^k).$$

Consider the quadratic transformation centered at x([14]). $p: P'_{n+1} \to P_{n+1}$ is given locally by $n+1$ projections

$$p_1(\eta_1, \ldots, \eta_{n+1}) = (\eta_1, \eta_1 \eta_2, \ldots, \eta_1 \eta_{n+1}),$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$p_{n+1}(\eta_1, \ldots, \eta_{n+1}) = (\eta_{n+1} \eta_1, \ldots, \eta_{n+1} \eta_n, \eta_{n+1}).$$

([14]) I.e., we blow up x and replace it by a P_n.

The total transform of W near x is given in pieces, for example by $f \circ p_1$ $= \eta_1^k + \eta_1^k \eta_2^k + \cdots + \eta_1^k \eta_{n+1}^k = \eta_1^k (1 + \eta_2^k + \eta_{n+1}^k)$. The proper transform is $f^1 = 1 + \eta_2^k$ $+ \cdots + \eta_{n+1}^k$, which is nonsingular. Moreover, we find that

$$p_1^* \omega = \frac{h \circ p \eta_1^n d\eta_1 \wedge \cdots \wedge d\eta_{n+1}}{f \circ p_1} = \frac{h \circ p \eta_1^{n-k} d\eta_1 \wedge \cdots \wedge d\eta_{n+1}}{f^1}.$$

Therefore, if $n \geq k$, the piece of the resolution given by p_1 imposes no adjoint conditions on ω. By symmetry, we see that the other pieces of the resolution give the same result. We conclude that W can be resolved in one step by $p: \boldsymbol{P}'_{n+1} \to \boldsymbol{P}_{n+1}$ and that no adjoint conditions are imposed, i.e. $p_* K_{\boldsymbol{P}_{n+1}}(W) = K_{\boldsymbol{P}_{n+1}}(W)$. Let ω be the globally defined rational $(n+1)$-form on $\boldsymbol{P}_{n+1}$ given by

$$\omega = (P dz_1 \wedge \cdots \wedge dz_{n+1})/Q$$

on a standard affine open $\boldsymbol{C}^{n+1} \subseteq \boldsymbol{P}_{n+1}$. Thus Q is an affine equation for W and P is a polynomial of degree $d - (n+2)$. The $p^*\omega$ is a globally defined rational $(n+1)$-form on $\boldsymbol{P}'_{n+1}$ with W' as simple polar locus. Since W' is nonsingular, the residue operation applied to $p^*\omega$ gives a holomorphic n-form on W'. Because of the explicit form of the residue operation given in (A), we see that the divisor of $R(p^*\omega)$ on $W' - \mathscr{E}$ is just the divisor $P \circ p = 0$.

Proof of Lemma 2. We first consider the case where $W \xrightarrow{\pi} \boldsymbol{P}_n$ is branched over D such that at most two branches of D meet at any point. Thus a local analytic equation for W is $f = z_{n+1}^d - z_1 z_2$. We blow up along the locus $\{z_1 = 0\}$ $\cap \{z_{n+1} = 0\}$[15]. Since $p^{-1}(P_{n+1}) \subseteq P_{n+1} \times \boldsymbol{P}_1$, the projection is given in two pieces:

$$p_A(\eta_1, \ldots, \eta_{n+1}) = (\eta_{n+1}\eta_1, \ldots, \eta_{n+1}),$$
$$p_B(\eta_1, \ldots, \eta_{n+1}) = (\eta_1, \eta_2, \ldots, \eta_1\eta_{n+1}).$$

A rational $(n+1)$-form on P_{n+1} with simple pole on W is represented on P_{n+1} by $\omega = (h dz_1 \wedge \cdots \wedge dz_{n+1})/f$. Now the total transform of f by p_A is $f \circ p_A = \eta_{n+1}^d$ $-(\eta_{n+1}\eta_1)\eta_2$ and the proper transform is $f^A = \eta_{n+1}^{d-1} - \eta_1\eta_2$. The pullback of ω is

$$p_A^* \omega = \frac{h \circ p_A \eta_{n+1} d\eta_1 \wedge \cdots \wedge d\eta_{n+1}}{f \circ p_A} = \frac{h \circ p_A d\eta_1 \wedge \cdots \wedge d\eta_{n+1}}{f^A}.$$

