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ON THE IDEAL STRUCTURE OF BANACH ALGEBRAS

BY

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ABSTRACT. For Banach algebras A in a class which includes all group and function algebras, we show that the family of ideals of A with the same hull is typically quite large, containing ascending and descending chains of arbitrary length through any ideal in the family, and that typically a closed ideal of A whose hull meets the Šilov boundary of A cannot be countably generated algebraically.

Guided by the results of [6], we explore here some consequences of the Cohen factorization theorem [3] for the ideal structure of a commutative Banach algebra A. If we denote the maximal ideal space of A by $\mathfrak{M}_A$ and define the zero set of an ideal I of A to be $Z(I) = \bigcap \{Z(\hat{f}) = \hat{f}^{-1}(0): f \in I\} \subset \mathfrak{M}_A$, we develop results which indicate that typically the class of ideals of A with the same zero set is quite large, containing ascending and descending chains of arbitrary length, and that typically a closed ideal of A whose zero set meets the Šilov boundary of A cannot be (algebraically) countably generated. For example, if G is a locally compact Abelian group, there is an ideal of $L^1(G)$ strictly between any two ideals $I \subsetneq J$ which have the same zero set whenever either I or J is closed (compare [17, 7.7.2]). Further, a closed ideal J of $L^1(G)$ can be countably generated only if $Z(J)$ is open-closed, and a maximal ideal can be countably generated only if G is finite (compare [11, 2.1], [5, Corollary 3]).

Algebraic applications of Cohen's theorem have been few to date (principally [4, 4.7]), but because the result transforms a purely analytic condition on A (existence of a bounded approximate identity) into a purely algebraic conclusion (factorization of elements), it provides a most appropriate tool for just such applications.

1. **Chains of ideals.** Let A be a commutative algebra(1) over a field F. An ideal of A will be a subspace over F closed under multiplication from A. If I and J are ideals, IJ will denote $\{fg: f \in I,\ g \in J\}$, not the ideal generated by this set. We begin with a purely algebraic remark.

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(1) All algebras discussed will be assumed commutative.

Lemma 1.1. *If $I \subset J$ are ideals of A and there are no ideals of A strictly between them, then either $AJ \subset I$ or $MJ \subset I$ where M is some maximal ideal of A containing I.*

Proof. Let M be the ideal $\{f \in A: fJ \subset I\}$. If $A = M$ we are done; if not, take any $g \in A \setminus M$. $kg \notin I$ for some $k \in J$, and $I + kgA + Fkg$ is an ideal of A between I and J which properly contains I. Thus $J = I + kgA + Fkg$, and for each $f \in A$ we may write $fk = i + kga + \alpha kg$, $i \in I$, $a \in A$, $\alpha \in F$; that is, $k(f - ga - \alpha g) \in I$. Since $k \notin I$, we also have $J = I + kA + Fk$, so that $(f - ga - \alpha g)J = (f - ga - \alpha g)I + k(f - ga - \alpha g)A + Fk(f - ga - \alpha g) \subset I$. This means $f - ga - \alpha g \in M$; that is, $A = M + gA + Fg$. Thus M is a maximal ideal of A.

Corollary 1.2. *Suppose $I \subsetneqq J$ are ideals of A contained in the same family of maximal ideals of A. Then if either $J^2 = J$ or I is semiprime, there is an ideal of A strictly between I and J.*

Proof. If not, 1.1 implies $J^2 \subset I$, since every maximal ideal of A which contains I also contains J. But then if $J^2 = J$, $J = I$; if I is semiprime, $J = I$ also.

Corollary 1.3. *Suppose X is locally compact and $I \subsetneqq J$ are ideals of $C_0(X)$ with the same zero set. If either I or J is semiprime, there is an ideal of $C_0(X)$ strictly between them.*

Proof. The maximal ideals of $C_0(X)$ correspond to the points of X, so that I and J are contained in the same maximal ideals. If J is semiprime, $J^2 = J$. For if $f \in J$, $|f|^2 = f\bar{f} \in J$, and hence $|f|$, $|f|^{1/2}$, $|f|^{1/4} \in J$ as well. But then $f = |f|^{1/2}(|f|^{1/4} \cdot |f|^{1/4} \text{ sign } f) \in J^2$. Apply 1.2.

Corollary 1.4. *If $I \subsetneqq J$ are semiprime ideals of $C_0(X)$ with the same zero set, J/I has no maximal or minimal ideals.*

A subset B of a Banach algebra is said to have approximate identity if there is a net $\{e_\alpha\}$ on B, uniformly bounded in norm, for which $\|e_\alpha f - f\| \to 0$ for each $f \in B$. A principal tool in our work is the following.

