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A NON-NOETHERIAN FACTORIAL RING[1]

BY

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ABSTRACT. This paper supplies a counterexample to the conjecture that factorial implies Noetherian in finite Krull dimension. The example is the integral closure of a three-dimensional Noetherian ring, and is the union of Noetherian domains, which are proven to be factorial by means of derivation techniques.

0. **Introduction.** This paper touches on the previously unexplored problem of when the factorial property implies the Noetherian property, in the category of commutative domains with unit.

Due to the abundant existence of non-Noetherian factorial rings in infinite Krull dimension, one restricts one's attention to rings of finite Krull dimension. However, this paper shows that one must make distinctions even finer than that of Krull dimension, finite or infinite, to properly treat factorial implies Noetherian. That is, there is a non-Noetherian factorial ring in dimension three.

1. **Notation.** We will retain the following notation for the remainder of the paper.

(i) "Dimension" means Krull dimension.

(ii) K is a field of characteristic 2 such that $[K: K^2]$ is countably infinite. $\{b_1, c_1, \cdots, b_k, c_k, \cdots\}$ is a 2-base for K over K^2.

(iii) $R^* = K[[x, y, z]]$, $R = K^2[[x, y, z]][K]$ where x, y, z are algebraically independent variables over K.

(iv) $d = \sum_{i=1}^{\infty} b_i y x^i + \sum_{i=1}^{\infty} c_i z x^i$.

For $N = 1, 2, \cdots$,

$$d_N = \sum_{i=0}^{\infty} b_{i+N} y x^i + \sum_{i=0}^{\infty} c_{i+N} z x^i, \quad e_N = \sum_{i=0}^{\infty} b_{i+N} x^i, \quad f_N = \sum_{i=0}^{\infty} c_{i+N} x^i.$$

For $N \neq 1$, $\alpha_N = \sum_{i=1}^{N-1} b_i y x^i + \sum_{i=1}^{N-1} c_i z x^i$, $\alpha_1 = 0$.

(v) T = the integral closure of $R[d]$ in its quotient field.

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(vi) $H_N = K^2[b_1, c_1, \cdots, b_{N-1}, c_{N-1}][[x, y, z]][e_N, f_N]$,
K_N = the quotient field of H_N,
$I_N = K^2[b_1, c_1, \cdots, b_{N-1}, c_{N-1}][[x, y, z]][d_N]$,
L_N = the quotient field of I_N,
$S_N = K^2[b_1, c_1, \cdots, b_{N-1}, c_{N-1}][[x, y, z]]$.

2. Some theorems by M. Nagata and P. Samuel.

Theorem 1 (Nagata). *$(R, (x, y, z))$ is a regular local Zariski ring, R^* the completion of R and $\sum_{i,j,k} a_{ijk} x^i y^j z^k \in R$ iff $\{a_{ijk}\}$ belongs to a finite field extension of K^2.*

Proof. See [1, p.206].

Theorem 2 (Nagata). *T is a three Krull-dimensional, non-Noetherian local ring with maximal ideal (x, y, z).*

Proof. See [1, p.208].

Theorem 3 (Samuel). *Let A be a UFD of characteristic $p \neq 0$, L its quotient field, Δ a derivation of L such that $\Delta(A) \subseteq A$, $L' = \mathrm{Ker}(\Delta)$, and A' the Krull ring $L' \cap A$. Define the logarithmic derivatives, D, of Δ relative to A as the additive subgroup of A consisting of elements of the form $\Delta t/t$, $t \in L$. The logarithmic derivatives of unity, D', are defined to be the subgroup of D consisting of those elements that can be written as $\Delta u/u$ where u is a unit of A. Then if $D = D'$, A' is a UFD.*

Proof. See [2, p.86].

Lemma A (Samuel). *Let L be a field of characteristic $p \neq 0$, Δ a derivation of L, L' the subfield $\mathrm{Ker}(\Delta)$. If $[L: L'] = p$, then there exists $a \in L'$ such that $\Delta^p = a\Delta$. (Δ^p is Δ composed with itself p times.)*

Proof. See [2, p.87].

Lemma B (Samuel). *With the same notation and hypotheses in the above theorem and lemma, so that an element t of A is a logarithmic derivative of Δ with respect to A, it is necessary and sufficient that $\Delta^{p-1}(t) = at - t^p$.*

Proof. See [2, p.88].

