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A SHEAF-THEORETIC DUALITY THEORY FOR CYLINDRIC ALGEBRAS

BY

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ABSTRACT. Stone's duality between Boolean algebras and Boolean spaces is extended to a dual equivalence between the category of all α-dimensional cylindric algebras and a certain category of sheaves of such algebras. The dual spaces of important types of algebras are characterized and applications are given to the study of direct and subdirect decompositions of cylindric algebras.

It is a thesis of this paper that certain sheaves serve adequately as the dual spaces of cylindric algebras in the same way that Boolean spaces serve as the dual spaces of Boolean algebras. This duality is described in §1. These results are established by algebraically imitating, with suitable cylindric algebra concepts, the sheaf duality theory for rings presented in R. S. Pierce's monograph [6]. These results also hold for other versions of algebraic logic such as polyadic algebras. In §2 the dual spaces of locally finite, representable, and regular algebras are characterized; §4 gives some applications to the decomposition theory for cylindric algebras.

Our study can be viewed in several ways. In algebraic logic, with each first-order theory Γ there is associated an algebraic structure (called an algebra of formulas) that describes certain aspects of Γ. Since the theory Γ can be determined from the set of all complete theories extending Γ, the following problem concerning the adequacy of algebraic logic arises. Assuming we know the algebra $\mathfrak{F}_\Delta$ associated with each complete (and consistent) theory Δ extending the theory Γ, how can we describe the algebra $\mathfrak{F}_\Gamma$ associated with Γ in terms of all the pairs $(\Delta, \mathfrak{F}_\Delta)$? This problem is similar to the one in algebraic geometry of describing the ring associated with an affine variety in terms of the local rings given at each point of the variety. In our situation, if we think of a theory Γ as being determined by the set X_Γ of all complete extensions of Γ and think of the algebra of formula $\mathfrak{F}_\Delta$ as being assigned to each point Δ of X_Γ, then our problem is of the same nature as the one in algebraic geometry mentioned above. This analogue with algebraic geometry is very close; in §3 we solve the logical prob-

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lem by just imitating the ring theoretic solution to the geometric one. The idea is to consider X_Γ as a topological space and to glue all of the $\mathfrak{F}_\Delta$'s $(\Delta \in X_\Gamma)$ together with a suitable topology to form a space $\mathcal{S}(\Gamma)$. This is just the construction of a sheaf $\mathcal{S}(\Gamma)$ of cylindric algebras over the space X_Γ. Then, the algebra $\mathfrak{F}_\Gamma$ is (up to isomorphism) the algebra $\Gamma(X_\Gamma, \mathcal{S}(\Gamma))$ of all continuous sections of the sheaf $(X_\Gamma, \mathcal{S}(\Gamma))$.

The analogue of the problem raised above concerning the adequacy of Boolean algebras for studying propositional theories has a well-known solution. Complete (and consistent) sentential theories extending a given theory Γ correspond to maximal ideals in the Boolean algebra $\mathfrak{F}_\Gamma$ and the topological Stone representation theorem tells us how to describe $\mathfrak{F}_\Gamma$ in terms of these ideals. The representation as all sections of a sheaf is a generalization to α-dimensional cylindric algebras of a sheaf-theoretic formulation of Stone's theorem. Not only does the duality between Boolean algebras and Boolean spaces extend but also a form of the correspondence between ideals in Boolean algebras and open subsets in the dual space.

From the viewpoint of universal algebra the representation of an algebra as all sections of a reduced sheaf is a certain subdirect decomposition in which (at least in the nice cases) the factors are directly indecomposable. This representation has the advantage that it is unique in a certain sense. A common universal algebra generalization of this aspect of both the duality results outlined here and the ring representation results in [6] can be found in [1].

0. **Preliminaries.** An α-dimensional cylindric algebra will be referred to as a CA_α and a Boolean algebra as a BA. Basic facts about CA_α's may be found in [3] and [4]. In this paper it is convenient to disallow the one element algebra; thus, it is assumed $0 \neq 1$ in all CA_α's. When a class of CA_α's is treated as a category the morphisms are the usual homomorphisms between members of the class.

A central position in this study of a CA_α $\mathfrak{A}$ is played by $Z(\mathfrak{A}) = \{x \in A: \Delta x = 0\}$ which forms a Boolean subalgebra of $\mathfrak{A}$. $Z(\mathfrak{A})$ plays the same role for CA_α's that the BA of central idempotents does for rings in [6]. The connection between cylindric ideals of a CA_α $\mathfrak{A}$ and BA ideals of $Z(\mathfrak{A})$ also plays a major role. For an ideal J of the BA $Z(\mathfrak{A})$, $\overline{J} = \{x \in A: x \leq y \text{ for some } y \in J\}$ is the smallest CA_α ideal of $\mathfrak{A}$ containing J; for an ideal I of $\mathfrak{A}$, $I \cap Z(\mathfrak{A})$ is a BA ideal of $Z(\mathfrak{A})$. Observe that $\overline{J} \cap Z(\mathfrak{A}) = J$ and $\overline{I \cap Z(\mathfrak{A})} \subseteq I$. An ideal I of $\mathfrak{A}$ is called *regular* if $\overline{I \cap Z(\mathfrak{A})} = I$ and a CA_α $\mathfrak{A}$ is *regular* if every ideal of $\mathfrak{A}$ is regular. It is known that every LCA_α is regular and it is obvious that every simple CA_α is regular. Regular algebras arose naturally during this investigation; they appear to be a very natural generalization of LCA_α's and share many of their properties. It is easy to show that a CA_α $\mathfrak{A}$ is regular if and only if every principal ideal of $\mathfrak{A}$ is generated by an element of $Z(\mathfrak{A})$. This condition is equivalent to: for every $x \in A$, $\Delta c_{(\Gamma)} x = 0$ for some finite subset Γ of α. For $\alpha < \omega$ every CA_α is regular; if $\alpha \geq \omega$ every regular CA_α is an RCA_α by 2.3 and 2.4 below.

