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HYPERSURFACES OF NONNEGATIVE CURVATURE IN A HILBERT SPACE

BY

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ABSTRACT. In this paper we prove the following generalizations of known theorems about hypersurfaces in $\mathbf{R}^n$: Let M be a hypersurface in a Hilbert space. (1) If on M the sectional curvature $K(\sigma)$ is nonnegative for every 2-plane section σ and if $K(\sigma) > 0$ for at least one σ, then M is the boundary of a convex body. (2) If $K(\sigma) = 0$ for all σ, then M is a hypercylinder. The main tool in these theorems is Smale's infinite dimensional Sard's theorem.

1. **Introduction.** Throughout this paper M denotes a *Riemannian Hilbert manifold*; that is, M is a C^∞ connected manifold, modelled on a separable Hilbert space, such that for each $x \in M$ there exists an inner product $\langle \cdot , \cdot \rangle_x$ in M_x, the tangent space of M at x, which varies differentiably with x. A precise definition may be found in [4]. M can be made into a metric space by letting the distance $d(x, y)$ between two points x and y be the infimum of the lengths of differentiable curves joining them. M is said to be *complete* if it is complete in this metric.

A C^∞ map ξ: $M \longrightarrow H$ of a Riemannian Hilbert manifold M into a Hilbert space H is an *immersion* if the differential $d\xi_x$: $M_x \longrightarrow H$ is one-to-one and $d\xi_x(M_x) \subset H$ is closed in H. If ξ is a homeomorphism onto its image, ξ is called an *embedding*.

An *isometric immersion* is an immersion ξ: $M \longrightarrow H$ such that $d\xi_x$: $M_x \longrightarrow H$ is an isometry at each point $x \in M$. If in addition $d\xi_x(M_x) \subset H$ has codimension one, we call M a *hypersurface* in H. In this case we do not assume that ξ is one-to-one.

Just as in the case of finite dimensional manifolds, the metric on M induces a unique covariant derivative, and we may define the curvature tensor, geodesics, and sectional curvatures. This has been done in [5] where it is also shown that every point has a convex neighbourhood and that completeness implies geodesic completeness.

So far, relatively little is known about the global differential geometry of Riemannian Hilbert manifolds, the most interesting exception being perhaps M. P. doCarmo's partial generalization of a theorem of Sacksteder ([1], [7]). In his paper, doCarmo shows that, if ξ immerses M isometrically in H as a hypersurface, and if at each $x \in M$ all sectional curvatures are positive and bounded away

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from zero by a positive constant δ_x depending only on x, then ξ is an embedding and $\xi(M)$ is the boundary of a convex body in H. The hypothesis of doCarmo's theorem is a good deal stronger than that used by Sacksteder, who for a hypersurface in $\mathbf{R}^n$ gets the same conclusion by assuming only that all sectional curvatures are nonnegative, and that there is at least one section at one point at which the sectional curvature is strictly positive.

In this paper we will present a method for reducing this kind of theorem about hypersurfaces in Hilbert space to theorems about hypersurfaces in $\mathbf{R}^n$. The idea is very simply to intersect the given hypersurface $\xi: M \longrightarrow H$ with finite dimensional linear submanifolds L of H. Using Smale's infinite dimensional version of Sard's theorem [8], we show in § 2 that for almost all choices of L, M and L are transverse (Theorems 2.6, 2.7). After proving some technical results in § 3, we show in § 4 that, if L and M are transverse, $\xi^{-1}(L)$ is a submanifold of M whose sectional curvatures have the same sign as the sectional curvatures of the corresponding sections of M.

In § 5 we apply our method to obtain the infinite dimensional version of Sacksteder's theorem with the weakest possible hypothesis (Theorem 5.1).

In § 6 we apply our method to obtain the infinite dimensional counterpart of a theorem of Pogorelov [6] and Hartman and Nirenberg [2] which states that, if the hypersurface M has zero sectional curvatures, $\xi(M)$ is a hypercylinder (Theorem 6.1).

2. **Transverse linear sections.** Throughout this paper H will denote a fixed separable Hilbert space, and M will denote a Riemannian Hilbert manifold immersed in H as a hypersurface by an isometric immersion ξ. We will use M_x to denote the tangent space to M at x and T_x to denote the linear subspace $d\xi(M_x) \subset H$.

By a *linear submanifold* of H we mean a set of points of the form

$$L + p = \{q + p \mid q \in L\}$$

where L is any linear subspace of H and p is a point in H. If $\{L_1, \cdots, L_k\}$ is a set of linear submanifolds of H we shall use $[L_1, \cdots, L_k]$ to denote its *linear hull*; that is,

$$[L_1, \cdots, L_k] = \left\{ p \in H \,\middle|\, p = \sum_{i=1}^{k} \lambda^i p_i,\ p_i \in L_i,\ \sum_{i=1}^{k} \lambda^i = 1 \right\}.$$

Occasionally we will also use $|[L_1, \cdots, L_k]|$ to denote its *convex hull*; that is,

$$|[L_1, \cdots, L_k]| = \left\{ p \in H \,\middle|\, p = \sum_{i=1}^{k} \lambda^i p_i,\ p_i \in L_i,\ \sum_{i=1}^{k} \lambda^i = 1,\ \lambda^i \geq 0 \right\}.$$

If $L_1, \cdots, L_k$ are all finite dimensional, the set $\{L_1, \cdots, L_k\}$ is said to be *in general position* whenever

$$\dim [L_1, \cdots, L_k] = \sum_{i=1}^{k} \dim L_i + (k - 1).$$

Throughout this paper we will use $G(H, k)$ to denote the space of all linear k-manifolds in H $(k < \infty)$, and if $N \in G(H, n)$, $n < k$, we will let $G(H, k, N)$ denote the space of all linear k-submanifolds of H that contain N. We regard $\emptyset$ as a linear (-1)-manifold in H; so we identify $G(H, k, \emptyset) = G(H, k)$.