3. T is a Krull ring which is a union of an ascending chain of Noetherian three dimensional nonregular UFD's.

Proposition 1. *T is a Krull ring.*

Proof. This follows from 33.10 of [1], as $R[d]$ is a Noetherian integral domain.

Lemma 2.1. *$W = R[e_1, f_1, \cdots, e_k, f_k, \cdots]$ is normal.*

Proof. Let $g \in W'$, the derived normal ring of W. Then $\exists p, q, r, s, t \in R$ such that $g = (p + qe_n + rf_n + se_nf_n)/t$ for some n as $R[e_1, f_1, \cdots, e_l, f_l] = R[e_l, f_l]$ and the squares of elements of R^* lie in R. As the coefficients of the terms of p, q, r, s and t together generate a finite extension of K^2 and because

$$e_n = b_n + b_{n+1}x + \cdots + b_{m-1}x^{m-n-1} + x^{m-n}e_m, \qquad m \geq n,$$

$$f_n = c_n + c_{n+1}x + \cdots + c_{m-1}x^{m-n-1} + x^{m-n}f_m, \qquad m \geq n,$$

we get g contained in the derived normal ring of H_N for some N, as $g^2 \in K^2[[x, y, z]]$.

$g \in W$ follows from the next lemma and $H_N \subseteq W$.

Lemma 2.2. *H_N is a regular local ring.*

Proof. Since H_N is a finite module extension if $K^2[[x, y, z]]$, it is three dimensional local.

We thus need only show its maximal ideal, m, is (x, y, z). Since $H_N \subseteq R^*$, an element of H_N is a nonunit if it has subdegree ≥ 1. The converse is also true since it is true of $K^2[[x, y, z]]$, and the squares of elements of H_N lie in $K^2[[x, y, z]]$ and units of H_N are such iff their squares are. Now let $\alpha \in m$. Thus $\alpha^2 \in (x, y, z)K^2[[x, y, z]]$. Thus, as a power series in R^*, α must be of subdegree one or greater. Thus

$$\alpha = k_0 + k_1s_1 + k_2s_2 + \cdots + k_qs_q$$

where s_i are various products of b_i's, c_i's $(i < N)$, e_N and f_N that are square-free, and $k_j \epsilon K^2[[x, y, z]]$.

Since the zero degree forms of the s_i are linearly independent over K^2, the subdegree of α is ≥ 1 iff the subdegree of each of the k_i's are ≥ 1, iff $k_i \in (x, y, z)K^2[[x, y, z]]$.

We conclude $m = (x, y, z)H_N$.

Lemma 2.3. *$1, e_N, f_N, e_Nf_N$ are linearly independent over the quotient field of R.*

Proof. It suffices to check independence over R. Let $r_i \in R$ such that

$$(*) \qquad r_1 + r_2e_N + r_3f_N + r_4e_Nf_N = 0.$$

$\exists M > 0$ such that $r_i \in S_M$ $\forall i$. Let $Q = \max\{N, M\}$. Then

$$e_N = b_N + b_{N+1}x + \cdots + b_{Q-1}x^{Q-N-1} + x^{Q-N}e_Q,$$

$$f_N = c_N + c_{N+1}x + \cdots + c_{Q-1}x^{Q-N-1} + x^{Q-N}f_Q.$$

Substituting these two equations into (*), we have a relation of the form

$$r_1' + r_2' e_Q + r_3' + r_4' e_Q f_Q = 0 \qquad (r_i' \in S_Q).$$

By the linear independency of the leading forms of $1, e_Q, f_Q, e_Q f_Q$ over S_Q, $r_i' = 0$ for all i. Then by the nature of these r_i', we see that $r_4 = 0$, hence $r_2 = r_3 = 0$. We conclude $r_i = 0 \; \forall i$. End of lemma.

Lemma 2.4. $R[d_1, \cdots, d_k, \cdots] = \{(v_1 + v_2 d)/x^l \mid v_i \in R,\ l \geq 0\} \cap R^*$.

Proof. If $(v_1 + v_2 d)/x^l = r^* \in R^*$ where $v_i \in R$ and $l \geq 1$ then $x^l r^* - v_2 d_l x^l = v_1 + \alpha_l v_2$. It follows since R is normal and R^* is integral over R that

$$v_1 + v_2 \alpha_l = x^l \cdot v \quad \text{where } v \in R.$$

Thus we get $r^* = v + v_2 d_l \in R[d_1, d_2, \cdots, d_k, \cdots]$.