The definition of a sheaf $(X, \mathcal{S}, \pi)$ of CA_α's and the basic elementary properties of sheaves and sections of sheaves can be obtained from Part I of [6] by replacing the word "ring" by "CA_α". In case π and/or X are understood, a sheaf of CA_α's is denoted by $(X, \mathcal{S})$ or simply $\mathcal{S}$ and it is called an α-space. $\Gamma(X, \mathcal{S})$ denotes the set of all continuous sections of $(X, \mathcal{S}, \pi)$; it is given the structure of a CA_α by considering it as a subalgebra of $\Pi_{x \in X} \mathcal{S}_x$ where $\mathcal{S}_x = \pi^{-1}x$ is the stalk over $x \in X$. If X is a Boolean space, i.e. a totally disconnected, compact Hausdorff space, then $\Gamma(X, \mathcal{S})$ is a subdirect product of $\{\mathcal{S}_x \colon x \in X\}$. If X is a topological space and a CA_α $\mathfrak{A}$ is given the discrete topology, then $\mathcal{S} = X \times A$ with the product topology is a sheaf of CA_α's over X with each stalk $\mathcal{S}_x = \{x\} \times \mathfrak{A}$ isomorphic to $\mathfrak{A}$. $\mathcal{S}$ is called the *trivial* $\mathfrak{A}$-*sheaf over* X. For this sheaf $\Gamma(X, \mathcal{S})$ is isomorphic to the CA_α of all continuous functions from X into $\mathfrak{A}$.

1. **Duality theory.** Contravariant functors between the category of all CA_α's and a certain category of α-spaces are described below. The basic properties of these functors and the duality they determine are established by replacing ring notions by suitable CA_α notions in the arguments given in Part I of [6].

An α-space $(X, \mathcal{S})$ is called *reduced* if X is a Boolean space and, for all $\sigma \in Z(\Gamma(X, \mathcal{S}))$ and $x \in X$, either $\sigma(x) = 0_x$ or $\sigma(x) = 1_x$. The following proposition is useful for identifying α-spaces as reduced. The condition (*) is not, in general, necessary (cf. Example 2.5).

Proposition 1.1. *If* $(X, \mathcal{S})$ *is an* α*-space where* X *is a Boolean space then* $(X, \mathcal{S})$ *is reduced if*

(*) $\mathcal{S}_x$ *is directly indecomposable for all* $x \in X$.

If $(X, \mathcal{S})$ *is a regular* CA_α *(in particular an* LCA_α*), then* (*) *is also necessary.*

Observe that a trivial $\mathfrak{A}$-sheaf $(X, \mathcal{S})$ is reduced if and only if X is a Boolean space and $\mathfrak{A}$ is directly indecomposable. Unlike the situation for rings, a subsheaf of a reduced sheaf of CA_α's is always reduced.

We now describe a functor that associates to each CA_α $\mathfrak{A}$ a reduced α-space $(X(\mathfrak{A}), \mathcal{S}(\mathfrak{A})) = \mathfrak{A}^d$ called the *dual space* of $\mathfrak{A}$. The base space $X(\mathfrak{A})$ is the usual Boolean dual space of the BA $Z(\mathfrak{A})$. To be concrete take $X(\mathfrak{A})$ as the set of all maximal ideals of $Z(\mathfrak{A})$; this becomes a Boolean space when we take the collection of all sets $N(y) = \{M \in X(\mathfrak{A}):\ y \notin M\}$ as a basis for the topology. For $M \in X(\mathfrak{A})$, let $\mathcal{S}_M(\mathfrak{A}) = \mathfrak{A}/\overline{M}$ (the stalk over M) and $\mathcal{S}(A) = \bigcup\{\mathcal{S}_M(\mathfrak{A}):\ M \in X(\mathfrak{A})\}$. Note that $\mathcal{S}_{M_1}(\mathfrak{A})$ and $\mathcal{S}_{M_2}(\mathfrak{A})$ are disjoint when $M_1 \neq M_2$. The projection $\pi:\ \mathcal{S}(\mathfrak{A}) \to X(\mathfrak{A})$ is defined for $s \in \mathcal{S}_M(\mathfrak{A})$ by $\pi(s) = M$. To describe the topology on $\mathcal{S}(\mathfrak{A})$ we need some auxiliary functions. For $a \in A$ we define a function σ_a: $X(\mathfrak{A}) \to \mathcal{S}(\mathfrak{A})$ by $\sigma_a(M) = a/\overline{M}$ for all $M \in X(\mathfrak{A})$. The topology given to $\mathcal{S}(\mathfrak{A})$ is the smallest topology for which all σ_a's $(a \in A)$ are open.

It turns out that $\mathfrak{A}^d = (X(\mathfrak{A}), \mathcal{S}(\mathfrak{A}))$ is a reduced α-space and that the mapping $\xi_{\mathfrak{A}}:\ \mathfrak{A} \to \Gamma(X(\mathfrak{A}), \mathcal{S}(\mathfrak{A}))$ defined by $\xi_{\mathfrak{A}}(a) = \sigma_a$ is an isomorphism. In particular, under $\xi_{\mathfrak{A}}$, an element $a \in Z(\mathfrak{A})$ corresponds with the characteristic function $\sigma_N \in \Gamma(X(\mathfrak{A}), \mathcal{S}(\mathfrak{A}))$ of the clopen subset $N = N(a)$ of $X(\mathfrak{A})$.

To make the correspondence $\mathfrak{A} \rightsquigarrow \mathfrak{A}^d$ into a functor we describe the dual $\lambda^d = (\overline{\lambda}, \lambda^0)$ of a homomorphism $\lambda:\ \mathfrak{A} \to \mathfrak{B}$. For $M \in X(\mathfrak{B})$ let $\overline{\lambda}(M) = \lambda^{-1}(M) \cap Z(\mathfrak{A})$; $\overline{\lambda}$ is a continuous map of $X(\mathfrak{B})$ into $X(\mathfrak{A})$. For $M \in X(\mathfrak{B})$ and $a \in A$ define $\lambda^0(M, a/\overline{\overline{\lambda}M}) = \lambda(a)/\overline{M}$. Then $\lambda^0(M, \cdot)$ is a homomorphism of $\mathcal{S}_{\overline{\lambda}M}(\mathfrak{A})$ into $\mathcal{S}_M(\mathfrak{B})$ and λ^d is a sheaf morphism of $\mathfrak{B}^d$ into $\mathfrak{A}^d$.