Now let $L_0 \in G(H, k)$ pass through the origin of H, let K be the orthogonal complement of L_0 and let $\Pi : H \longrightarrow L_0$ and $\overline{\Pi} : H \longrightarrow K$ be the orthogonal projections. Let

$$V(L_0) = \{L \in G(H, k) | \ \Pi | L \text{ is one-to-one}\}.$$

It is clear that these sets cover $G(H, k)$. We may then define a coordinate map on $V(L_0)$ depending on a fixed basis $v_1, \cdots, v_k$ for L_0:

$$\phi : V(L_0) \longrightarrow K^{k+1}$$

is defined by

$$\phi(L) = (\overline{\Pi}(\Pi|L)^{-1}(0), \overline{\Pi}(\Pi|L)^{-1}(v_1), \cdots, \overline{\Pi}(\Pi|L)^{-1}(v_k)). \tag{1}$$

The following lemma may be proved by a straightforward computation:

Lemma 2.1. *$G(H, k)$ is a C^∞ manifold with the structure given by* (1). *Moreover, if $\emptyset \neq N_1 \subset N_2$, $G(H, k, N_2)$ is an embedded C^∞ submanifold of $G(H, k, N_1)$; and $G(H, k, N_1)$, of $G(H, k)$.*

Whenever for $L \in G(H, k)$ and $\xi(x) \in L$ we have $(L - \xi(x)) + T_x = H$ we will say that L and M are *transverse* at x (or at $\xi(x)$).

For any $N \in G(H, n)$, $n < k$, let $S(H, k, N) \subset H^{k-n}$ be the set of $(k-n)$-tuples $(p_{n+1}, \cdots, p_k) \in H^{k-n}$ such that $\{N, p_{n+1}, \cdots, p_k\}$ is in general position. We will also write $S(H, k)$ for $S(H, k, \emptyset)$. This will be standard notation throughout this paper. We also introduce $\Lambda : S(H, k, N) \longrightarrow G(H, k, N)$ defined by

$$\Lambda(p_{n+1}, \cdots, p_k) = [N, p_{n+1}, \cdots, p_k]. \tag{2}$$

The following lemma is easy to prove:

Lemma 2.2. *If $N \in G(H, n)$, $k < n$, then $S(H, k, N)$ is a fibre bundle over $G(H, k, N)$ with projection Λ and a fibre of dimension $k(k-n)$. Thus $\Lambda : S(H, k, N) \longrightarrow G(H, k, N)$ is a Fredholm map of index $k(k-n)$.*

A C^∞ map Λ between C^∞ manifolds is called a Fredholm map if at each point the image of $d\Lambda$ is closed and if dim ker $d\Lambda$ and codim im $d\Lambda$ are finite. The index at a point is then given by

$$\text{index } \Lambda = \dim\ker d\Lambda - \operatorname{codim}\operatorname{im} d\Lambda. \tag{3}$$

Corollary 2.3. *Let $L \in G(H, k, N)$ contain points $p_1, \cdots, p_r$. Then for a given $\delta > 0$ there exists a neighbourhood V of L in $G(H, k, N)$ so that, if $L' \in V$, L' contains points $p_1', \cdots, p_r'$ with $\|p_j' - p_j\| < \delta$, $j = 1, \cdots, r$.*

Finally, throughout this paper we will also let $S_1(H, k, N) \subset M \times H^{k-n-1}$ denote the set of $(k-n)$-tuples $(x, p_{n+2}, \cdots, p_k) \in M \times H^{k-n-1}$ so that

$$(\xi(x), p_{n+2}, \cdots, p_k) \in S(H, k, N).$$

With $S_1(H, k, N)$ we associate the mapping $\Lambda_1 : S_1(H, k, N) \longrightarrow G(H, k, N)$ defined by

$$\Lambda_1 = \Lambda \circ (\xi \times (\mathrm{id}_H)^{k-n-1}). \tag{4}$$

Lemma 2.4. *The maps $\Lambda_1 : S_1(H, k, N) \longrightarrow G(H, k, N)$ are smooth Fredholm maps.*

Proof. Clearly, letting $L_0 = \Lambda_1(x, p_{n+2}, \cdots, p_k) - p$ where $p \in N$, or $p \in \Lambda_1(x, p_{n+2}, \cdots, p_k)$ if $N = \emptyset$, and letting K, Π, $\overline{\Pi}$ be defined just as before, we have, for $u \in T_x$, $u_j \in H$, $j = n+2, \cdots, k$,

$$d\Lambda_1(u, u_{n+2}, \cdots, u_k) = (\overline{\Pi} d\xi(u), \overline{\Pi}(u_{n+2}), \cdots, \overline{\Pi}(u_k)).$$

Hence,

$$\ker d\Lambda_1 = (T_x \cap L_0) \times L_0^{k-n-1}, \qquad \operatorname{im} d\Lambda_1 = (\overline{\Pi} T_x) \times K^{k-n-1}, \tag{5}$$

so that

$$\begin{aligned} \text{index } d\Lambda_1 &= \dim\ker d\Lambda_1 - \operatorname{codim}\operatorname{im} d\Lambda_1 \\ &= (k(k-n-1) + k - \epsilon) - (1 - \epsilon) = k(k-n) - 1. \end{aligned}$$

Q.E.D.

Corollary 2.5. *If $L \in G(H, k, N)$ is transverse to M at $x_1, \cdots, x_s$ then for given $\delta > 0$ there exists a neighbourhood V of L so that, if $L' \in V$, L' contains points $\xi(x_1'), \cdots, \xi(x_s')$ with $x_i' \in U_\delta(x_i)$, $i = 1, \cdots, s$, where $U_\delta(x_i)$ denotes the open geodesic δ-ball about x_i.*

Proof. For each $i = 1, \cdots, s$ we have $L = \Lambda_1(x_i, q_1, \cdots, q_{n_i})$ for some $q_1, \cdots, q_{n_i} \in H$, so that by (5) there is a neighbourhood V_i of L so that $L' \in V_i \Rightarrow L'$ contains a point $\xi(x_i')$, $x_i' \in U_\delta(x_i)$. The corollary follows if we take $V = \bigcap_{i=1}^r V_i$. Q.E.D.