Therefore p_A^* has the proper transform f^A as simple polar locus. Doing the same calculation for p_B, we find $f^B = \eta_1^{d-1}\eta_{n+1}^d - \eta_2$ and

$$p_B^* \omega = (h \circ p_B d\eta_1 \wedge \cdots \wedge d\eta_{n+1})/f^B,$$

so that p_B^* has a pole only along W'. Note that f^B is smooth whereas f^A is still singular. However, the ramification of f^B has dropped by one. Therefore we

---

[15] The point is that we blow up along an algebraic subvariety of codimension 2 in $\boldsymbol{P}_{n+1}$. To determine the effect of this globally defined transformation, it is enough to see what happens locally at each point.

conclude that the process can be repeated d times to obtain a sequence of quadratic transformations

$$P_{n+1} \xleftarrow{\quad p_1 \quad} Q_1 P_{n+1} \xleftarrow{\quad p_2 \quad} Q_2 P_{n+1} \xleftarrow{\quad p_3 \quad} \cdots \xleftarrow{\quad p_d \quad} Q_d P_{n+1}$$

$$W \xleftarrow{\quad p_1 \quad} W_1 \xleftarrow{\quad p_2 \quad} W_2 \xleftarrow{\quad p_3 \quad} \cdots \xleftarrow{\quad p_d \quad} W_d$$

such that at each stage a rational $(n+1)$-form on $Q_i P_{n+1}$ with W_i as simple polar locus pulls back to give a rational differential form on $Q_{i+1} P_{n+1}$ with a pole only along W_{i+1}. In fact, the above diagram is given locally by

$$
\begin{array}{ccccc}
z_{n+1}^d - z_1 z_2 & \xleftarrow{\ B\ } & z_{n+1}^{d-1} - z_1 z_2 & \xleftarrow{\ B\ } & \cdots z_{n+1} - z_1 z_2 \xleftarrow{\ B\ } 1 - z_1 z_2 \\
{\scriptstyle A}\uparrow & & {\scriptstyle A}\uparrow & & {\scriptstyle A}\uparrow \\
z_1^{d-1} z_{n+1}^d - z_2 & & z_1^{d-2} z_{n+1}^{d-1} - z_2 & & z_{n+1} - z_2
\end{array}
$$

At each stage of the resolution, the computation of the adjoint conditions is the same as above. Moreover, each transform on the second row is a nonsingular piece, whereas the last piece on the first row is nonsingular, so the process terminates with a variety W_d having no singularities lying above $\{z_1 = 0\} \cap \{z_{n+1} = 0\}$ in the original W. Applying the same process to the finite number of components in the singular locus yields Lemma 2 in the special case.

Proof of Lemma 2 in general. Suppose that V_0 is obtained from P_{n+1} by a succession of quadratic transforms with nonsingular centers. Let $W_0 \subseteq V_0$ be the proper transform of W, where W is our standard cyclic cover. Let Σ_0 be a nonsingular variety of codimension 2 in V_0 such that for $x \in \Sigma_0$, W_0 has the local equation $f = z_1^{a_1} z_2^{a_2} \cdots z_l^{a_l} z_{n+1}^{a_{n+1}} - z_{l+1} \cdots z_k$. Thus W is given by such local equations, where $l = 0$. Define the weight of W_0 along Σ_0 by

$$\rho(W_0, \Sigma_0) = \min(k - l, a_1 + a_2 + \cdots + a_l + a_{n+1}).$$

Note that if $\rho(W_0, \Sigma_0) = 1$, then W_0 is nonsingular along Σ_0. Suppose we can prove that the map $p: V_1 \to V_0$ obtained by blowing up along Σ_0 satisfies

(a) $p(W_1, \Sigma_1) < p(W_0, \Sigma_0)$, where W_1, Σ_1 are the proper transforms of W_0, Σ_0, respectively.

(b) W_1 is given by local equations as above.

(c) $p_*(\Omega^{n+1}(W_1)) = \Omega^{n+1}(W_0)$, i.e. $p: V_1 \to V_0$ imposes no adjoint conditions.

Then iteration of this process progressively decreases the weight of the singularities of the transforms W_i and no adjoint conditions are imposed at any stage. Applying this to the irreducible components of the singular locus of W establishes the lemma, since the weights all go to zero after a finite number of steps.