The functor from the category of all reduced α-spaces (and sheaf morphisms) to the category of CA_α's is easier to describe. For an α-space $\mathcal{O} = (X, \mathcal{S})$ let $\mathcal{O}^* = \Gamma(X, \mathcal{S})$; if $\Lambda = (\lambda, \phi)$ is a sheaf morphism from the α-space $(X, \mathcal{S})$ into $(Y, \mathcal{R})$, the dual $\Lambda^*:\ \Gamma(Y, \mathcal{R}) \to \Gamma(X, \mathcal{S})$ of Λ is defined by requiring $(\Lambda^*\sigma)(x) = \phi(x, \sigma(\lambda x))$ for $x \in X$ and $\sigma \in \Gamma(Y, \mathcal{R})$.

The following theorem justifies calling $\mathfrak{A}^d$ the dual space of $\mathfrak{A}$. (Cf. Theorem 6.6 in [6].)

Theorem 1.2. *The correspondences* $\mathfrak{A} \rightsquigarrow \mathfrak{A}^d$ $(\lambda \rightsquigarrow \lambda^d)$ *and* $\mathcal{O} \rightsquigarrow \mathcal{O}^*$ $(\Lambda \rightsquigarrow \Lambda^*)$ *are contravariant functors between the category of all CA_α's and the category of all reduced α-spaces. Further, there exist natural isomorphisms*

$$\xi_{\mathfrak{A}}:\ \mathfrak{A} \cong (\mathfrak{A}^d)^*, \qquad \eta_{\mathcal{O}}:\ \mathcal{O} \cong (\mathcal{O}^*)^d$$

showing that the categories are dual equivalent.

The isomorphism $\xi_{\mathfrak{A}}$ in 1.2 is the function defined above.

The preceding theorem raises the problem of finding the dual of various cylindric notions. We mention a few. For a CA_α homomorphism $\lambda:\ \mathfrak{A} \to \mathfrak{B}$ we always have $\lambda(Z(\mathfrak{A})) \subseteq Z(\mathfrak{B})$. For λ onto $\mathfrak{B}$ we say that λ is a *conformal epi* if $\lambda(Z(\mathfrak{A})) = Z(\mathfrak{B})$. A regular CA_α $\mathfrak{A}$ (in particular an LCA_α) has the property that for any $\mathfrak{B}$ and λ mapping $\mathfrak{A}$ onto $\mathfrak{B}$, λ is a conformal epi. An example of a CA_α without this property is given in 2.5. The dual of the notion of conformal epi is very nice—it is a sheaf morphism (λ, ϕ) where λ is one-to-one and ϕ restricted to each stalk is onto. The proof is similar to the one given for rings [6].

Also as with rings we can extend to CA_α's a form of the correspondence between ideals of BA's and open subsets of Boolean spaces. Let $\mathrm{Id}^R(\mathfrak{A})$ denote the set of all regular ideals of $\mathfrak{A}$. $\mathrm{Id}^R(\mathfrak{A})$ is a sublattice of the lattice $\mathrm{Id}(\mathfrak{A})$ of all ideals of $\mathfrak{A}$. For $\sigma \in \Gamma(X, \mathfrak{S})$ we call the closed subset $\|\sigma\| = \{x \in X: \sigma(x) \neq 0_x\}$ the support of σ. For any subset U of X define $J[U] = \{\sigma \in \Gamma(X, \mathfrak{S}): \|\sigma\| \subseteq U\}$ and for any subset J of $\Gamma(X, \mathfrak{S})$ define $U[J] = \bigcup\{\|\sigma\|: \sigma \in J\}$. The following is proved in the same way as the corresponding theorem for rings (see 9.3 of [6]).

Theorem 1.3. *The function* $J \mapsto U[J]$ *is an isomorphism from* $\mathrm{Id}^R(\Gamma(X, \mathfrak{S}))$ *onto the lattice of all open subsets of* X; *its inverse is the function mapping* U *to* $J[U]$.

Using 1.3 the problem of characterizing the class of ideal lattices of certain CA_α's can be reduced to the same problem for BA's. The latter problem is fairly easy.

Corollary 1.4. *If* K *denotes the class of all BA's,* LCA_α*'s, regular* CA_α*'s, or (if* $\alpha < \omega$*)* CA_α*'s and* L *is a lattice, then the following are equivalent.*

(i) L *is a complete, compactly generated pseudo BA in which an element is compact if and only if it is complemented*;

(ii) L *is isomorphic to* $\mathrm{Id}(\mathfrak{A})$ *for some* $\mathfrak{A}$ *in* K;

(iii) L *is isomorphic to* $\mathrm{Id}^R(\mathfrak{A})$ *for some* CA_α $\mathfrak{A}$.

Parts of 1.4 are mentioned in [3]; (i) does not characterize the class of ideal lattices of CA_α's when $\alpha \geq \omega$.

2. **Duals of** $\mathbf{LCA}_\alpha$ **and** $\mathbf{RCA}_\alpha$**'s**. In this section we are interested in describing the dual spaces of LCA_α's, RCA_α's and regular CA_α's. If $\mathfrak{A}$ is an LCA_α and $(X, \mathfrak{S})$ its dual space, then $\mathfrak{A} \cong \Gamma(X, \mathfrak{S})$, so 1.1 implies each stalk $\mathfrak{S}_x$ is directly indecomposable and hence simple since $\mathfrak{S}_x$ is an LCA_α. This suggests that we look for the duals of LCA_α among the following type of α-spaces.

We say an α-space $(X, \mathfrak{S})$ is *regular* if X is a Boolean space and $\mathfrak{S}$ is a sheaf of simple CA_α's (i.e. $\mathfrak{S}_x$ is simple for all $x \in X$). By 1.1, a regular α-space is reduced.

Before characterizing the dual space of an LCA_α we need to observe an additional property.

Lemma 2.1. *If* $(X, \mathfrak{S})$ *is the dual space of an* LCA_α $\mathfrak{A}$, *then for every* $s \in \mathfrak{S}$ *there is a neighborhood* T *of* s *such that* $\Delta t \subseteq \Delta s$ *for every* $t \in T$.

Proof. It is enough to show that if $\sigma_x \in \Gamma(X(\mathfrak{A}), \mathfrak{S}(A))$ and $M \in X(\mathfrak{A})$, there is a neighborhood $\mathfrak{N}$ of M for which $\Delta\sigma_x(N) \subseteq \Delta\sigma_x(M)$ for all $N \in \mathfrak{N}$. Since

Δx is finite, $z = \Sigma\{c_\kappa x \oplus x\colon \kappa \in \Delta x \sim \Delta\sigma_x(M)\}$ is in $\mathfrak{A}$. Thus, $N(-c_{(\Delta z)}z)$ is the desired neighborhood of M.