Theorem 2.6. *Let $N \in G(H, n)$, $n < k$, be transverse to M. Then*

$$\{L \in G(H, k, N) \mid L \text{ not transverse to } M\}$$

is of the first category in $G(H, k, N)$.

Proof. Note that $L \in G(H, k, N)$ fails to be transverse to M if and only if it is a singular value for $\Lambda_1 : S_1(H, k, N) \longrightarrow G(H, k, N)$. Since Λ_1 is a Fredholm map, this set of singular values is of the first category in $G(H, k, N)$ (see [8, Theorem 1.3]). Q.E.D.

In the case $N = p$, the condition that N and M be transverse is that $p \notin M$. We have the following stronger theorem for this case:

Theorem 2.7. *For any* $p \in H$, *the set*

$$\{L \in G(H, k, p) | L \text{ not transverse to } M\}$$

is of the first category in $G(H, k, p)$.

Proof. As in the preceding theorem,

$$X = \{L \in G(H, k, p) | L \text{ not transverse to } M - \xi^{-1}(p)\}$$

is of the first category. It remains to show that the set

$$Y = \{L \in G(H, k, p) | L \text{ not transverse to } M \text{ at } p\}$$

is of the first category. Since M is immersed, the set $\xi^{-1}(p) \subset M$ is discrete; since M is separable, $\xi^{-1}(p)$ must then be countable, say $\xi^{-1}(p) = \{x_1, x_2, \cdots\}$. Now M and L fail to be transverse at p if and only if $L \subset T_{x_j} + p$ for some integer j. Let Y_j be the set of $L \in G(H, k, p)$ such that $L \subset T_{x_j} + p$. This is clearly a nowhere dense set. Hence $Y = \bigcup_{j=1}^{\infty} Y_j$ is of the first category. Q.E.D.

3. **The connected components of** $\xi^{-1}(L)$. If $x \in M$, let $U_\delta(x)$ denote the geodesic δ-ball about x, $U_\delta(x) = \{y \in M | d(y, x) < \delta\}$. The purpose of this section is to prove the following proposition:

Proposition 3.1. *Let* $x_1, \cdots, x_r \in M$. *Then there is an integer* k *and an* $L \in G(H, k)$ *so that* $x_1, \cdots, x_r$ *belong to the same connected component of* $\xi^{-1}(L)$. *Moreover, for each* $\delta > 0$ *there is a neighbourhood* V *of* L *so that*

$$L' \in V \Rightarrow \xi^{-1}(L') \cap U_\delta(x_i) \neq \emptyset \quad \text{for } i = 1, \cdots, r,$$

and so that, if $y_i \in \xi^{-1}(L') \cap U_\delta(x_i)$, $i = 1, \cdots, r$, *then* $y_1, \cdots, y_r$ *belong to the same connected component of* $\xi^{-1}(L')$.

To prove it we need a lemma. Let $\nu(x)$ denote a unit normal to $\xi(M)$ at $x \in M$.

Lemma 3.2. *Each point* $z_0 \in M$ *has a neighbourhood* $U \subset M$ *together with a neighbourhood* B *of* $\nu(z_0)$ *so that, for any* $z_1, z_2 \in U$ *and* $q \in \xi(z_0) + B$, z_1 *and* z_2 *lie in the same connected component of* $\xi^{-1}[\xi(z_1), \xi(z_2), q]$, *and* $[\xi(z_1), \xi(z_2), q]$ *and* M *are transverse at* z_1 *and* z_2.

Proof. Let W be a neighbourhood of $\nu(z_0)$ in $S = \{v \in H \mid \|v\| = 1\}$. For $\mu \in W$ let $\Pi_\mu : H \longrightarrow T_{z_0}$ be the projection given by

$$q = \Pi_\mu(q) + \lambda\mu, \qquad \lambda \in \mathbf{R}.$$

The lemma is proved if we can show that, for W small enough and for z_1, z_2 belonging to a suitably small neighbourhood U of z_0, the line segment joining $\Pi_\mu(\xi(z_1))$ and $\Pi_\mu(\xi(z_2))$ is the $(\Pi_\mu \circ \xi)$-image of a continuous curve in M.

Clearly $\Pi_\mu(q)$ depends differentiably on $(\mu, q) \in W \times H$. Define the differentiable map $\theta : W \times M \longrightarrow W \times T_{z_0}$ by

$$\theta(\mu, z) = (\mu, \Pi_\mu(\xi(z) - \xi(z_0))).$$

Since $d\theta$ is then given by

$$d\theta = \mathrm{id}_H \times d\xi : H \times M_{z_0} \longrightarrow H \times T_{z_0},$$

there is an open geodesic ϵ-ball $B'_\epsilon \subset S$ with center $\nu(z_0)$ and a neighbourhood U' of $z_0 \in M$ so that $\theta : B'_\epsilon \times U' \longrightarrow B'_\epsilon \times T_{z_0}$ is a diffeomorphism onto its image. In particular, for each $\mu \in B'_\epsilon$,

$$\Pi_\mu \circ (\xi - \xi(z_0)) : U' \longrightarrow T_{z_0}$$

is a diffeomorphism. Choose U' so that, for some $r > 0$,

$$\Pi_{\nu(z_0)}(\xi(U') - \xi(z_0)) = D(r),$$

the open r-ball about $0 \in T_{z_0}$, with r so small that, for $z \in U'$, $\delta > 0$,

$$\|\Pi_{\nu(z_0)}(\xi(z) - \xi(z_0)) - \xi(z) + \xi(z_0)\| < \delta.$$

Now put

$$U = (\Pi_{\nu(z_0)} \circ (\xi - \xi(z_0)) | U')^{-1}(D(r - 2\delta \tan \epsilon))$$

and let $\mu \in B'_\epsilon$. Let

$$p_t = (1 - t)(\xi(z_1) - \xi(z_0)) + t(\xi(z_2) - \xi(z_0)), \qquad 0 \le t \le 1,$$

and let l_t, $0 \le t \le 1$, be the line $l_t(s) = s\mu + p_t$, $s \in \mathbf{R}$. It is enough to show that l_t intersects $\xi(U') - \xi(z_0)$.