To prove the above claim, we consider the local equation $f = z_1^{a_1} z_2^{a_2} \cdots z_l^{a_l} z_{n+1}^{a_{n+1}} - z_{l+1} \cdots z_k$. Blowing up along Σ_0, we get locally

$$f \circ p_A = \eta_{n+1}(\eta_1^{a_1} \eta_2^{a_2} \cdots \eta_l^{a_l} \eta_{n+1}^{a_{n+1}} - \eta_{l+1} \cdots \eta_k) = \eta_{n+1} f^A,$$

$$f \circ p_B = \eta_{l+1}(\eta_1^{a_1} \eta_2^{a_2} \cdots \eta_l^{a_l} \eta_{n+1}^{a_{n+1}-1} - \eta_{l+2} \cdots \eta_k) = \eta_{l+1} f^B.$$

Notice that the proper transforms have the correct form and that in each case, the weight drops by one. If $\omega = (h dz_1 \wedge \cdots \wedge dz_{n+1})/f$ is a rational $(n+1)$-form, we find that

$$p_A^* \omega = (h \circ p_A d\eta_1 \wedge \cdots \wedge d\eta_{n+1})/f^A,$$

$$p_B^* \omega = (h \circ p_B d\eta_1 \wedge \cdots \wedge d\eta_{n+1})/f^B,$$

so that no adjoint conditions are imposed. This completes the proof of Lemma 2.

Proof of Theorem 1'. We can now show that the desingularization W' of the standard cyclic cover W is of general type. For this we use the lemma:

LEMMA. *Let g_K be the map determined by the global sections of $K_{W'}$. Suppose that g_K has no base points*([16]) *and that* $\dim (\mathrm{im}\, g_K) = \dim W'$. *Then W' is of general type.*

Proof. Since g_K has no base points, we have a holomorphic map $g_K : W' \to W_1 \subseteq P_N$. Here W_1 is an irreducible, though generally singular variety of dimension $n = \dim W$. By construction, if H is a hyperplane section of W_1, the $g_K^{-1}(H)$ is a divisor belonging to $K_{W'}$. Now let $\pi : W_1 \to P_n$ be a generic linear projection. If H_0 is a hyperplane in P_n, then $\pi^{-1}(H_0)$ is a hyperplane section of W_1. Now

$$\dim H^0(P_n, [H_0]^k) = \binom{n+k}{k} = O(k^n).$$

Therefore, since independent sections of $[H_0]^k$ pull back to independent sections of $K_{W'}^k$ by $(\pi \circ g_K)^{-1}$, we see that W' is of general type.

To show that g_K has no base points, we must show that there is a section $\phi \in \Gamma(K_{W'})$ such that $\phi(x) \neq 0$, where $x \in W$ is arbitrary. We first let x be a simple point of W'. Now the singularities of W' are of codimension two, so that we can find a homogeneous polynomial P_0 of degree $d - (n+2)$ such that $P_0(x) \neq 0$. Let

$$\omega_0 = (P_0 dz_1 \wedge \cdots \wedge dz_{n+1})/Q$$

be the corresponding rational differential form. Since no adjoint conditions are imposed, $R(\pi^* \omega_0)$ gives a holomorphic n-form on W' which does not vanish at x. (Recall that outside the exceptional set $\mathscr{E}$, the divisor of $R(\pi^* \omega_0)$ is just $\pi^{-1}(P_0 = 0)$.) Therefore g_K has no base points lying above simple points of W. Now suppose that $x \in W$ is a singular point. We can perform the same construction, but in general $R(\pi^* \omega_0)$ may vanish on $\pi^{-1}(x)$. In our case, however, this does not happen. Referring to the calculation of the adjoint conditions, we see that

$$\pi^*(h dz_1 \wedge \cdots \wedge dz_{n+1})/f = (h \circ \pi dz_1 \wedge \cdots \wedge dz_{n+1})/f^\pi,$$

where f^π is the proper transform of the local defining equation f. Therefore, no new zeros are introduced by π^*. Hence $R(\pi^* \omega_0) \neq 0$ on $\pi^{-1}(x)$, and so g_K is without base points.

([16]) By this we mean that g_K is everywhere defined. If for all $x \in W'$ there is a section $\phi \in \Gamma(K_{W'})$ such that $\phi(X) \neq 0$, then g_K has no base points.