Theorem 1.5. *If I and J are closed ideals in a Banach algebra and either has approximate identity, then $IJ = I \cap J$.*

Proof. If I has approximate identity, apply Cohen's theorem to obtain, for each $z \in I \cap J \subset I$, $x, y \in I$ such that $z = xy$ where y is chosen from the closed ideal generated by z. Since $I \cap J$ is closed, this means $y \in I \cap J \subset J$; hence $I \cap J = IJ$. Because $IJ = JI$, this also obtains when J has approximate identity.

Theorem 1.6. *Let A be a Banach algebra with approximate identity. Suppose $I \subsetneqq J$ are ideals of A with $Z(I) = Z(J)$, and that for each $p \in Z(I)$,*

$I_p = \{f \in A: \hat{f}(p) = 0\}$ *has approximate identity. Then if either* I *or* J *is closed in* A, *there is an ideal of* A *strictly between them.*

Proof. 1.5 implies that $A^2 = A$, and from this it follows that every maximal ideal M of A is regular; that is, A/M has identity. For $A^2 \not\subset M$, so that $(A/M)^2 \neq 0$; since A/M is a simple commutative algebra, this means it is actually a field [15, 2.1.5, p. 45]. Thus every maximal ideal of A is closed and those which contain I and J are of the form I_p, $p \in Z(I)$.

Suppose there are no ideals between I and J. Then by 1.1, $AJ \subset I$ or $MJ \subset I$ for some maximal ideal M containing I (and hence J). If J is closed, then since M and A have approximate identity, 1.5 implies that $J = J \cap A = AJ$ and $J = J \cap M = MJ$, so that in either case above, $J = I$, a contradiction. If I is closed, then again because A and M have approximate identities, we see that $J \subset \overline{AJ} \subset I$ or $J \subset \overline{MJ} \subset I$, hence $I = J$.

Corollary 1.7. *With* I *and* J *as above, the algebra* J/I *does not have the ascending chain condition on ideals when* J *is closed, and does not have the descending chain condition when* I *is closed.*

Corollary 1.8 **(compare** [17, 7.7.2, p. 183]**).** *Let* G *be a locally compact Abelian group. If* $I \subsetneq J$ *are ideals of* $L^1(G)$ *with the same zero set, and if one of them is closed, then there is an ideal of* $L^1(G)$ *strictly between them.*

Proof. $L^1(G)$ has approximate identity, as does each of its maximal ideals [14, p. 151]. Apply 1.6.

The maximal ideal space of $L^1(G)$ is of course the dual group Γ, which we may also view as a subset of the maximal ideal space of the measure algebra $M(G)$. For $\gamma \in \Gamma$, let K', K be compact neighborhoods of γ in Γ with $K' \subset \operatorname{int} K$, and choose a bounded net $\{j_\alpha\}$ on $L^1(G)$, indexed by the neighborhoods $\{U_\alpha\}$ of γ contained in K', for which $\operatorname{supp}\hat{j}_\alpha \subset K$, $\hat{j}_\alpha|U_\alpha = 1$ and $\|j_\alpha * g\|_1 \to 0$ when $\hat{g}(\gamma) = 0$ (cf. [14, p. 151]). Then notice $\mu_\alpha = \delta_0 - j_\alpha \in M(G)$ is a bounded approximate identity for the maximal ideal $I_\gamma = \{\mu \in M(G): \hat{\mu}(\gamma) = 0\}$ (here δ_0 denotes point mass of 0). For if $h \in L^1(G)$ is chosen so that $\hat{h}|K = 1$, then, for each $\mu \in I_\gamma$, $\|\mu_\alpha * \mu - \mu\| = \|j_\alpha * \mu\|_1 = \|j_\alpha * h * \mu\|_1 \to 0$. From 1.6 we therefore conclude

Corollary 1.9. *If* $I \subsetneq J$ *are ideals in* $M(G)$ *with* $Z(I) = Z(J) \subset \Gamma$, *there is an ideal of* $M(G)$ *strictly between them whenever either is closed.*

The assumption in 1.8 that I or J is closed cannot be deleted. For suppose there is an $f \in L^1(G)$ whose zero set $Z(\hat{f})$ is exactly a nonisolated point $p \in \Gamma$. Such an f will exist whenever Γ is nondiscrete and second countable. Set $I = f * I_p$ and $J = f * L^1(G)$. Certainly $Z(I) = p = Z(J)$ and $I \subsetneq J$. For if $g \in L^1(G)\setminus I_p$,