Let us call an α-space $(X, \mathcal{S})$ *locally finite* if (i) each stalk is an LCA_α, and (ii) every $s \in \mathcal{S}$ has a neighborhood T in which $\Delta t \subseteq \Delta s$ for all $t \in T$. Under assumption (i) it can be shown that (ii) is equivalent to the apparently stronger condition that every $s \in \mathcal{S}$ has a neighborhood in which every element has the same dimension set as s.

Theorem 2.2. *An α-space $(X, \mathcal{S})$ is the dual space of an LCA_α if and only if $(X, \mathcal{S})$ is a locally finite regular α-space. Consequently the functors in* 1.2 *give a duality between LCA_α's and locally finite regular α-spaces.*

Proof. In view of 1.1 and 2.1, it suffices to show that $\Gamma(X, \mathcal{S})$ is an LCA_α whenever $(X, \mathcal{S})$ is a regular, locally finite α-space. Suppose we have such a $(X, \mathcal{S})$ and $\sigma \in \Gamma(X, \mathcal{S})$. It follows that for each $x \in X$ there is a neighborhood N_x of x such that $\Delta\sigma(y) \subseteq \Delta\sigma(x)$ for all $y \in N_x$. Applying the partition property (see [6, p. 12]), there exist a finite number of clopen sets N_i for $i < n$ which partition X and such that, for each $i < n$, $N_i \subseteq N_{x_i}$ for some $x_i \in X$. Now $\Delta\sigma \subseteq \bigcup\{\Delta\sigma(x_i)\colon\ i < n\}$ which is finite. It follows that $\Gamma(X, \mathcal{S})$ is an LCA_α.

Another important class of CA_α's is the class RCA_α of all representable CA_α's. It is well known that this is an equational class, so, on the strength of general facts about sheaves over Boolean spaces, the following holds.

Theorem 2.3. *$(X, \mathcal{S})$ is the dual space of an RCA_α if and only if it is a reduced α-space of RCA_α's. Consequently, the functors in* 1.2 *give a duality between RCA_α's and reduced α-spaces of RCA_α's.*

For $\alpha < \omega$ every CA_α is locally finite; so every reduced α-space is regular. This is not true for $\alpha \geq \omega$. It is known from [3] that, for $\alpha \geq \omega$, simple CA_α's are representable; thus, the class of regular α-spaces lie between the class of regular locally finite α-spaces and the class of reduced α-spaces of RCA_α's. By the following result regular CA_α's are the duals of regular α-spaces and consequently are representable if $\alpha \geq \omega$.

Theorem 2.4. *$(X, \mathcal{S})$ is the dual of a regular CA_α if and only if it is a regular α-space. Consequently* 1.2 *gives a duality between regular CA_α's and regular α-spaces.*

The proof of 2.4 is similar to that of 1.11 in [6] using in the appropriate places the property of simple CA_α's $\mathcal{S}_x$ that for every $0 \neq a \in \mathcal{S}_x$, $c_{(F)}a = 1$ for some finite subset F of α. From the proof of 2.4 it follows that $\|\sigma\|$ is clopen whenever $\sigma \in \Gamma(X, \mathcal{S})$ where $(X, \mathcal{S})$ is regular. Consequently, when $(X, \mathcal{S})$ is a regular α-space (in particular, the dual of an LCA_α) the topology on $\mathfrak{S}$ is Hausdorff.

For $\alpha \geq \omega$, there are simple CA_α's which are not LCA_α's (see [3]); hence, the class of regular CA_α's properly includes the class of LCA_α's. A regular CA_α is clearly semisimple. The example below (with $\mathfrak{C}$ simple) shows that a semisimple CA_α does not have to be regular and, moreover, the stalks of the dual space of a semisimple CA_α do not even need to be directly indecomposable. (Cf. 1.1 and remark on conformal epi's following 1.2.)

Example 2.5. Let $\mathfrak{C}$ be a directly indecomposable CA_α, $\alpha \geq \omega$, and let I be the set of all finite subsets of α. Choose an ultrafilter F on I such that $\{\Delta \in I\colon \Delta \supseteq \Gamma\} \in F$ for all $\Gamma \in I$. We claim

(1) The epimorphism $\lambda\colon {}^I\mathfrak{C} \twoheadrightarrow {}^I\mathfrak{C}/F$ induced by the ultrafilter F is not conformal. (Consequently, ${}^I\mathfrak{C}$ is not regular.)

We must construct an element with dimension set 0 in the ultrapower ${}^I\mathfrak{C}/F$ that is not the image of an element in $Z({}^I\mathfrak{C})$. Choose a one-to-one $\sigma\colon I \longrightarrow I$ such that, for all $\Gamma \in I$, $\Gamma \cap \sigma(\Gamma) = 0$ and $\sigma(\Gamma)$ contains at least two elements. Define $f \in {}^I\mathfrak{C}$ by $f(\Gamma) = d_{\sigma(\Gamma)} = \Pi\{d_{\kappa\lambda}\colon \kappa, \lambda \in \sigma(\Gamma)\}$ for $\Gamma \in I$. Since, for each $\kappa < \alpha, \{\Gamma \in I\colon \Gamma \supseteq \{\kappa\}\} \subseteq \{\Gamma \in I\colon c_\kappa f(\Gamma) = f(\Gamma)\} \in F$, $\Delta(f/F) = 0$.

Also observe that $f/F \neq 0/F$ and $f/F \neq 1/F$. This is true since $f(\Gamma) = d_{\sigma(\Gamma)} \neq 0^{\mathfrak{C}}$ and $f(\Gamma) \neq 1^{\mathfrak{C}}$ for all $\Gamma \in I$. To show the λ in (1) is not conformal it is enough to show $f/F \notin \lambda(Z({}^I\mathfrak{C}))$. For $g \in Z({}^I\mathfrak{C})$, $\Delta g = \bigcup\{\Delta g(\Gamma)\colon \Gamma \in I\} = 0$ and $Z(\mathfrak{C}) = \{0, 1\}$ so $\{\Gamma \in I\colon g(\Gamma) = 0\} \cup \{\Gamma \in I\colon g(\Gamma) = 1\} = I \in F$. Thus, one of the two sets above belongs to F implying that $g/F = 0/F$ or $g/F = 1/F$. Consequently, $f/F \notin (Z({}^I\mathfrak{C}))$ and (1) holds.