To see that $\dim(\text{im } g_K) = \dim W'$, we pick a simple point $x \in W$, and select forms of degree $d - (n+2)$, $P_0, \ldots, P_n$, such that $P_0(x) \neq 0$ and the divisors $(P_i = 0)$ meet transversely at x for $i = 1, \ldots, n$. We can do this because $[H]^{d-(n+2)}$ is very ample. Now let $\phi_i = R(\pi^* \omega_i)$, where ω_i is the rational differential determined by P_i. Among the coordinate functions of g_K are $\phi_1/\phi_0, \ldots, \phi_n/\phi_0$. The fact that the divisors $(P_i = 0)$ meet transversely at x shows that the ϕ_i/ϕ_0 give local coordinates there. Therefore the Jacobian of g_K has maximal rank at x, proving the claim. This completes the proof of Theorem 1'.

REMARKS. (1) One can compute the dimension of $H^0(W', K_{W'}^m)$ explicitly to show that W is of general type. The proof is not hard and is essentially contained in the calculation of the adjoint conditions.

(2) The fact that we can find a polynomial P of degree $d - (n+2)$ avoiding the singularities of W shows that there is a holomorphic n-form ϕ on W' which does not meet $\mathscr{E}$. Therefore the canonical bundle $K_{W'}$ restricted to any component of $\mathscr{E}$ is trivial. This shows that $\mathscr{E}$ is collapsed by all the pluricanonical maps g_{K^m}.

IV. Miscellaneous remarks and examples.

(1) The classical little Picard theorem can be obtained by the above techniques. To see this, let W be a five-sheeted branched cover of $\boldsymbol{P}_1$, branched with ramification index 4 at 0, 1, and ∞. Such a Riemann surface is guaranteed by the Riemann existence theorem[17]. Alternatively, one may desingularize the algebraic curve determined by the affine equation $z^5 = x(x-1)$. If ρ is the total ramification index of W, the Hurwitz formula gives $\rho = 2(g(W) - n + 1)$, where n is the number of sheets. Hence we find $g(W) = 2$. From Riemann surface theory we know that K_W^3 is very ample. Now let $f : \boldsymbol{C} \to \boldsymbol{P}_1 - \{0, 1, \infty\}$. We lift to $f' : \boldsymbol{C} \to W$, and apply the slightly strengthened version of Griffiths' canonical algebraic varieties theorem to conclude that f', and hence f, is constant.

We remark that this method fails to show that $f : \boldsymbol{C}^n \to \boldsymbol{P}_n - \{n+2 \text{ hyperplanes in general position}\}$ is totally degenerate.

(2) As another example, we prove the classical theorem that any holomorphic $f : \boldsymbol{C} \to \mathscr{E} - x$ is constant, where $\mathscr{E}$ is an elliptic curve and $x \in \mathscr{E}$ is an arbitrary point. Let $\mathscr{E}_1 \xrightarrow{\pi_1} \mathscr{E}$ be a two-sheeted unramified cover of $\mathscr{E}$. By the Hurwitz formula, $\mathscr{E}_1$ is an elliptic curve. Now let $W \xrightarrow{\pi_2} \mathscr{E}_1$ be a two-sheeted branched cover of $\mathscr{E}_1$ branched at $\{y_1, y_2\} = \pi^{-1}(x)$[18]. Now the Hurwitz formula $\rho = 2(g' - ng + n - 1)$

---

[17] This says that a Riemann surface can be constructed by knowing how the sheets interchange around the branch points. To be explicit, let $A_1, \ldots, A_k$ generate a transitive permutation group on $\{1, \ldots, n\}$. Suppose that $A_1 \cdot A_2 \cdots A_k = 1$. Then there is a Riemann surface W which is an n-sheeted branched cover of $\boldsymbol{P}_1$, branched at points $x_1, \ldots, x_n$. The effect of going around x_i is described by A_i.

[18] In general, if V is a complex manifold, $D \subseteq V$ a smooth divisor, there is a k-sheeted cyclic cover of V branched along D if and only if $c_1([D])$ is divisible, i.e. there is an integral cohomology class β such that $k\beta = c_1([D])$. (See J. Wavrick, *Deformations of branched coverings of complex manifolds*, Amer. J. Math. **90** (1968), 929.)

shows that $g' = g(W) = 2$. By lifting f from $\mathscr{E} - x$ to $\mathscr{E}_1 - \{y_1, y_2\}$ and then to W_1 and then applying Griffiths' theorem to W, we see that f must be constant.

REMARK. Using the same trick for "multiplying the Chern class of D" we may show that if A is an abelian variety and D is a smooth divisor such that the intersection number $(D^n) > 0$, then any holomorphic map $f: C^n \to A - D$ is totally degenerate.