If $h \in R[d_1, d_2, \cdots, d_k, \cdots]$ then $h \in R^*$ and $\exists k$ such that $h \in R[d_k]$ as $R[d_1, d_2, \cdots, d_p] = R[d_p]$. Thus $h = v_1 + v_2 d_k$, $v_i \in R$. Thus

$$h = [(x^k v_1 + v_2 \alpha_k) + v_2 d]/x^k.$$

Thus $h \in \{(v_1 + v_2 d)/x^l \mid v_i \in R,\ l \geq 0\} \cap R^*$. End of lemma.

Proposition 2. $R[d_1, d_2, \cdots, d_k, \cdots] = T$.

Proof.

$\subseteq$: T normal, and R^* integrally closed and integral over R imply $R^* \cap$ quotient field $R[d] = T$. Since $d_N = (d + \alpha_N)/x^N$, $N = 1, 2, \cdots$, $d_N \in R^* \cap$ quotient field $R[d]$. Thus $d_N \in T$.

$\supseteq$: Let $h \in T$. By Lemma 2.1, $\exists k$ such that $h \in R[e_1, f_1, \cdots, e_k, f_k]$. Thus $\exists N$ such that $x^N h \in R[e_1, f_1]$. So $x^N h = a_0 + a_1 e_1 + a_2 f_1 + a_3 e_1 f_1$ where $a_i \in R$. Also $x^N h = a + a'd$ where $a, a' \in$ the quotient field of R. Thus by Lemma 2.3,

$$(*) \qquad a = a_0, \quad a' yx = a_1, \quad a' zx = a_2 \quad \text{and} \quad 0 = a_3.$$

By Theorem 1, R is a UFD so we can write $a' = w_1/w_2$, $(w_1, w_2) = 1$; $w_i \in R$. Then y and z being distinct primes of R allow us to conclude from (*) that w_2 divides x in R. Thus

$$x^{N+1} h = x a_0 + (x w_2^{-1}) w_1 d \in R[d].$$

Thus by Lemma 2.4, $h \in R[d_1, \cdots, d_k, \cdots]$. End of Proposition 2.

Let $N > 1$; we are to prove that I_N is a UFD. But first some lemmas.

Lemma 3.1. *H_N is a UFD.*

Proof. Follows from Lemma 2.2.

Lemma 3.2. $[K_N : L_N] = 2$.

Proof. $f_N \in L_N[e_N]$ and $H_N = I_N[f_N, e_N]$ implies $L_N[e_N] = K_N$. Thus e_N being square integral over L_N implies $[K_N : L_N] = 2$, as $K_N \neq L_N$.

Thus $\{e_N\}$ is a 2-base for K_N over L_N. Define the following L_N-derivation of K_N: $\Delta(e_N) = zf_N$.

Lemma 3.3. $\operatorname{Ker}(\Delta) = L_N$ *and* $\Delta(H_N) \subseteq H_N$.

Proof. $K_N \supsetneq \operatorname{Ker}(\Delta) \supseteq L_N$ and Lemma 3.2 imply $\operatorname{Ker}(\Delta) = L_N$. $\Delta(H_N) \subseteq H_N$ is easily verified by checking the action of Δ on f_N and $e_N f_N$.

Lemma 3.4. I_N *is integrally closed.*

Proof. Let $h \in$ integral closure of I_N. By Lemma 3.1, $h \in H_N$, so

$$h = t_1 + t_2 e_N + t_3 f_N + t_4 e_N f_N, \qquad t_i \in S_N.$$

Since $h \in L_N$,

$$h = a + a' d_N, \qquad a, a' \in \text{the quotient field of } S_N.$$

By Lemma 2.3, $t_1 = a$, $t_2 = ya'$, $t_3 = za'$, $t_4 = 0$. Letting $a' = w/v$, where $w, v \in S_N$, we have $t_2 v = wy$ and $t_3 v = wz$. Since S_N is a UFD in which y and z are relatively prime, we get v divides w in S_N. Thus $h = t_1 + a'd_N$, $t_1, a' \in S_N$. Thus $h \in I_N$ and I_N is integrally closed.

Lemma 3.5. $\operatorname{Ker}(\Delta) \cap H_N = I_N$.