Let Π_0 be the function on H given by

$$p = \Pi_{\nu(z_0)}(p) + \Pi_0(p)\nu(z_0).$$

Then

$$\|\Pi_{\nu(z_0)}(l_t(s)) - \Pi_{\nu(z_0)}(p_t)\| < \tan \epsilon \, |\Pi_0(l_t(s)) - \Pi_0(p_t)|.$$

Let f be the C^∞ function defined on $(\Pi_{\nu(z_0)})^{-1}(D(r))$ by

$$f(p) = \Pi_0(p) - \Pi_0\{\{\xi \circ (\Pi_{\nu(z_0)} \circ (\xi - \xi(z_0))|U')^{-1} \circ \Pi_{\nu(z_0)}\}(p) - \xi(z_0)\}.$$

Restrict this function to l_t to give $f \circ l_t : \mathbf{R} \to \mathbf{R}$. We claim that $f \circ l_t$ is defined for $|s| \leq 2\delta/\cos\epsilon$; for then

$$\|\Pi_{\nu(z_0)}(l_t(s)) - \Pi_{\nu(z_0)}(p_t)\| \leq 2\delta \tan \epsilon,$$

which combines with $\Pi_{\nu(z_0)}(p_t) \in D(r - 2\delta \tan \epsilon)$ to give $\Pi_{\nu(z_0)}(l_t(s)) \in D(r)$. But

$$|\Pi_0(p_t)| < \delta,$$

$$|\Pi_0(l_t(\pm 2\delta/\cos\epsilon)) - \Pi_0(p_t)| > 2\delta,$$

$$|\Pi_0\{\xi \circ [\Pi_{\nu(z_0)} \circ (\xi - \xi(z_0))|U']^{-1} \circ \Pi_{\nu(z_0)}\}(l_t(2\delta/\cos\epsilon)) - \xi(z_0)| < \delta$$

together yield

$$f \circ l_t(2\delta/\cos\epsilon) > 0, \qquad f \circ l_t(-2\delta/\cos\epsilon) < 0.$$

By the intermediate value theorem this shows that l_t intersects $\xi(U') - \xi(z_0)$. Q.E.D.

Proof of the proposition. By joining the points $x_1, \cdots, x_r$ with a curve in M and covering the curve by finitely many neighbourhoods of the kind discussed in Lemma 3.2, we can arrange to have the following situation: $\{x_1, \cdots, x_r\}$ is included in a finite set $\{z_1, \cdots, z_s\} \subset M$ such that, for each $i = 1, \cdots, s-1$, z_i and z_{i+1}, together with a point $q_i \in H$ and neighbourhoods $U_i \subset M$ containing z_i and z_{i+1} and $B_i \subset H$ containing q_i, satisfy the conclusion of Lemma 3.2. That is, if

$$L_i = [\xi(z_i'), \xi(z_{i+1}'), q_i'], \qquad z_i', z_{i+1}' \in U_i, \ q_i' \in B_i,$$

then z_i' and z_{i+1}' lie in the same connected component of $\xi^{-1}(L_i')$, and L_i' and M are transverse at z_i' and z_{i+1}'. Clearly, if we put $L_i = [\xi(z_i), \xi(z_{i+1}), q_i]$, $i = 1, \cdots, s-1$, and $L = [L_1, \cdots, L_{s-1}]$, then $x_1, \cdots, x_r$ lie in the same connected component of $\xi^{-1}(L)$. By Corollary 2.5, L has a neighbourhood V so that $L' \in V$ implies that

$$L' \cap \xi(U_i \cap U_{i-1} \cap U_\delta(z_i)) \neq \emptyset.$$

Take

$$z_i' \in U_i \cap U_{i-1} \cap U_\delta(z_i) \cap \xi^{-1}(L'), \qquad i = 1, \cdots, s.$$

Then $z_1', \cdots, z_s'$ are all in the same connected component of $\xi^{-1}(L')$. Among these points $z_1', \cdots, z_s'$ we can find the points $y_1, \cdots, y_r$ to satisfy the conclusion of the proposition. Q.E.D.

4. **The curvature of a transverse linear section.** In this section we first show that, if L is transverse to M, $\xi^{-1}(L)$ is a submanifold of M (Proposition 4.1). We then show that the sign of a sectional curvature of $\xi^{-1}(L)$ is the same as the sign of the corresponding sectional curvature of M (Proposition 4.2). We complete the section by showing that in some cases the sectional curvatures of $\xi^{-1}(L)$ depend smoothly on L (Lemma 4.3).

Proposition 4.1. *Let $L \in G(H, k)$ be transverse to M. Then $\xi^{-1}(L)$ is an (embedded) C^∞ submanifold of M.*

Proof. It suffices to find Hilbert spaces E_1, E_2, $E = E_1 \oplus E_2$, and open sets $U_1 \subset E_1$, $U_2 \subset E_2$ and a chart $\Psi : E = E_1 \oplus E_2 \supset U_1 \times U_2 \longrightarrow M$ centered at $x \in M$ so that

$$\Psi(U_1 \times 0) = \Psi(U_1 \times U_2) \cap \xi^{-1}(L).$$

Let $(E \supset U, \phi)$ be any chart for M centered at x. Let K be the orthogonal complement of L and note that

$$L + d(\xi\phi)_0(E) = L + T_x = H$$

since M and L are transverse. Hence, if $\overline{\Pi} : H \longrightarrow K$ is the orthogonal projection, $\overline{\Pi} \circ d(\xi\phi)_0 : E \longrightarrow K$ is onto. Let $N \subset E$ be the kernel of $\overline{\Pi} \circ d(\xi\phi)_0$ and P its orthogonal complement. Then the partial derivative with respect to P at $0 \in E$ is given by

$$d_2(\overline{\Pi}\xi\phi)_0 = \overline{\Pi} \circ (d(\xi\phi)_0 | P) : P \longrightarrow K.$$