$f * g \in J \backslash I$, since otherwise $f * g = f * h$ for some $h \in I_p$, and taking transforms and dividing out $\hat{f}$, we obtain $\hat{g} = \hat{h}$ on $\Gamma \backslash \{p\}$; hence $g = h \in I_p$, a contradiction. However there is no ideal K of $L^1(G)$ strictly between I and J. For if K is any ideal of $L^1(G)$ contained in J and properly containing I, and if $k \in K \backslash I$ with $k = f * h$, $h \in L^1(G) \backslash I_p$, we have $L^1(G) = I_p + h * L^1(G) + Ch$, because I_p is a maximal ideal. Thus $J = f * L^1(G) = f * I_p + k * L^1(G) + Ck \subset K$.

If A is a function algebra on a compact space X, let ∂A denote its Choquet boundary (cf. [2]). Similar to 1.9 we have

Corollary 1.10. *Suppose $I \subsetneqq J$ are ideals of a function algebra with $Z(I) = Z(J) \subset \partial A$. If either I or J is closed, there is an ideal strictly between them.*

Proof. For $p \in X$, the maximal ideal $I_p = \{f \in A;\ f(p) = 0\}$ has approximate identity iff $p \in \partial A$ [2, 1.6.3, 2.3.4]. We may therefore apply 1.6.

If A has unique representing measures on X, then $\partial A = X$, and 1.10 applies whenever $Z(I) = Z(J) \subset X$. 1.10 also provides new ideals even in function algebras where all the closed ideals are known. For example, in the disc algebra A, every nonzero closed ideal is of the form $FI_K = \{Fg\colon g \in A,\ g|K = 0\}$ where F is an inner function and K is a closed set of zero measure on the circle T [13, p. 85]. If $p \in T = \partial A$, $I = (z - p)A$ is an ideal of A, proper and dense in I_p, and $Z(I) = p = Z(I_p)$. Thus although there are no closed ideals between I and I_p, 1.10 yields an infinite ascending chain of ideals of A between them. More generally if we set

$$u_\alpha(z) = (z - p)\exp(\alpha(z + p)/(z - p)),$$

then every closed ideal of A with zero set p is of the form $J_\alpha = \overline{u_\alpha A}$, $\alpha \geqq 0$, [13, p. 88]. But $I_\alpha = u_\alpha A$ is then proper and dense in J_α, $Z(I_\alpha) = Z(J_\alpha)$, and we may obtain a chain of (nonclosed) ideals between them. This example also reveals the necessary role of the approximate identity in 1.6: if p is in the open unit disc, the family of ideals of A which have p as zero set is the discretely descending chain $(z - p)^n A$, $n = 1, 2, \cdots$, [9, p. 235].

Actually there is an abstract reason why the disc algebra has a dense chain of ideals at every maximal ideal on its boundary.

Proposition 1.11. *If E is a nonopen peak set for a function algebra A, there are infinite ascending and descending chains of ideals of A whose zero sets are E and which are all densely contained in $I_E = \{f \in A\colon f|E = 0\}$.*

Proof. Choose an $h \in A$ which peaks to 1 on E, and set $f = 1 - h$ and $I_n = f^n I_E$, $n = 0, 1, \ldots$; $Z(I_n) = Z(f^n) = Z(f) = E$ and $I_{n+1} \subsetneqq I_n \subset I_E$. For if $I_{m+1} = I_m$ for some m, $f^{m+1} \in I_m = I_{m+1}$; hence $f^{m+1} = f^{m+1}g$, $g \in I_E$, so that $\chi_{E'} = g \in A$ and E is open-closed in X, a contradiction. Induction shows that each I_n is dense in $I_0 = I_E$. For assuming I_n is dense, $I_{n+1} = (1-h)I_n$ will be dense if $(1-h)I_E$ is. But for fixed $g \in I_E$, $(1-h^k)g = (1-h)$ $\cdot (1 + h + \cdots + h^{k-1})g \in (1-h)I_E$ and $\|(1-h^k)g - g\|_\infty = \|h^k g\|_\infty \to 0$. Since I_E has approximate identity, 1.5 implies $I_E^2 = I_E$. 1.2 then yields an infinite ascending chain of ideals of A, $\{J_n\}$, between I_1 and $I_0 = I_E$.