Proof. Follows from Lemmas 3.3, 3.4 and H_N being integral over I_N.

Lemma 3.6. $\Delta^2 = y\Delta$.

Proof. Follows from an easy calculation; see Lemma A.

From here on D denotes the logarithmic derivatives of H_N with respect to Δ. D' denotes the logarithmic derivatives of unity of H_N with respect to Δ. (See Theorem 3.)

Lemma 3.7. *Let* $t \in D$. *Then if* t *is a unit of* H_N, $t \in D'$.

Proof. By Lemmas B, 3.1, 3.2 and 3.6, $\Delta t = t^2 + yt$. As $y \in \operatorname{Ker}\Delta$, $\Delta(t + y)/(t + y) = t$. As $t + y$ is a unit of H_N, $t \in D'$.

Lemma 3.8. $(D \cap (z, y)H_N) \cup (D \cap \operatorname{Units}(H_N)) = D$.

Proof. Let $h \in D$. We shall show if $h \notin \operatorname{Units}(H_N)$, then $h \in (z, y)H_N$. Assume then that $h \in D \setminus \operatorname{Units}(H_N)$. Then

$$h = v_1 + v_2 e_N + v_3 f_N + v_4 e_N f_N, \qquad v_i \in S_N.$$

As in Lemma 3.7, $\Delta(h) = yh + h^2$. This is equivalent to, by Lemma 2.3,

$$(*) \qquad v_4 z f_N^2 + y v_1 = v_1^2 + v_3^2 f_N^2 + v_4^2 e_N^2 f_N^2, \qquad v_2 = 0.$$

By an easy reasoning, none of the v_i are regular in x. Also, as h is a nonunit in H_N, it is a nonunit in R^*. This follows from $q \in R^* \Rightarrow q^2 \in H_N$. Thus we conclude $v_i \in (z, y)R^*$.

We finally conclude $h \in (Z, Y)H_N$ by the following

Lemma 3.8.1. $(z, y)R^* \cap S_N \subseteq (z, y)S_N$.

Proof. Let $\alpha \in (z, y)R^* \cap S_N$. Then as S_N is a power series ring, $\alpha \in (z, y)S_N$.

Lemma 3.9. $D' \supseteq D \cap (z, y)H_N$.

Proof. Let $Q = \{(v, w) | v, w \in H_N \text{ and } vz + wy \in D \cap (z, y)H_N\}$. Define γ, α, β: $Q \to H_N$ as follows:

Let $(r', s') \in Q$ and let $t = r'z + s'y$. Then

$$t = r\Delta e_N + s\Delta f_N \quad \text{where } r = r'/f_N, \ s = s'/f_N.$$

By Lemma B and the proof of Lemma 3.7, it follows that

$$(\Delta r + r^2 \Delta e_N)\Delta e_N = (\Delta s + s^2 \Delta f_N)\Delta f_N.$$

Since Δe_N, Δf_N are relatively prime in H_N, a UFD by Lemma 3.1, $\exists$ unique $b \in H_N$ such that

$$(1) \qquad \Delta s = s^2 \Delta f_N + b\Delta e_N.$$

By derivations of (1), using Lemma 3.6, one gets $\Delta b \Delta f_N = 0$, so $\Delta b = 0$.

Now let $\beta((r', s')) = 1 + re_N + sf_N + (rs + b)e_N f_N$, and rewrite

$$\beta((r', s')) = k_0 + k_1 e_N + k_2 f_N + k_3 e_N f_N, \qquad k_i \in S_N.$$

Also let

$$r' = r_1 + r_2 e_N + r_3 f_N + r_4 e_N f_N,$$

$$s' = s_1 + s_2 e_N + s_3 f_N + s_4 e_N f_N, \qquad r_i, s_i \in S_N,$$

and let r_i^0, s_i^0 be the constant terms of the power series r_i, s_i, respectively. Define $\alpha((r', s'))$ to be the constant term of k_0. Thus $\alpha((r', s')) = 1 + r_4^0 b_N^2 + s_1^0 + r_1^0 s_4^0 b_N^2 + r_2^0 s_3^0 b_N^2 + r_3^0 s_2^0 b_N^2 + r_4^0 s_1^0 b_N^2$ (we have used Lemma 3.5 and that $\Delta b = 0$) and define $\gamma((r', s')) = 1 + b_N^2 r_4^0$. Now let $t_0 \in D \cap (z, y)H_N$, $t_0 = r'z + s'y$. One has

$$\Delta\beta((r', s')) = \beta((r', s'))t_0.$$

One would like to have $\beta((r', s'))$ a unit, for then $t_0 \in D'$, but this is not necessarily so. However, we have $\alpha((r', s')) \neq 0$ implies $\beta((r', s'))$ is a unit by the reasoning contained in Lemma 2.2. So suppose $\alpha((r', s')) = 0$.