By the inverse function theorem (see [4]) there exist closed subspaces E_1 and E_2 of $E = E_1 \oplus E_2$ with open sets $U_1 \subset E_1$, $U_2 \subset E_2$ containing the origin and a diffeomorphism h of $U_1 \times U_2$ onto a nieghbourhood of 0 contained in U so that $\overline{\Pi} \circ \xi \circ \phi \circ h : U_1 \times U_2 \longrightarrow K$ is a projection onto U_2. Put $\Psi = \phi \circ h$. Then for $(x_1, x_2) \in U_1 \times U_2$ we have

$$\xi \circ \Psi(x_1, x_2) \in L \iff \overline{\Pi}\xi\phi h(x_1, x_2) = 0 \iff x_2 = 0.$$

That is,

$$\Psi(U_1 \times 0) = \Psi(U_1 \times U_2) \cap \xi^{-1}(L). \qquad \text{Q.E.D.}$$

Proposition 4.2. *Let $L \in G(H, k)$ be transverse to M. For a plane section σ tangent to $\xi^{-1}(L) \subset M$ let $\overline{K}_\sigma$ and K_σ denote the sectional curvatures of $\xi^{-1}(L)$ at σ and M at σ respectively. Then either $\overline{K}_\sigma$ and K_σ are both zero, or else they are both nonzero and have the same sign.*

Proof. Let R and $\overline{R}$ be the curvature tensors on M and $\xi^{-1}(L)$ respectively, and let ∇ and $\overline{\nabla}$ be the connections on H and L, and let ν and $\overline{\nu}$ be (local) unit normal fields to $\xi(M)$ in H and to $\xi(\xi^{-1}(L))$ in L. For $u, v \in (\xi^{-1}(L))_x \subset M_x$ we have the following form of Gauss's Theorema Egregium:

$$\langle R(u, v)v, u\rangle = \langle \nabla_u u, \nu\rangle \langle \nabla_v v, \nu\rangle - \langle \nabla_u v, \nu\rangle^2, \tag{6}$$

$$\langle \overline{R}(u, v)v, u\rangle = \langle \overline{\nabla}_u u, \overline{\nu}\rangle \langle \overline{\nabla}_v v, \overline{\nu}\rangle - \langle \overline{\nabla}_u v, \overline{\nu}\rangle^2. \tag{7}$$

Let $x \in \xi^{-1}(L)$ and let $\sigma \subset (\xi^{-1}(L))_x$ be a plane section. We may choose u and v to be vector fields tangent to M in an M-neighbourhood U of x so that σ is generated by $u(x)$ and $v(x)$ and so that u and v are tangent to $\xi^{-1}(L)$ at all points of $\xi^{-1}(L) \cap U$, and so that $[u, v] = 0$ on $\xi^{-1}(L) \cap U$. If K is the orthogonal complement of L we have $\nu = \nu_1 + \nu_2$, $\nu_1 \in L$, $\nu_2 \in K$, and because T_x and L are transverse, $\|\nu_1\| \neq 0$. Then $\overline{\nu} = \nu_1/\|\nu_1\|$, so that, by (7),

$$\begin{aligned}(1/\|\nu_1\|)^2\{\langle \overline{\nabla}_u u, \nu_1\rangle \langle \overline{\nabla}_v v, \nu_1\rangle - \langle \overline{\nabla}_u v, \nu_1\rangle^2\} \\ = (1/\|\nu_1\|)^2\{\langle \nabla_u u, \nu\rangle \langle \nabla_v v, \nu\rangle - \langle \nabla_u v, \nu\rangle^2\} = (1/\|\nu_1\|)^2 \langle R(u, v)v, u\rangle,\end{aligned}$$

since by the choice of u and v, $\nabla_u u$, $\nabla_v v$ and $\nabla_u v$ are automatically tangent to L. Q.E.D.

Remark. The fact that L is finite dimensional is not used in the proofs of Propositions 4.1 or 4.2. Both results are true if L is any closed linear submanifold of H, transverse to M.

Lemma 4.3. *Suppose that $\{\xi(x), p_1, \cdots, p_k\}$ is in general position and that $L = [\xi(x), p_1, \cdots, p_k]$ is transverse to M at $x \in \xi^{-1}(L)$ and that at least one sectional curvature at x in $\xi^{-1}(L)$ is positive. Then for some $\delta > 0$ this remains true at y for $[\xi(y), q_1, \cdots, q_k]$ whenever $y \in U_\delta(x)$ and $q_j \in B_\delta(p_j)$, $j = 1, \cdots, k$.*

Proof. If δ is small enough, $\{\xi(y), q_1, \cdots, q_k\}$ is automatically again in general position. Then

$$\{\Lambda_1(y, q_1, \cdots, q_k) - \xi(y)\} \cap T_y$$

is the orthogonal complement in $\Lambda_1(y, q_1, \cdots, q_k) - \xi(y)$ of the projection of the unit normal $\nu(y)$ on $\Lambda_1(y, q_1, \cdots, q_k) - \xi(y)$. That projection is given by

$$\overline{\nu}(y, q_1, \cdots, q_k) = \sum_{j=1}^{k} \langle \nu(y), q_j - \xi(y)\rangle g^{jk}(q_k - \xi(y)),$$

where $((g^{jk})) = ((g_{jk}))^{-1}$, and $g_{jk} = \langle q_j - \xi(y), q_k - \xi(y)\rangle$. This shows that $\overline{\nu}(y, q_1, \cdots, q_k)$ varies smoothly with $(y, q_1, \cdots, q_k)$. In particular, if δ is small enough,