(3) We can prove the following generalization of the big Picard theorem:

PROPOSITION. *Let V be a nonsingular hypersurface of degree $d \geq n + 3$ in P_n. Let f be a nondegenerate holomorphic map $f: \Delta^* \times \Delta^{n-1} \to P_n - V$. Then f extends to a holomorphic map $f: \Delta^n \to P_n$.*

Proof. Let W be the usual canonical algebraic variety which is a d-sheeted cyclic cover of P_n branched along V. Consider the diagram

$$
\begin{array}{ccccc}
\Delta^* \times \Delta^{n-1} & \xrightarrow{\ f_1\ } & W - V & \longrightarrow & W \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\Delta^* \times \Delta^{n-1} & \xrightarrow{\ f\ } & P_n - V & \longrightarrow & P_n
\end{array}
$$

where $p(z_1, \ldots, z_n) = (z_1^d, z_2, \ldots, z_n)$ and f_1 is a lifting of f. By Griffiths' theorem, f_1 extends to a holomorphic map $\bar{f}_1: \Delta^n \to W$. This implies (by commutativity of the diagram) that f is locally bounded. Thus f extends to $\bar{f}$ by the Riemann extension theorem.

REMARK. The classical big Picard theorem can also be proved by these methods. In the argument above we let W be the five-sheeted cyclic cover constructed in (1). Since K_W^3 is very ample, we know that the lifting of $f: \Delta^* \to P_1 - \{0, 1, \infty\}$ to $f': \Delta^* \to W$ extends as a rational map. Now for one complex variable, there are no points of indeterminacy, so f' is holomorphic. The argument now proceeds as before.

The above theorem has several useful corollaries.

COROLLARY 1. *Let $f: A \to P_n - V$ be holomorphic, where A is an open algebraic variety and V is smooth and of degree $d \geq n + 3$. Then f extends to a holomorphic map $\bar{f}: \bar{A} \to P_n$, for any smooth compactification $\bar{A}$.*

Proof. Immediate, using the above proposition. Any smooth compactification works because the local extensions are holomorphic, so we do not have to worry about points of indeterminacy.

COROLLARY 2. *Let V_1, V_2 be two smooth hypersurfaces of degree $d \geq n + 3$ in P_n. Then $P_n - V_1$ is biholomorphic to $P_n - V_2$ if and only if there is a projective linear automorphism of P_n which carries V_1 onto V_2.*

Proof. Let $f\colon \boldsymbol{P}_n - V_1 \to \boldsymbol{P}_n - V_2$ and $g\colon \boldsymbol{P}_n - V_2 \to \boldsymbol{P}_n - V_1$ be maps exhibiting the isomorphism. Thus $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. By Corollary 1, f extends to a holomorphic map $\bar{f}\colon \boldsymbol{P}_n \to \boldsymbol{P}_n$, and we obviously have $\bar{f}(V_1) = V_2$. We have a similar extension $\bar{g}$ of g. Now $\bar{g} \circ \bar{f}$ restricts to the identity map on the open set $\boldsymbol{P}_n - V_1$, so $\bar{g} \circ \bar{f} = \mathrm{id}_{\boldsymbol{P}_n}$. The same reasoning shows that $\bar{f} \circ \bar{g} = \mathrm{id}_{\boldsymbol{P}_n}$, hence $\bar{f}$ is a biholomorphic map of $\boldsymbol{P}_n$ into itself. Now the only holomorphic automorphisms of $\boldsymbol{P}_n$ are the projective linear automorphisms, which completes the proof.

COROLLARY 3. *The automorphism group of $\boldsymbol{P}_n - V$ is finite.*

Proof. Any automorphism β of $\boldsymbol{P}_n - V$ lifts to an automorphism of $W - V$, where W is the standard d-fold cyclic cover of $\boldsymbol{P}_n$ branched along V. The same argument as in the proof of Corollary 2 shows that β extends to an automorphism $\bar{\beta}$ of W. Now the automorphism group of a canonical algebraic variety is always finite (see [7]).