Case I. Assume $\gamma((r', s')) \neq 0$. Then $\alpha((r', s' + 1)) \neq 0$. So if $t' = r'z + (s' + 1)y$ then $t' \in D \cap (Z, Y)H_N$ using Lemmas B and 3.6. So $\beta((r', s' + 1))$ is a unit. As $\Delta\beta((r', s' + 1)) = \beta((r', s' + 1))t'$, then

$$\Delta(\beta((r', s' + 1)) \cdot f_N) = \beta((r', s' + 1)) \cdot f_N \cdot t_0$$

and $\beta((r', s' + 1)) \cdot f_N$ is a unit. Thus $t_0 \in D'$.

Case II. Assume $\gamma((r', s')) = 0$. Let $t' = (r' + e_N f_N / e_N^2)z + s'y$. Then $t' \in D \cap (Z, Y)H_N$, using Lemmas B and 3.6. Note that as the constant term of $1/e_N^2$ is $1/b_N^2$,

$$\gamma((r' + e_N f_N / e_N^2, s')) \neq 0.$$

Thus since either $\alpha((r' + e_N f_N / e_N^2, s')) \neq 0$ or, by Case I, $\exists$ a unit $u \in H$ such that $\Delta u = ut'$, then $\Delta(e_N u) = e_N u t_0$ and $e_N u$ is a unit. Thus $t_0 \in D'$. End of lemma.

Lemma 3.10. $D' = D$.

Proof. Follows from Lemmas 3.7, 3.8, 3.9.

Proposition 3. *I_N is a UFD.*

Proof. Follows from Theorem 3 and Lemmas 3.1, 3.3, 3.5 and 3.10.

Remark. In proving Lemma 3.9, some techniques of P. Samuel were used that are found in Lemma 3 of [3].

Proposition 4. *T is a UFD.*

Proof. In view of Proposition 1, we need only show the minimal primes of T are prinicipal.

Let P denote a minimal prime of T. By Proposition 1, $\exists\ \alpha \in T$ such that $PTP = \alpha \cdot TP$. As $d_i \in I_j$, $i \leq j$, and $T = R[d_1, d_2, \cdots, d_k, \cdots]$ (Proposition 2), we see that $\exists M$ such that $\alpha \in I_M \subseteq T$. Since T is an integral extension of I_M, and by Lemma 3.4, $P' = P \cap I_M$ is a minimal prime of I_M. Thus, by Proposition 3, P' is principal. Let β generate P'. Since the squares of elements of T lie in I_M, no two primes can contract to the same prime in I_M. Thus β is not contained in any other minimal prime of T.

As $\alpha \in P'$, $\exists\ q \in T$ such that $\alpha = q \cdot \beta$. Thus $\beta \cdot TQ = PTQ$, for every minimal prime Q, of T.

By the following lemma, β generates P.

Lemma 4.1. *Let R be a Krull domain such that $\beta, \delta \in R$. Let β divide δ in R_p $\forall$ minimal prime P. Then β divides δ in R.*

Proof. As $(\beta R: \partial R)$ is divisorial, $(\beta R: \delta R) \cap R$ is such. As $(\beta R: \delta R) \cap R$ is integral, it must contain 1 or be contained in some height 1 prime. As the latter is not possible, $1 \in (\beta R: \partial R)$. Thus β divides δ in R. (See [3, pp. 1, 7].)

We are now able to state

Theorem. *Let K be a field of characteristic two such that $[K: K^2]$ is countably infinite. Let $\{b_i, c_i\}_{i=1}^{\infty}$ be a two-base for K over K^2. Let $R = K^2[[x, y, z]][K]$,*

$$d = \sum_{i=1}^{\infty} b_i y x^i + c_i z x^i.$$

Then T = the integral closure of $R[d]$ is a three dimensional non-Noetherian quasi-local factorial ring.

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