$$\|\overline{\nu}(y, q_1, \cdots, q_k)\| \neq \emptyset,$$

so that $\Lambda_1(y, q_1, \cdots, q_k)$ and M are transverse at y. It also follows that $\mu = \overline{\nu}/\|\overline{\nu}\|$ is smooth in its arguments. But a set of generators for $\{\Lambda_1(y, q_1, \cdots, q_k) - \xi(y)\} \cap T_y$ is then given by

$$w_j(y, q_1, \cdots, q_k) = q_j - \xi(y) - \langle q_j - \xi(y), \mu\rangle\mu, \qquad j = 1, \cdots, k,$$

which also vary smoothly. Hence, so do the sectional curvatures

$$K(u, v) = \langle \bar{R}(u, v)v, u\rangle / \{\|u\|^2\|v\|^2 - \langle u, v\rangle^2\}$$

when, for $a^i, b^j \in \mathbf{R}$,

$$u = u(y, q_1, \cdots, q_k) = \sum_{i=1}^{k} a^i w_i(y, q_1, \cdots, q_k),$$

$$v = v(y, q_1, \cdots, q_k) = \sum_{j=1}^{k} b^j w_j(y, q_1, \cdots, q_k).$$

Hence, if a^i and b^j are chosen in such a way that

$$K(u(x, p_1, \cdots, p_k), v(x, p_1, \cdots, p_k)) > 0,$$

then also

$$K(u(y, q_1, \cdots, q_k), v(y, q_1, \cdots, q_k)) > 0,$$

if, for a small enough $\delta > 0$,

$$y \in U_\delta(x), \qquad q_j \in B_\delta(p_j), \quad j = 1, \cdots, k. \qquad \text{Q.E.D.}$$

5. **Hypersurfaces with positive sectional curvatures.** In this section we prove the following generalization of the theorem of Sacksteder [7, Theorem (*)] which we shall refer to as (S) for convenience.

Theorem 5.1. *Let M be a complete Riemannian Hilbert manifold with nonnegative sectional curvatures and with at least one point at which at least one sectional curvature is strictly positive. Let $\xi : M \to H$ be an isometric immersion of M as a hypersurface of a Hilbert space H. Then*

(i) *ξ is an embedding.*

(ii) *$\xi(M) \subset H$ is the boundary of a convex body in H; in particular, ξ embeds M topologically as a closed subset of H.*

(iii) *M is homeomorphic to $H \times S^n$ for some integer $n \geq 0$.*

Proof. We shall let x_0 and $\sigma \subset M_{x_0}$ be a point and a plane section at x_0 at which the sectional curvature is strictly positive.

(a) Suppose first that ξ is not one-to-one. Say $\xi(y_1) = \xi(y_2) = q \in H$. By Proposition 3.1 there is an $L \in G(H, l, q)$ transverse to M at x_0 together with a neighbourhood V of L in $G(H, l, q)$ so that, if $L' \in V$, L' contains $\xi(x)$ for some $x \in U_\epsilon(x_0)$, and y_1, y_2, x lie in the same connected component of $\xi^{-1}(L')$. We may assume that $d\xi(\sigma) + \xi(x_0) \subset L$, and then by Lemma 4.3 we may assume that at $x \in \xi^{-1}(L')$ there is a plane section of strictly positive curvature. By Theorem 2.7 we may assume L' and M to be transverse. But then we can apply (S) to the connected component of $\xi^{-1}(L')$ containing x, to get a contradiction to $\xi(y_1) = \xi(y_2)$.

(b) We now show that no line intersecting $\xi(M)$ transversally twice intersects it again. For suppose there is a line that intersects M transversally at $\xi(x_1)$ and $\xi(x_2)$ and intersects M again at $\xi(x_3)$. Then for y_1 sufficiently near x_1 on M, the line $[\xi(y_1), \xi(x_3)]$ will also be transverse to M at y_1 and at a point y_2 near x_2. By applying Proposition 3.1, Corollary 2.5, Theorem 2.7, and Lemma 4.3 in that order, we may suppose that $[\xi(y_1), \xi(x_3)]$ is contained in a linear submanifold $L \in G(H, l, \xi(x_3))$ transverse to M, so that, for some x near x_0, y_1, y_2, y_3, x are contained in the same connected component X of $\xi^{-1}(L)$, and so that at least one sectional curvature of $\xi^{-1}(L)$ at x is strictly positive. But then $\xi : X \longrightarrow L$ satisfies (S), contradicting the assumption that $[\xi(y_1), \xi(y_2)]$ meets $\xi(\xi^{-1}(L))$ a third time.

(c) To show now that ξ is an embedding it is enough to show that each $y_0 \in M$ has an arbitrarily small neighbourhood U so that, for some neighbourhood $V \subset H$ of $\xi(y_0)$, $\xi^{-1}(V) = U$.

If $\xi(M)$ is contained in $T_{y_0} + \xi(y_0)$, this is trivial. Otherwise, let f be defined on $M - y_0$ by

$$f(y) = \|\xi(y) - \xi(y_0)\|.$$

Let $y \in M$ be such that $\xi(y) \notin T_{y_0} + \xi(y_0)$. By following the integral trajectory α of $-\operatorname{grad} f$ and using (a) and the completeness of M we see that α must contain a point y_1 such that $\xi(y_1) \notin T_{y_0} + \xi(y_0)$ and such that M and $[\xi(y_0), \xi(y_1)]$ are transverse at y_1. Choose neighbourhoods U of y_0 and W of y_1 such that whenever $y_0' \in U$, $y_1' \in W$; then $[\xi(y_0'), \xi(y_1')]$ and M are transverse at y_1' and y_0'. Put

$$v = \{t\xi(y_1') + (1-t)\xi(y_0') \mid t \in (-1, 1),\ y_0' \in U,\ y_1' \in W\}.$$

If V contained points $\xi(y)$, $y \notin U$, (b) would be contradicted. This completes the proof of (i).