Implicit in the above proof of (1) is a description of the dual space of a product of directly indecomposable CA_α's. We state the general result.

Proposition 2.6. *Suppose* $\mathfrak{A} = \Pi_{i\in I}\mathfrak{B}_i$, $\mathfrak{B}_i$ *directly indecomposable* CA_α*'s. Then*

(a) $Z(\mathfrak{A}) = {}^I\{0, 1\}$ *and* $X(\mathfrak{A})$ *is the Stone space of* ${}^I2$.

(b) *Since there is a biunique correspondence between the maximal ideals of* $Z(\mathfrak{A})$ *and the ultrafilters on* I, *the stalk* $\mathfrak{S}_M(\mathfrak{A})$ *of* $\mathfrak{A}^d$ *over* $M \in X(\mathfrak{A})$ *is just the ultraproduct* $\Pi_{i\in I}\mathfrak{B}_i/F$ *where* F *is the ultrafilter on* I *corresponding to* M.

3. **Sheaves and theories.** In this section we give an interpretation of the dual space of the algebra of formulas associated with a standard first order theory. It is this interpretation that yields a solution to the question posed in the introduction. The interpretation can obviously be extended to theories in other languages; however, we restrict ourselves to a standard first order language L with equality in which we have variables v_i for $i < \omega$ and each predicate symbol has finite rank. The set $Fmla_L$ of formulas is defined by recursion in

the usual way. Sentences are formulas without free variables; the set of L-sentences is denoted by $Sent_L$. A set Γ of L-sentences is a *theory* if for every $\phi \in Sent_L$, $\Gamma \vdash \phi$ implies $\phi \in \Gamma$. A theory Γ is *complete* if Γ is consistent and either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$ for every L-sentence ϕ. For information on the relationship between languages and CA_α's see [4].

An LCA_ω $\mathfrak{F}^L_\Gamma$, called the algebra of formulas of Γ, can be associated with an L-theory Γ. For an L-theory Γ define

$$\Xi^L_\Gamma = \{(\phi, \psi):\ \phi, \psi \in Fmla_L,\ \Gamma \vdash \phi \leftrightarrow \psi\}.$$

The elements of $\mathfrak{F}^L_\Gamma$ are Ξ^L_Γ-equivalence classes of L-formulas; the cylindric operations on $\mathfrak{F}^L_\Gamma$ are the natural quotient operations induced on Ξ^L_Γ-equivalence classes by the analogous logical operations on $Fmla_L$. For brevity, we denote the Ξ^L_Γ-class of a formula ϕ by $[\phi]_\Gamma$.

It is easily seen that $\mathfrak{F}^L_\Gamma$ is an LCA_ω and that $Z(\mathfrak{F}^L_\Gamma) = Sent_L/\Xi^L_\Gamma$. For an L-theory Γ let X_Γ be the set of all complete theories of L extending Γ. For $\Delta \in X_\Gamma$ let $M[\Delta] = \{[\neg\phi]_\Gamma:\ \phi \in \Delta\}$ and for $M \in X(\mathfrak{F}^L_\Gamma)$ let $\Delta[M] = \{\phi \in Sent_L:$ $[\neg\phi]_\Gamma \in M\}$. For a theory Γ, define $\mathfrak{S}(\Gamma) = \bigcup_{\Delta \in X_\Gamma} \{\Delta\} \times \mathfrak{F}^L_\Delta$ (the disjoint union of $\{\mathfrak{F}^L_\Delta:\ \Delta \in X_\Gamma\}$).

Theorem 3.1. *There exist suitable topologies on X_Γ and $\mathfrak{S}(\Gamma)$ such that $(X_\Gamma, \mathfrak{S}(\Gamma))$ is (up to sheaf isomorphism) the dual space of $\mathfrak{F}^L_\Gamma$.*

Proof. It is well known (see [4]) that the correspondences $\Delta \rightsquigarrow M[\Delta]$ and $M \rightsquigarrow \Delta[M]$ are inverse one-to-one functions between X_Γ and $X(\mathfrak{F}^L_\Gamma)$. Using these functions X_Γ can be made into a Boolean space. The relation $f =$ $\{([\phi]_\Gamma, [\phi]_\Delta):\ \phi \in Fmla_L\}$ is a homomorphism of $\mathfrak{F}^L_\Gamma$ onto $\mathfrak{F}^L_\Delta$. The ideal I of $\mathfrak{F}^L_\Gamma$ associated with f is $I = \{[\phi]_\Gamma:\ \Delta \vdash \neg\phi\}$; thus, $I \cap Z(\mathfrak{F}^L_\Gamma) = M[\Delta]$. Since LCA_ω's are regular, $I = \overline{M[\Delta]}$ and $\mathfrak{F}^L_\Delta \cong \mathfrak{F}^L_\Gamma/\overline{M[\Delta]}$. The disjoint union of the above isomorphisms gives a one-to-one function h from $\mathfrak{S}(\mathfrak{F}^L_\Gamma)$ onto $\mathfrak{S}(\Gamma)$; transferring the topology to make h a homeomorphism, $(X_\Gamma, \mathfrak{S}(\Gamma))$ becomes an α-space and the maps $\Delta \rightsquigarrow M[\Delta]$ and h give rise to a sheaf isomorphism of $(X_\Gamma, \mathfrak{S}(\Gamma))$ onto $(X(\mathfrak{F}^L_\Gamma), \mathfrak{S}(\mathfrak{F}^L_\Gamma))$.

For completeness we describe the topologies on X_Γ and $\mathfrak{S}(\Gamma)$ arising in the above proof. Note that these topologies are quite natural and do not depend on the duality theory. A basis for the topology on X_Γ is the collection of all sets $N(\phi) = \{\Delta \in X_\Gamma:\ \phi \in \Delta\}$ for $\phi \in Sent_L$. A basis for the desired topology on $\mathfrak{S}(\Gamma)$ is the collection of all sets $B_{\phi,\psi} = \{(\Delta, [\phi]_\Delta):\ \psi \in \Delta,\ \Delta \in X_\Gamma\}$ where $\phi \in Fmla_L$ and $\psi \in Sent_L$.