(4) We close by noting that there are many particular cases in which very strong degeneracy theorems can be obtained by essentially one-variable techniques. For example, let $|H(d)|$ be the linear system of hypersurfaces of degree d on $\boldsymbol{P}_n$. Pick $f_0, f_1 \in |H(d)|$ to be independent. Then the set of hypersurfaces $\{\lambda f_0 + \mu f_1\}$ forms a pencil. We set $f(\alpha) = f_0 + \alpha f_1$ and $f(\infty) = f_0$. Notice that all the $f(\alpha)$ intersect along the base locus $B = f(0) \cap f(\infty)$. Now the pencil $\mathscr{P}$ gives a holomorphic map $\pi\colon \boldsymbol{P}_n - \{f(0), f(1), f(\infty)\} \to \boldsymbol{P}_1 - \{0, 1, \infty\}$ by sending $x \to [f_0(x), f_1(x)]$. If $g\colon \boldsymbol{C} \to \boldsymbol{P}_n - \{f(0), f(1), f(\infty)\}$ is holomorphic, we conclude by Picard's theorem applied to $\pi \circ g$ that the image of g lies in some fiber $f(\alpha)$. By applying this argument to all complex lines through the origin in $\boldsymbol{C}^m$, we see that the image of $g\colon \boldsymbol{C}^m \to \boldsymbol{P}_n - \{f(0), f(1), f(\infty)\}$ lies in a fiber.

Using this observation, we see that by using several pencils in general position we can require $\mathrm{im}\,(g)$ to lie in the transverse intersection of several fibers. In this way we can progressively decrease the dimension of $\mathrm{im}\,(g)$ and in all cases $\mathrm{im}\,(g)$ lies in an algebraic subvariety. We can also drop the dimension of $\mathrm{im}\,(g)$ by increasing the degree of the pencil. For example, let $\mathscr{P}$ be a pencil of generically nonsingular cubic curves in $\boldsymbol{P}_2$. There will be a certain number of singular fibers $C(\alpha_1), \ldots, C(\alpha_k)$. If $k < 3$ we add more nonsingular fibers to get $k \geq 3$. Now let $f\colon \boldsymbol{C} \to \boldsymbol{P}_2 - (C(\alpha_1) \cdots C(\alpha_k))$. Then $\mathrm{im}\,(f)$ lies in $C(\alpha) - C(\alpha_1)$, where $\alpha = \alpha_i$, $i = 1, \ldots, k$. In other words, f maps into an elliptic curve minus at least one point, and hence is constant.

REMARKS. (1) The above examples are not essentially deep, since they use only one complex variable techniques. However, they lead one to ask whether there are conditions on a divisor $D \subseteq \boldsymbol{P}_n$ such that for $f\colon \boldsymbol{C}^{n-1} \to \boldsymbol{P}_n - D$, $\mathrm{im}\,(f)$ lies in a divisor, preferably a divisor in the complete linear system determined by D.

(2) The above examples show how the requirement of general position makes things harder. In fact, if we look at a configuration of more than three lines in $\boldsymbol{P}_2$, then either they are in general position or they contain three elements of a pencil.

BIBLIOGRAPHY

1. S. S. Chern, *Differential geometry, its past and future*, Internat. Congress Math., Nice, France, 1970.

2. P. Griffiths, *Hermitian differential geometry, Chern classes, and positive vector bundles*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 185–251. MR **41** #2717.

3. ———, *Holomorphic mappings into canonical algebraic varieties*, Ann. of Math. **93** (1971), 439–458.

4. ———, *On the periods of certain rational integrals*. I, II, Ann. of Math. (2) **90** (1969), 460–495; ibid., 496–541. MR **41** #5357.

5. H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368. MR **25** #583.

6. F. Hirzebruch, *Topological methods in algebraic geometry*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 9, Springer-Verlag, Berlin, 1962; English transl., Die Grundlehren der math. Wissenschaften, Band 131, Springer-Verlag, New York, 1966. MR **25** #1155; MR **34** #2573.

7. S. Kobayashi, *On the automorphism group of a certain class of algebraic manifolds*, Tôhoku Math. J. (2) **11** (1959), 184–190. MR **22** #3014.

8. K. Kodaira, *On holomorphic mappings of polydiscs into compact complex manifolds* (to appear).

9. P. Lelong, *Plurisubharmonic functions and differential forms*, Gordon and Breach, New York, 1969.

10. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966. MR **35** #1420.

11. H. Wu, *The equidistribution theory of holomorphic curves*, Ann. of Math. Studies, no. 64, Princeton Univ. Press, Princeton, N. J., 1970.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

Current address: Department of Mathematics, Stanford University, Stanford, California 94305