For example, we obtain such ideals in the disc algebra whenever E is a nonempty closed set of measure zero on the circle [13, p. 81]. Likewise, if A is a logmodular or hypo-Dirichlet [1, 3.1] function algebra on a metrizable space X, then we obtain dense descending and ascending chains of primary ideals at every nonisolated point of X.

Chains of ideals may be constructed in other ways. If A is a Banach algebra and E is a closed subset of its maximal ideal space, set $F_E = \{f \in A: \hat{f}$ vanishes on a neighborhood of $E\}$, $K_E = \{f \in F_E: \hat{f} \in C_c(\mathfrak{M}_A)\}$ and $I_E = \{f \in A: \hat{f}|E = 0\}$. Certainly $K_E \subset F_E \subset I_E$ are ideals of A; if A is regular, $Z(K_E) = E = Z(I_E)$ and every ideal of A whose zero set is E lies between K_E and I_E [14, 25D, p. 84].

Theorem 1.12 (**compare** [6, 4.5]). *If J is an ideal of $L^1(G)$ strictly between K_E and I_E, there are ideals $\underline{J}$ and $\overline{J}$ of $L^1(G)$ for which*

$$K_E \subset \underline{J} \subset J \subset \overline{J} \subset I_E$$

with all inclusions proper.

Proof. Since I_E is a closed ideal of $L^1(G)$ and $Z(J) = Z(I_E)$, the existence of $\overline{J}$ follows from 1.8. Since K_E is semiprime, 1.2 yields $\underline{J}$.

By repeated use of 1.12 we see that through any such J we may thread infinite ascending and descending chains of ideals of $L^1(G)$ whose zero sets are E. Notice this happens whether or not E is a set of spectral synthesis (in which case, of course, every ideal in the chain will be dense in I_E). In particular, infinite ascending and descending chains between K_E and I_E will exist whenever $K_E \neq I_E$ (compare [6, 1.11, 1.12]). If G is second countable, every closed set in Γ is of the form $\hat{f}^{-1}(0)$ for some $f \in L^1(G)$. So that if Γ is also connected, we have $K_E \neq I_E$ (and hence chains of ideals) for every nonempty proper closed set $E \subset \Gamma$. Of course, in any group this is also true whenever E is a set of nonsynthesis. For $E = \emptyset$, we obtain infinite ascending and descending chains of dense ideals in $L^1(G)$ whenever G is second countable and nondiscrete. In view of 1.9, similar results hold for $M(G)$.

2. **Finitely generated ideals.** Motivated by [6, 2.6] and [5, Corollary 3], we

combine 1.5 and a theorem due to Gleason [10, 2.1] to show here that closed ideals in Banach algebras can rarely be generated algebraically by a finite number of their elements (2.3).

Lemma 2.1. *If A is a regular Banach algebra with identity and J is a closed ideal of A, then the algebra $J + C$ is also regular.*

Proof. Standard arguments [8, p. 12] show that $J + C$ is a Banach algebra, and that restriction $r: \mathfrak{M}_A \to \mathfrak{M}_{J+C}$ is an identification which collapses $Z(J)$ to a point. Thus if F is a closed set in $\mathfrak{M}_{J+C}$ not containing a point $p \in \mathfrak{M}_{J+C}$, either $r^{-1}(F) \cup Z(J)$ and $r^{-1}(p)$ or $r^{-1}(F)$ and $Z(J) = r^{-1}(p)$ are disjoint compact subsets of $\mathfrak{M}_A$. In the first case, normality of $\mathfrak{M}_A$ and regularity of A yield a $g \in A$ whose transform vanishes on a neighborhood of $r^{-1}(F) \cup Z(J)$ and is identically 1 on $r^{-1}(p)$ [14, 25C, p. 84]. $g \in K_{Z(J)} \subset J + C$ [14, 25D] then separates p and F in $\mathfrak{M}_{J+C}$. In the second case, we select a $g \in A$ with transform vanishing on a neighborhood of $Z(J)$ which is 1 on $r^{-1}(F)$; again $g \in J + C$ serves to separate F and p.