The duality result 1.2 and 3.1 allows us to determine $\mathfrak{F}^L_\Gamma$ in terms of $\{\mathfrak{F}^L_\Gamma:\ \Delta \in X_\Gamma\}$ solving the question posed in the introduction.

Corollary 3.2. *For an L-theory* Γ, $\mathfrak{F}_\Gamma^L \cong \Gamma(X_\Gamma, \mathcal{S}(\Gamma))$.

If $\mathcal{S}$ is an α-space over X and $Y \subseteq X$, then $\mathcal{S}|Y = \pi^{-1}(Y)$ is a sheaf over Y called the *restriction* of $(X, \mathcal{S})$ to Y. If $(X, \mathcal{S})$ is a reduced α-space, Y a closed subset of X, and every $s \in \mathcal{S}$ has a neighborhood T such that $\Delta t \subseteq \Delta s$ for all $t \in T$, then a function extension argument shows that $(Y, \mathcal{S}|Y)$ is a reduced α-space. If $(X, \mathcal{S})$ is a regular α-space and Y is a closed subset of X then $(Y, \mathcal{S}|Y)$ is regular. In particular, either of the two statements above can be used to show that the restrictions of a locally finite regular α-space to a closed subset gives a locally finite regular α-space. The following result shows that a restriction of $(\mathfrak{F}_\Gamma^L)^d$ is the dual space of another L-theory. Denote by Ω the L-theory consisting of all logically valid L-sentences.

Proposition 3.3. *For a language L, the dual spaces of algebras of formulas associated with L-theories correspond to restrictions of* $(X_\Omega, \mathcal{S}(\Omega))$ *to closed subsets of* X_Ω.

Proof. It is well known that for any L-theory Γ there is a conformal epi λ: $\mathfrak{F}_\Omega^L \to \mathfrak{F}_\Gamma^L$ given by $\lambda([\phi]_\Omega) = [\phi]_\Gamma$ for every formula ϕ. The dual morphism $\lambda^d = (\bar{\lambda}, \lambda^0)$: $(X_\Gamma, \mathcal{S}(\Gamma)) \to (X_\Omega, \mathcal{S}(\Omega))$ is easy to describe: for $\Delta \in X_\Gamma$ and $(\Delta, [\phi]_\Delta) \in \mathcal{S}(\Omega)_\Delta$, $\bar{\lambda}(\Delta) = \Delta$ and $\lambda^0(\Delta, (\Delta, [\phi]_\Delta)) = (\Delta, [\phi]_\Delta) \in \mathcal{S}(\Gamma)_\Delta$. Thus, λ^d is an isomorphism of $(X_\Gamma, \mathcal{S}(\Gamma))$ onto the restriction of $(X_\Omega, \mathcal{S}(\Omega))$ to the closed subset X_Γ of X_Ω. Moreover, every restriction of $(X_\Omega, \mathcal{S}(\Omega))$ to a closed subset Y of X_Ω is (up to isomorphism) the dual space of an algebra of formulas; for if $\Gamma = \bigcap Y$, then $Y = X_\Gamma$ since Y is closed and the dual space of $\mathfrak{F}_\Gamma^L$ is isomorphic to $(Y, \mathcal{S}(\Omega)|Y)$.

4. **Applications to the decomposition theory of** $\mathbf{CA}_\alpha$**'s.** In this section we give some easy applications of sheaf theory to the study of decompositions of CA_α's. Trivial sheaves will be used to construct direct and subdirect decompositions with specific properties. An extensive study of the properties of direct and subdirect decompositions of CA_α's can be found in [3]. Several of the results to follow are new; for those which are known, the proofs and/or viewpoint is different from [3].

In [3] it is proved that every CA_α has the refinement property. Consequently, a direct decomposition of a CA_α into directly indecomposable factors is unique (up to isomorphism). In [2] Hanf proved that various pathological direct decompositions can exist for BA's. Theorem 4.2 shows that these decompositions can exist for nondiscrete CA_α's (LCA_α's) as well. The following lemma is essentially due to Jónsson [5]; its proof (in a nonsheaf theoretic form) and 4.2 can be found in [3]. We include it here because it is a nice application of the trivial sheaf construction. In the following $\mathfrak{B}|a$ denotes the relativized BA and

$\mathfrak{A}|fa$ denotes the relativized CA_α.

Lemma 4.1. *If $\mathfrak{B}$ is a BA and $\mathfrak{C}$ is a directly indecomposable CA_α, there is a CA_α $\mathfrak{A}$ and an isomorphism $f\colon \mathfrak{B} \rightarrowtail\!\!\!\twoheadrightarrow Z(\mathfrak{A})$ such that the following hold:*

(i) *for all $a, b \in B$, $\mathfrak{B}|a \cong \mathfrak{B}|b$ iff $\mathfrak{A}|fa \cong \mathfrak{A}|fb$;*

(ii) *if $\mathfrak{C}$ is an LCA_α, so is $\mathfrak{A}$;*

(iii) *for any variety V of CA_α's, $\mathfrak{A} \in V$ iff $\mathfrak{C} \in V$.*

Proof. Given $\mathfrak{B}$ and $\mathfrak{C}$ as above let X be the Stone space of $\mathfrak{B}$ and $(X, X \times \mathfrak{C})$ the trivial $\mathfrak{C}$-sheaf over X. Then $\mathfrak{A} = \Gamma(X, X \times \mathfrak{C})$ is the desired algebra. The function f is defined by $f(b) = \sigma_{N(b)}$ (the characteristic function of $N(b)$) for $b \in B$. It follows from §1 that f is an isomorphism. Parts (ii) and (iii) are obvious from the properties of sheaves so it is enough to verify (i). If $\mathfrak{A}|fa \cong \mathfrak{A}|fb$, then $Z(\mathfrak{A}|fa) \cong Z(\mathfrak{A}|fb)$; thus $\mathfrak{B}|a \cong \mathfrak{B}|b$ follows. Now suppose $\mathfrak{B}|a \cong \mathfrak{B}|b$. Then there is a homeomorphism mapping $N(a)$ one-to-one onto $N(b)$; this homeomorphism induces an isomorphism of the trivial $\mathfrak{C}$-sheaf $(N(a), N(a) \times \mathfrak{C})$ over $N(a)$ onto the trivial $\mathfrak{C}$-sheaf $(N(b), N(b) \times \mathfrak{C})$ over $N(b)$. Since $N(a)$ is a clopen subset of X, the restriction mapping gives an isomorphism $\mathfrak{A}|fa \cong \Gamma(N(a), N(a) \times \mathfrak{C})$. Similarly, $\mathfrak{A}|fb \cong \Gamma(N(b), N(b) \times \mathfrak{C})$. It follows that $\mathfrak{A}|fa \cong \mathfrak{A}|fb$ as desired.