(d) We will now show that the normal bundle over M is trivial. If this is not the case, there is a curve $\gamma(t)$ in M so that $\gamma(0) = \gamma(1) = y_0$ and such that, if $\nu(\gamma(t))$ is a continuous lift of γ to the unit normal bundle, $\nu(\gamma(0)) = -\nu(\gamma(1))$. Use (c) to cover M by an open covering $\{U_\alpha\}$ so that $U_\alpha = \xi^{-1}(V_\alpha)$ for a connected open set $V_\alpha \subset H$. Let $V = U_\alpha V_\alpha$. Since $\gamma([0, 1])$ is compact there is an $\epsilon > 0$ so that $\beta(t) = \xi(\gamma(t)) + \epsilon\nu(\gamma(t)) \in V - \xi(M)$, $0 \le t \le 1$. Approximate β by a piecewise linear curve in $V - \xi(M)$ with the same end points, and combine it with the segment $\{\xi(\gamma(0)) + \epsilon t\nu(\gamma(0)) \mid -1 \le t \le 1\}$. This gives a closed piecewise linear curve in V that intersects $\xi(M)$ transversely precisely once at $\xi(y_0)$. But then by Proposition 3.1, Theorem 2.7, and Lemma 4.3 there is an $L \in G(H, l, \xi(x_0))$ transverse to M with the following properties: (1) for some y near y_0, y and x_0 belong to the same connected component of $\xi^{-1}(L)$; (2) at least one sectional curvature of $\xi^{-1}(L)$ at x_0 is strictly positive; and (3) $V \cap L$ contains a piecewise linear curve that crosses $\xi(\xi^{-1}(L))$ precisely once at $\xi(y)$. But this contradicts (S).

(e) Let C be the convex hull of $\xi(M)$. We want to show that $\xi(M) \subset \partial C$. Suppose to the contrary that $\xi(y_0) \in \operatorname{int} C$. Then C, and therefore also M, lies on both sides of $T_{y_0} + \xi(y_0)$. Using precisely the same argument as that given in (c) there must be open sets $W_1, W_2 \subset M$ and a neighbourhood U of y_0 with the following property: If $y_1' \in W_1$ and $y_2' \in W_2$, and $y_0' \in U$, then $\xi(y_1')$ and $\xi(y_2')$ lie on opposite sides of $T_{y_0'} + \xi(y_0')$, and $[\xi(y_0'), \xi(y_1')]$ and $[\xi(y_0'), \xi(y_2')]$ are transverse to M at y_0' and y_1' and at y_0' and y_2' respectively. By Proposition 3.1, Theorem 2.7 and Lemma 4.3 we may assume that $[\xi(y_0'), \xi(y_1')]$ and $[\xi(y_0'), \xi(y_2')]$ are included in an $L \in G(H, l, \xi(x_0))$ which is transverse to M and such that x_0, y_0', y_1', y_2' are contained in the same connected component of $\xi^{-1}(L)$ and so that $\xi^{-1}(L)$ has at least one section of strictly positive curvature at x_0. We get a contradiction now if we apply (S) to this connected component.

(f) $\xi(M)$ is an open subset of ∂C. It is clear that, since one sectional curvature is strictly positive at x_0, C has a nonempty interior. Let p be any interior point. Join p to a point $\xi(y_0) \in \xi(M)$. The line $[p, \xi(y_0)]$ must be transverse to M at y_0, for otherwise $\xi(y_0)$ could not be a boundary point of C. Hence for some neighbourhood $U \subset M$ of y_0 this remains true for $[p, \xi(y)]$, $y \in U$. Thus the joins of p to the points $\xi(y)$, $y \in U$, form an open pencil of rays at p which must intersect ∂C in an open set, which clearly can be none other than $\xi(U)$.

(g) To prove that $\xi(M) = \partial C$ we borrow the following argument from [1]: Suppose $p \in \partial C - \xi(M)$. By [3, p. 31], ∂C is connected, and thus there exists a rectifiable curve $\theta(t)$ in ∂C, $t \in [0, 1]$ with $\theta(0) = \xi(y_0)$ and $\theta(1) = p$. Since $\xi(M)$ is open in ∂C there must be some $t_0 \in (0, 1]$ so that $\theta(t) \in \xi(M)$ if $t < t_0$, while $\theta(t_0) \notin \xi(M)$. Let $\{t_i\}$ be a sequence approaching t_0 from below. Let a_i be the arclength of θ between $\theta(0)$ and $\theta(t_i)$. Then $\{a_i\}$ converges and

$$d(\theta(t_i), \theta(t_j)) \leq |a_i - a_j|$$

shows that $\{\xi^{-1}\theta(t_i)\}$ is a Cauchy sequence in M that does not converge, contradicting the completeness of M. The proof of (ii) is now also complete.

(h) Part (iii) is now a direct consequence of [3, p. 31, Proposition 1.7]. Q.E.D.

6. Hypersurfaces with zero sectional curvatures.

In this section we prove the following generalization of the theorem of Hartman and Nirenberg [2, Theorem III] which we shall refer to as (H) throughout this section.

Theorem 6.1. *Let M be a complete Riemannian Hilbert manifold with zero sectional curvatures. Let $\xi: M \longrightarrow H$ be an isometric immersion of M as a hypersurface of a Hilbert space H. Then $\xi(M)$ is a cylinder of the form $\beta \times A$ where β is a plane curve and A is the closed subspace of H of codimension two orthogonal to the plane containing β.*

Proof. Choose $x_0 \in M$ and let L be a plane through $\xi(x_0)$ transverse to M. If for each such L the component of $\xi^{-1}(L)$ containing x_0 maps into a line, the theorem is finished. For then $\xi^{-1}(L)$ must be a geodesic, and thus for a dense subset of the unit tangent space $M_{x_0}(1)$, the corresponding geodesics would be mapped into lines by ξ. By the continuity of ξ this would then be true for all $v \in M_{x_0}(1)$, making $\xi(M)$ a hyperplane in H.

Thus, we assume for the rest of the proof that we have a plane L_0 through x_0 transverse to M so that the component of $\xi^{-1}(L_0)$ containing x_0 maps into a unit speed curve α which is not a line.