Theorem 2.2. *If A is a regular Banach algebra with identity and J is a closed, finitely generated ideal of A with approximate identity, then $Z(J)$ is open-closed in $\mathfrak{M}_A$.*

Proof. J is the maximal ideal in $J + C$ which corresponds to the point $r(Z(J))$ in $\mathfrak{M}_{J+C}$. Because J has approximate identity, it is finitely generated as an ideal of $J + C$. For if $J = f_1A + \cdots + f_nA$, $f_i \in J$, 1.5 yields g_i, $h_i \in J$ for which $f_i = g_ih_i$, so that $J = g_1h_1A + \cdots + g_nh_nA = g_1(J + C) + \cdots + g_n(J + C)$. Because $J + C$ is regular (2.1), its Šilov boundary is $\mathfrak{M}_{J+C}$, and Gleason's theorem [11, 2.1] combined with the maximum modulus principle for analytic varieties implies that $r(Z(J))$ is isolated in $\mathfrak{M}_{J+C}$. Thus $Z(J) = r^{-1}(r(Z(J)))$ is open-closed in $\mathfrak{M}_A$.

Corollary 2.3. *If $\mathfrak{M}_A$ is connected, 0 and A are the only closed ideals of A with approximate identity which can be finitely generated.*

Actually, 2.2 holds whether or not A has identity.

Theorem 2.4. *If A is a regular Banach algebra and J is a closed, finitely generated ideal of A with approximate identity, then $Z(J)$ is open-closed in $\mathfrak{M}_A$.*

Proof. Set $\tilde{A} = A \times C$ and make $\tilde{A}$ into a Banach algebra with identity (0, 1) via $(f, \alpha)(g, \beta) = (fg + \alpha g + \beta f, \alpha\beta)$ and $\|(f, \alpha)\| = \|f\| + |\alpha|$. The embedding $h \to \tilde{h}$ $[\tilde{h}(x, \alpha) = h(x) + \alpha]$ is a homeomorphism $\mathfrak{M}_A \to \mathfrak{M}_{\tilde{A}}$, and $\mathfrak{M}_{\tilde{A}} \setminus \mathfrak{M}_A$ is a single point ∞ corresponding to the maximal ideal $A = A \times 0$. Thus $\mathfrak{M}_{\tilde{A}}$ is the one-point compactification of $\mathfrak{M}_A$, so that $\tilde{A}$ is a regular algebra also. For if F

is a closed set in $\mathfrak{M}_{\tilde{A}}$ not containing ∞, F is a compact subset of $\mathfrak{M}_A$, and there is an $f \in A$ with $\tilde{h}(f, 0) = h(f) = 1$ for all $\tilde{h} \in F$ [14, 25C]. Since $\infty(f, 0) = 0$, $(f, 0)$ separates ∞ and F. If F is a closed set of $\mathfrak{M}_{\tilde{A}}$ not containing $p \in \mathfrak{M}_A$, we choose $f \in A$ with $\hat{f}(p) = 1$ and $\operatorname{supp} \hat{f} \subset \mathfrak{M}_A$ disjoint from $F \cap \mathfrak{M}_A$. Then $(f, 0)$ separates p and F in $\mathfrak{M}_{\tilde{A}}$. Certainly $\tilde{J} = J \times 0$ is a closed, finitely generated ideal of $\tilde{A}$ with approximate identity. So $\{\infty\} \cup Z(J) = Z(\tilde{J})$ is open-closed in $\mathfrak{M}_{\tilde{A}}$ (2.2); hence $Z(J) = Z(\tilde{J}) \cap \mathfrak{M}_A$ is open(-closed) in $\mathfrak{M}_A$.

For example in the regular algebra $L^1(G)$, the closed ideal I_E will have approximate identity whenever E is a closed subgroup in the dual group Γ. We conclude from 2.4 that, for instance, I_Z is not finitely generated in $L^1(R)$. Actually the only finitely generated, closed ideal in $L^1(R)$ is the zero ideal (cf. §3). This does not follow from 2.4 however. For the approximate identity assumption here is quite restrictive. If we let $\mathfrak{R}(\Gamma)$ denote the ring of subsets of Γ generated by the cosets of all subgroups of Γ, we have

Lemma 2.5. *An ideal J in $L^1(G)$ has approximate identity iff $Z(J) \in \mathfrak{R}(\Gamma)$.*

Proof. Suppose J has an approximate identity $\{e_\alpha\}$ bounded by M. For each $\gamma \in \Gamma \setminus Z(J)$, there is an $f \in J$ whose Fourier transform is 1 at γ. Since $\hat{e}_\alpha(\gamma)\hat{f}(\gamma) \to \hat{f}(\gamma)$, we see that $\hat{e}_\alpha \to \chi_{Z(J)'}$ pointwise on Γ. If we view each e_α as a measure on $\bar{G}$, the Bohr compactification of G, then as a net in the weakly compact ball of measures $\{\mu \in M(\bar{G}): \|\mu\| \le M\}$, $\{e_\alpha\}$ must have a subnet $\{e_{\alpha_\beta}\}$ converging weakly to some $\mu \in M(\bar{G})$. Since each $\gamma \in \Gamma$ extends to a continuous function on $\bar{G}$, $\hat{e}_{\alpha_\beta}(\gamma) \to \hat{\mu}(\gamma)$ pointwise on Γ. Thus $\hat{\mu} = \chi_{Z(J)'}$ and μ is an idempotent measure on $\bar{G}$. By Cohen's theorem [17, 3.1.3, p. 60], $Z(J) \in \mathfrak{R}(\Gamma)$.