As a consequence of 4.1 and Hanf's result [2] the following result is immediate.

Theorem 4.2. *For any α there are nondiscrete LCA_α's $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ such that*

(i) *$\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{B} \times \mathfrak{B}$ and $\mathfrak{A} \not\cong \mathfrak{A} \times \mathfrak{B}$;*

(ii) *$\mathfrak{A}|\mathfrak{C}|\mathfrak{A}$ and ${}^2\mathfrak{A} \cong {}^2\mathfrak{C}$ but $\mathfrak{A} \not\cong \mathfrak{C}$.*

Moreover, a nondiscrete CA_α $\mathfrak{A}$ may be chosen to belong to any given variety and not to another.

The various other pathological kinds of decompositions established by Hanf can also be extended to nondiscrete CA_α's. For other uses of 4.1 see [3].

Next we turn to the study of subdirect decompositions. We restrict ourselves to the following unique decomposition properties.

Definition 4.3. (i) A CA_α $\mathfrak{A}$ has the *strict unique irredundant subdirect decomposition property* if whenever $\langle K_i\colon i \in I\rangle$ and $\langle L_j\colon j \in J\rangle$ are two systems of ideals of $\mathfrak{A}$ such that

(1) $\bigcap_{i \in I} K_i = \{0\} = \bigcap_{j \in J} L_j$;

(2) $\mathfrak{A}/K_i$ and $\mathfrak{A}/L_j$ are subdirectly indecomposable for $i \in I$, $j \in J$;

(3) $K_i \nsubseteq K_{i'}$ and $L_k \nsubseteq L_{j'}$, for $i, i' \in I$, $j, j' \in J$ with $i \neq i'$ and $j \neq j'$;

then $|I| = |J|$ and there is a one-to-one function f of I onto J such that $K_i = L_{f(i)}$ for all $i \in I$.

(ii) A CA_α $\mathfrak{A}$ has the *unique irredundant subdirect decomposition property* if whenever $\mathfrak{A} \cong | \subseteq_{sd} \Pi_{i\in I}\mathfrak{B}_i$ and $\mathfrak{A} \cong | \subseteq_{sd} \Pi_{j\in J}\mathfrak{C}_j$ with $\mathfrak{B}_i$ and $\mathfrak{C}_i$ subdirectly indecomposable such that for no $I' \subset I$ and no $J' \subset J$ is $\mathfrak{A} \cong | \subseteq_{sd} \Pi_{i\in I'}\mathfrak{B}_i$ and $\mathfrak{A} \cong | \subseteq_{sd} \Pi_{j\in J'}\mathfrak{C}_j$, then $|I| = |J|$ and there is a one-to-one function f of I onto J such that $\mathfrak{B}_i \cong \mathfrak{C}_{f(i)}$ for all $i \in I$.

It is easily seen, on general algebraic grounds, that 4.3(i) implies 4.3(ii) and that 4.3(i) is implied by the strict refinement property. In [3] only 4.3(ii) is considered. It is shown in 2.4.42 that the unique subdirect decomposition property 4.3(ii) holds for a CA_α when only finitely many factors are involved (i.e. 4.3(ii) with the additional assumption that $|I|, |J| < \omega$). The proof of 2.4.42 actually establishes a stronger result. Namely, it is shown that any two finite systems of ideals in a CA_α have a strict refinement. The general algebraic implications mentioned above then establish 4.3(ii) in case $|I|, |J| < \omega$. In 4.4 we show that 4.3(ii) holds with restrictions for a reasonably large class of CA_α's including all LCA_α's. We then give a few examples to show that 43(i) fails very often even for LCA_α's.

Proposition 4.4. *If $\mathfrak{A}$ is a regular CA_α, then $\mathfrak{A}$ has the unique subdirect decomposition property* 4.3(ii).

Proof. Suppose $\langle \mathfrak{B}_i: i \in I\rangle$ and $\langle \mathfrak{C}_j: j \in I\rangle$ are two systems of subdirectly indecomposable CA_α's each giving a subdirect decomposition of $\mathfrak{A}$ as in the hypothesis of 4.3(ii). Thus, there exist two systems of ideals $\langle K_i: i \in I\rangle$ and $\langle L_j: j \in J\rangle$ of $\mathfrak{A}$ satisfying conditions (1)–(3) in 4.3(i) such that $\mathfrak{A}/K_i \cong \mathfrak{B}_i$ and $\mathfrak{A}/L_j \cong \mathfrak{C}_j$. For $i \in I$ and $j \in J$ let $M_i = K_i \cap Z(\mathfrak{A})$ and $N_j = L_j \cap Z(\mathfrak{A})$. Conditions (1)–(3) or 4.3(i) imply that $\overline{M}_i = K_i, \{M_i: i \in I\}$ is a dense subset of $X(\mathfrak{A})$ and that $M_i \neq M_{i'}$ whenever $i \neq i'$. Similar facts are true about the N_j's. The notion of irredundancy used in 4.3(ii) is so strong we can prove the following.

(1) Every M_i (and similarly N_j) is an isolated point of $X(\mathfrak{A})$.

By the irredundancy condition on the $\mathfrak{B}_i$'s and the regularity of the ideals, $\{M_i: i \in I, i \neq i_0\}$ is not dense in $X(\mathfrak{A})$. Thus, there is a clopen subset N such that $M_{i_0} \in N$ and N is disjoint from $\{M_i: i \in I, i \neq i_0\}$. Since $X(\mathfrak{A})$ is Hausdorff and $\{M_i: i \in I\}$ is dense, $\{M_i\} = N$; thus, M_{i_0} is isolated.