Let $d\xi(v_0)$ be the velocity of α at x_0, $v_0 \in M_{x_0}(1)$. If $L \in G(H, 3, L_0)$ is transverse to M, (H) tells us that $\xi(\xi^{-1}(L))$ is a two dimensional cylinder generated by α and $d\xi(v)$ for some $v \in M_{x_0}(1)$. Let $K_{x_0} \subset M_{x_0}$ be the linear span of all vectors v obtained this way. Clearly, $d\xi(K_{x_0})$ is independent of the position of x_0 on α. We want to show that if $v \in K_{x_0}$, then

$$\xi\gamma_v(t) = \xi(x_0) + td\xi(v),$$

where $\gamma_v(t)$ denotes the geodesic in M with $\gamma_v'(0) = v$. Note that

$$v = \sum_{i=1}^{n} a^i v_i, \qquad v_i \in M_{x_0}(1),$$

where v_i is obtained as above as the generator of a cylinder $\xi(\xi^{-1}(L_i))$, $L_i \in G(H, 3, L_0)$ transverse to M. We may assume $v_1, \cdots, v_n$ linearly independent, and let $L' = [L_1, \cdots, L_n]$. Let $\{L_j'\}_{j=1}^{\infty}$ be an approximating sequence for L', $L_j' \in G(H, n+2, L_0)$ transverse to M. Such a sequence exists by Theorem 2.6. For each j, the connected component of $\xi^{-1}(L_j')$ containing x_0 maps into an $(n+1)$-dimensional cylinder generated by $d\xi(v_0)$ and an n-space $d\xi(K_j)$, $K_j \subset M_{x_0}$. Select an orthonormal basis for K_j:

$$v_{j1}, \cdots, v_{jn} \in K_j \cap M_{x_0}(1).$$

Let $w_{j1}, \cdots, w_{jn}$ be the orthogonal projections of the vectors $d\xi(v_{j1}), \cdots, d\xi(v_{jn})$ on $L' - \xi(x_0)$. For large j these vectors will be linearly independent and will be close to unit lenght. Hence, by choosing a subsequence if necessary, we may arrange that, as $j \longrightarrow \infty$,

$$d\xi(v_{ji}) \longrightarrow w_i \in L' - \xi(x_0).$$

But then $\{v_{ji}\}_{j=1}^{\infty}$ is a Cauchy sequence in $M_{x_0}(1)$, whence there are vectors $u_i \in M_{x_0}(1)$, $i = 1, \cdots, n$, so that $v_{ji} \longrightarrow u_i$ as $j \longrightarrow \infty$. Clearly then $d\xi(u_i) = w_i$. But then

$$\xi \circ \gamma_{\sum a^i u_i}(t) = \lim_{j\to\infty} \xi \circ \gamma_{\sum a^i v_{ji}}(t)$$

$$= \lim_{j\to\infty} \{\xi(x_0) + ta^i d\xi(v_{ji})\} = \xi(x_0) + ta^i w_i.$$

Thus the vectors $w_1, \cdots, w_n$ generate an n-dimensional linear subspace J of $L' - \xi(x_0)$ with the property $J + \xi(x_0) \subset \xi(M)$. For any value of $i = 1, \cdots, n$,

$$\dim(L_i - \xi(x_0)) + \dim J = n + 3 = \dim(L' - \xi(x_0)) + 1,$$

whence $J \cap \{L_i - \xi(x_0)\}$ must be at least one dimensional. In fact, since $J \subset \xi(M) - \xi(x_0)$ and since the connected component of $\xi^{-1}(L_i)$ containing x_0 satisfies (H), it follows that $J \cap \{L_i - \xi(x_0)\}$ is precisely the line generated by $d\xi(v_i)$. Thus $d\xi(v_1), \cdots, d\xi(v_n)$ will also generate J, whence, for $v \in K_{x_0}$, $\xi \circ \gamma_v$ is a straight line.

If $v \in \overline{K}_{x_0}$, $\xi \circ \gamma_v$ is a straight line. For let $v_i \in K_{x_0}$ be such that $v_i \longrightarrow v$ as $i \longrightarrow \infty$. Then

$$\begin{aligned} \xi \circ \gamma_v(t) &= \lim_{i \to \infty} \xi \circ \gamma_{v_i}(t) \\ &= \lim_{i \to \infty} \{\xi(x_0) + td\xi(v_i)\} = \xi(x_0) + td\xi(v). \end{aligned}$$

It also follows that $v_0 \notin \overline{K}_{x_0}$, for otherwise $\xi \circ \gamma_{v_0}$ would be a straight line, which would mean $\xi \circ \gamma_{v_0} \subset L_0$. But then $\xi \circ \gamma_{v_0}(t) = \alpha(t)$ which contradicts our assumption that α is not a straight line.

We want to prove now that v_0 and $\overline{K}_{x_0}$ together span M_{x_0}. Suppose there is a vector $v \in M_{x_0}(1)$ orthogonal to both v_0 and $\overline{K}_{x_0}$. Let $\epsilon > 0$. By Theorem 2.6 there is a linear manifold $L \in G(H, 3, L_0)$ transverse to M and so that there is a vector

$$w \in (d\xi)^{-1}(L - \xi(x_0)) \cap M_{x_0}(1), \qquad \langle w, v\rangle^2 > 1 - \epsilon^2 > 0.$$

By (H), w may be expressed as

$$w = av_0 + bw', \qquad w' \in \overline{K}_{x_0} \cap M_{x_0}(1), \qquad a, b \in \mathbf{R}.$$

But then $\langle v, v_0\rangle = 0 = \langle v, w'\rangle$ implies also $\langle v, w\rangle = 0$ which contradicts $|\langle v, w\rangle| > 0$.

Theorem 6.1 now follows if we put $A = d\xi(\overline{K}_{x_0})$, B = the orthogonal complement of A, and β the orthogonal projection of α on B. Q.E.D.

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