Conversely, the arguments of [18, 1.7, 2.6] imply that if $Z(J) \in \mathfrak{R}(\Gamma)$, there is a constant K so that given $\epsilon > 0$ and $f_1, \cdots, f_n \in I_E$, there is an $e \in F_E$ for which

$$(\#) \qquad \|e\|_1 \le K \text{ and } \|e * f_i - f_i\|_1 < \epsilon, \qquad i = 1, 2, \cdots, n.$$

But if $\epsilon > 0$ and $f_1, \cdots, f_n \in J$ are given, set $\delta = \min\{1, \epsilon/(2 \max \|f_i\|_1)\}$, choose e as above for $\epsilon/2$, pick $u \in L^1(G)$ with $\|u\|_1 = 1$ and $\|e - u * e\|_1 < \delta/2$ [17, 1.1.8], and then find $v \in L^1(G)$ such that $\hat{v}$ has compact support and $\|u - v\|_1 < \delta/(2\|e\|_1 + 1)$ [17, 2.6.6]. Then

$$\|e * v - e\|_1 < \delta, \qquad \|e * v\|_1 \le \|e\|_1(\|u - v\|_1 + \|u\|_1) \le 2\|e\|_1 \le 2K,$$

and for each i,

$$\|e * v * f_i - f_i\|_1 \le \|f_i\|_1 \|e * v - e\|_1 + \|e * f_i - e\|_1 < \epsilon.$$

But $e * v \in K_E \subset J$, and we conclude that (#) holds with $2K$ in place of K and e choosen from J. Thus J has approximate identity.

In particular, $Z(J)$ must be a set of synthesis if J is to have approximate identity [18, 2.6]. Further, if J is a proper nonzero ideal and Γ is connected, $Z(J)$ must be contained in a finite union of cosets of proper closed subgroups of G; in fact for $G = R$, $Z(J)$ must differ from some finite union of arithmetic progressions by at most a finite set [18, 1.5, 1.6].

Furthermore it is not clear that 2.4 holds if the approximate identity assumption on J is deleted — the principal difficulty being that ideals J finitely generated over A need not be finitely generated over $J + C$. Consider, for example, a semisimple Banach algebra A with 1 and a nonisolated point p in its Šilov boundary Γ_A. Suppose there is an $f \in A$ with $Z(\hat{f}) = \{p\}$. Such f will exist if A is regular and selfadjoint, and $\mathfrak{M}_A - \{p\}$ is σ-compact. Let $J = fA$ and suppose $J = g_1(J + C) + \cdots + g_n(J + C)$ for some $g_i = fh_i \in J$. For each $g \in A$,

$$fg = \sum_{i=1}^{n} f^2 h_i k_i + \alpha_i f h_i, \qquad k_i \in A,\ \alpha_i \in C.$$

Taking transforms, dividing out $\hat{f}$ and using the assumptions on p and A, we conclude that each $g \in A$ can be written $g = g' + \sum_{i=1}^n \alpha_i h_i$, $g' \in J$, $\alpha_i \in C$; that is, $\{h_i + J\}$ spans the vector space A/J. In particular, the subspace I_p/J is finite dimensional with some basis $\{s_1 + J, \cdots, s_k + J\}$. We conclude that $\{f, s_1, \cdots, s_k\}$ generates I_p as an ideal of A. But this contradicts Gleason's theorem since p is nonisolated in Γ_A.