Since isolated points of $X(\mathfrak{A})$ correspond to maximal principal ideals of $Z(\mathfrak{A})$, it follows from (1) that, for every $i \in I$, M_i is the ideal generated by the dual atom a_i of $Z(\mathfrak{A})$. Since $\bigcap_{i\in I}\overline{M}_i = \{0\}$, it follows that $\{a_i: i \in I\}$ is the set of all dual atoms of $Z(\mathfrak{A})$. Similarly, for each $j \in J$, N_j is the principal ideal generated by a dual atom b_j of $Z(\mathfrak{A})$ and $\{b_j: j \in J\}$ is the set of all dual atoms of $Z(\mathfrak{A})$. Since $a_i \neq a_i'$ for $i \neq i'$ and $b_j \neq b_{j'}$ for $j \neq j'$, it follows that

$|I| = |J|$ and there is a one-to-one function f of I onto J such that $N_{f(i)} = M_i$ for all $i \in I$. Since $\mathfrak{A}$ is regular $K_i = L_{f(i)}$ and so $\mathfrak{B}_i \cong \mathfrak{C}_{f(i)}$ for all $i \in I$ as desired.

Corollary 4.5. *Every LCA_α (and in particular, every CA_α if $\alpha < \omega$) has the unique subdirect decomposition property* 4.3 (ii).

Along with the uniqueness problem for subdirect decompositions there is also the question of the existence of subdirect decompositions of $\mathfrak{A}$ into subdirectly indecomposable CA_α's which is irredundant in the strong sense of 4.3 (ii). The proof of 4.4 shows that if $\mathfrak{A}$ is regular the existence of such a subdirect decomposition implies $Z(\mathfrak{A})$ is atomic. It is easily seen that this condition is not sufficient. In fact, for CA_0's, i.e. BA's, such an irredundant decomposition exists for $\mathfrak{A}$ if and only if $\mathfrak{A}$ is finite.

As mentioned above 4.3 (i) holds for every CA_α under the additional assumption that $|I|, |J| < \omega$. The following examples show this is not true for arbitrary I and J.

Proposition 4.6. *There exist a CA_α $\mathfrak{A}$ and two systems of ideals $\langle K_i$: $i \in I\rangle$ and $\langle L_j$: $i \in I\rangle$ satisfying* (1)–(3) *in* 4.3 (i) *such that, for some $j \in I$, $L_j \notin \{K_i$: $i \in I\}$. Thus, the unique subdirect decomposition property* 4.3 (i) *fails.*

Proof. Let $\mathfrak{B}$ be a subdirectly indecomposable CA_α that is not simple and let $X = {}^\omega 2$ be the Stone space of the free BA on ω generators. The algebra $\mathfrak{A}$ we want is $\mathfrak{A} = \Gamma(X, \mathfrak{S})$ where $\mathfrak{S}$ is the trivial $\mathfrak{B}$-sheaf over X. Results involved in the proof of the duality theorem (Theorem 1.2) allow us to write down subdirect representations of $\mathfrak{A}$. Namely, for $x \in X$, let $M_x = \{\sigma \in Z(\mathfrak{A})$: $\sigma(x) = 0_x\}$; the system of ideals $\langle \overline{M}_x$: $x \in X\rangle$ satisfies (1)–(3) above. For the system of ideals $\langle K_i$: $i \in I\rangle$ let $I = X$ and $K_x = \overline{M}_x$ for $x \in X$. Fix $j \in I$ and define the ideal L_i for $i \neq j$ by $L_i = \overline{M}_i$; let $L_j = \{\sigma \in A; \sigma(j) \in N\}$ where N is some maximal ideal of the stalk $\mathfrak{S}_j \cong \mathfrak{B}$. Clearly L_j is an ideal such that $L_j \supset M_j$. For all $i \in I$, $\mathfrak{A}/L_i$ is subdirectly indecomposable and, since X is Hausdorff, $L_x \nsubseteq L_y$ for $x, y \in X$, $x \neq y$. Before proving that $\bigcap_{i \in I} L_i = \{0\}$, observe the following consequence of the partition property.

(4) For any $\sigma \in A$ there exists a partition $\{N_i$: $i < n\}$ of X into a finite number of clopen subsets such that $\mathrm{pr}_1 \circ \sigma$ is constant on each N_i (pr_1 is the natural projection of $X \times \mathfrak{B}$ onto $\mathfrak{B}$).

Now suppose $0 \neq \sigma \in A$. By (4) there exist a clopen subset N of X such that $\sigma(x) \neq 0_x$ for all $x \in N$. Since $\{j\}$ is not open there is an $x \in N$, $x \neq j$. Since $\sigma(x) \neq 0_x$, $\sigma \notin \overline{M}_x = L_x$. Thus $\bigcap_{x \in X} L_x = \{0\}$. The two systems of ideals $\langle K_x$: $x \in X\rangle$ and $\langle L_x$: $x \in X\rangle$ in $\mathfrak{A}$ satisfy 4.6.

Instead of using, for X, the Cantor space ${}^\omega 2$ we could have used any nondiscrete Boolean space; we must only find some $j \in X$ such that $\{j\}$ is not open. A disadvantage of 4.6 is that the construction will not give an LCA_α $\mathfrak{A}$ since directly indecomposable LCA_α's are simple. The next result can yield LCA_α's and indicates how $|I|$ and $|J|$ may differ.

Proposition 4.7. *There exist a* CA_α $\mathfrak{A}$ *(which can be chosen an* LCA_α *if desired) and two systems* $\langle K_i:\ i \in I\rangle$ *and* $\langle L_j:\ j \in J\rangle$ *of ideals of* $\mathfrak{A}$ *satisfying conditions* (1)–(3) *in* 4.3(i) *but for which* $|I| \neq |J|$.

Proof. Let $\mathfrak{B}$ be a subdirectly indecomposable CA_α and $X = {}^\omega 2$. Choose $\mathfrak{A} = \Gamma(X, \mathcal{S})$ where $\mathcal{S}$ is the trivial $\mathfrak{B}$-sheaf over X. Note $\mathfrak{A}$ is an LCA_α if $\mathfrak{B}$ is. The system of ideals $\langle \overline{M}_x:\ x \in X\rangle$ satisfies (1)–(3). X has a countable dense subset J. The system of ideals $\langle \overline{M}_x:\ x \in J\rangle$ of $\mathfrak{A}$ also satisfies (1)–(3); (1) is a consequence of J being dense and property (4) from the proof of 4.6. The result follows since $|X| > |J|$.

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