On the positive side, this method yields

Theorem 2.6. *If J is a finitely generated, closed ideal of finite codimension in a Banach algebra A, then $Z(J) \cap \Gamma_A$ is open-closed in Γ_A.*

Proof. Suppose first that the result is valid for every A with identity. For $\tilde{A} = A \times C$ and $\tilde{J} = J \times 0$, there is a vector space isomorphism $\tilde{A}/A \oplus A/J \approx \tilde{A}/\tilde{J}$. Hence $\dim \tilde{A}/\tilde{J} = \dim \tilde{A}/A + \dim A/J = 1 + \dim A/J < \infty$. Since $\tilde{J}$ is a finitely generated, closed ideal in $\tilde{A}$, $Z(\tilde{J}) \cap \Gamma_{\tilde{A}}$ is open in $\Gamma_{\tilde{A}}$. But the canonical embedding $\mathfrak{M}_A \to \mathfrak{M}_{\tilde{A}}$ carries Γ_A onto a subset of $\Gamma_{\tilde{A}}$ so that $Z(J) \cap \Gamma_A = Z(\tilde{J}) \cap \Gamma_A$ is open in Γ_A.

Now suppose A has identity. For each $p \in Z(J) \cap \Gamma_A$, I_p/J is a finite dimensional vector space with some basis $\{s_1 + J, \cdots, s_n + J\}$. But if $f_1, \cdots, f_m$ generate J as an ideal over A, this means $I_p = f_1 A + \cdots + f_m A + s_1 A + \cdots + s_n A$, a finitely generated ideal of A. By Gleason's theorem, $\{p\}$ is open in Γ_A; hence $Z(J) \cap \Gamma_A$ is open-closed in Γ_A.

Thus in a regular algebra with connected maximal ideal space, no proper closed ideal of finite codimension can be finitely generated. For example, in the algebra $C^r[0, 1]$ of C^r functions on $[0, 1]$ with the C^r norm, no closed ideal I

whose zero set is finite can be finitely generated. For if $Z(I)$ is finite and we set $E_i = \bigcap\{Z(f^{(i)}): f \in I\}$, then it follows easily from Whitney's theorem [20, Theorem 1, p. 636] that $I = \{f \in C^r[0, 1]: f^{(i)}|E_i = 0\}$. Because $E_0 \supset E_1 \supset \cdots \supset E_r$, if we set $r = \max\{i: p \in E_i\}$ and $I^r_p = \{f \in C^r[0, 1]: f^{(i)}(p) = 0 \text{ for } 0 \leq i \leq r\}$ for each $p \in E_0$, we have $I = \bigcap\{I^r_p: p \in E_0\}$. But this yields a vector space isomorphism

$$C^r[0, 1]/I \approx \bigoplus_{p \in E_0} C^r[0, 1]/I^r_p,$$

and since E_0 is finite and each summand is finite dimensional (cf. [9, p. 206]), I has finite codimension. Actually, recent results of Roth [16], established with more direct methods, imply that 0 and $C^r[0, 1]$ are the only closed ideals in this algebra that can be finitely generated. In view of this, one might conjecture that 2.6 holds for a closed, finitely generated ideal of any codimension. In the next section we show that at least for group and function algebras, this is indeed the case.

3. **Countably generated ideals in group and function algebras.** Besides improving 2.6 for these algebras (3.1, 3.9), we provide, for compactly generated LCA groups G, a complete description of the countably generated, closed ideals of $L^1(G)$ (3.7).

Theorem 3.1. *Let G be a locally compact Abelian group with dual Γ. If J is a closed, countably generated ideal of $L^1(G)$, then $Z(J)$ is open-closed in Γ.*

Proof. Suppose instead that $\partial Z(J) \neq \emptyset$. If $\eta_0 \in \partial Z(J)$, the map $f \to \overline{\eta}_0 f$ $[\overline{\eta}_0 f(x) = \overline{\eta_0(x)} f(x)]$ is a Banach algebra automorphism of $L^1(G)$ which carries J onto a closed, countably generated ideal whose zero set is $Z(J) - \eta_0$. We may therefore assume $0 \in \partial Z(J)$.

J is the smallest ideal of $L^1(G)$ containing some sequence $\{w_n\}$. Since $0 \in Z(J)$ and I_0 has approximate identity, $J = I_0 * J$ (1.5). If we choose $h_i \in I_0$ so that $\|h_i\|_1 \leq 1/2^i$ and $w_i \in h_i * J$, then J is the sum of its subspaces $\{h_i * J\}$. Let $\bigoplus_1^\infty J$ denote the direct sum of countably many copies of J, and J_n the subspace of sequences whose entries following the nth one are all zero. J_n is a Banach space with the norm $\|\{f_i\}\| = \max_{1 \leq i \leq n}\{\|f_i\|_1\}$, and with the final topology induced by the inclusions $J_n \hookrightarrow \bigoplus_1^\infty J$, $\bigoplus_1^\infty J$ is the strict inductive limit of the J_n's. The mapping $T: \bigoplus_1^\infty J \to J$ given by $T(\{f_i\}) = \Sigma_i h_i * f_i$ is linear, continuous and surjective. A theorem of Dieudonné and Schwartz [7, Theorem 1, p. 72] implies T is open. In particular, if U denotes the set of all sequences in $\bigoplus_1^\infty J$ whose entries have norm no larger than 1, there is a $\delta > 0$ so that $B(0, \delta) \subset T(U)$. It follows that there is a constant $M > 0$ such that given $g \in J$, there are $g_1, \cdots, g_p \in J$ with

$$(*) \qquad g = \sum_{i=1}^{p} h_i * g_i \quad \text{and} \quad \|g_i\|_1 \le M\|g\|_1, \qquad i = 1, 2, \cdots, p.$$

Since $\|h_i\|_1 \le 1/2^i$, there is an n so large that $\Sigma_{i=n+1}^{\infty} M\|h_i\|_1 < \frac{1}{2}$. Set $\epsilon = 1/2Mn$, $\delta = \epsilon/(2 \max_{1 \le i \le n}\{1 + \|h_i\|_1\})$ and find a compact set $K \subset G$ so large that the integral of each $|h_i|$ over the complement of K is less than δ. $W = \{\gamma \in \Gamma: |1 - (x, \gamma)| < \delta,\ x \in K\}$ is an open neighborhood of 0 in Γ. Since $0 \in \partial Z(J)$, we may choose a $\gamma_0 \in W \cap Z(J)^c$, and then a compact neighborhood V of 0 so that $V \subset W$ and $\gamma_0 + V - V \subset W \cap Z(J)^c$. Choose $s, t \in L^2(G)$ whose Plancherel transforms are χ_V and $\chi_{\gamma_0 - V}$ respectively, and define $k(x) = s(x)t(x)/m(V)$ where m denotes Haar measure on Γ. Then [17, 2.6.1] $\hat{k}$ is 0 off the compact set $\gamma_0 + V - V$, so that $k \in K_{Z(J)} \subset J$, and $1 = \hat{k}(\gamma_0) \le \|k\|_1 \le \{m(\gamma_0 - V)/m(V)\}^{\frac{1}{2}} = 1$, so that $\|k\|_1 = 1 = \hat{k}(\gamma_0)$. Further, since $\gamma_0 - V$ and V are contained in W, and the transform of each h_i vanishes at 0, the computations of [17, p. 50] yield $\|h_i * k\|_1 < \epsilon$ for $i = 1, 2, \cdots, n$.

Let $\alpha_i = |\hat{h}_i(\gamma_0)|/\hat{h}_i(\gamma_0)$ when $\hat{h}_i(\gamma_0) \ne 0$, 1 otherwise, and let $t_i = \alpha_i h_i$. Since α_i has modulus 1, the series $\Sigma_{i=1}^{\infty} t_i * k$ is absolutely convergent and defines an element g in the closed ideal J. $(*)$ yields $f_1, \cdots, f_p \in J$ with $g = \Sigma_{i=1}^{p} h_i * f_i$, and setting $g_i = f_i/\alpha_i$ we obtain $g = \Sigma_{i=1}^{p} t_i * g_i$ and $\|g_i\|_1 \le M\|g\|_1$. In particular,

$$\sum_{i=1}^{\infty} |\hat{h}_i(\gamma_0)| = \sum_{i=1}^{\infty} \hat{t}_i(\gamma_0)\hat{k}(\gamma_0) = \hat{g}(\gamma_0)$$

$$= \sum_{i=1}^{p} |\hat{h}_i(\gamma_0)|\,\hat{g}_i(\gamma_0) \le \sum_{i=1}^{p} |\hat{h}_i(\gamma_0)|\,|\hat{g}_i(\gamma_0)|.$$

But then if $|\hat{g}_i(\gamma_0)| < 1$ for each $i = 1, 2, \cdots, p$, this inequality forces $\hat{h}_i(\gamma_0) = 0$ for all i. Since J is the vector space sum of the $h_i * J$'s, this implies $\gamma_0 \in Z(J)$, a contradiction. We conclude that $|\hat{g}_j(\gamma_0)| \ge 1$ for some j. Thus

$$1 \le \|\hat{g}_j\|_\infty \le \|g_j\|_1 \le M\|g\|_1$$

$$\le \sum_{i=1}^{n} M|\alpha_i|\,\|h_i * k\|_1 + \sum_{i=n+1}^{\infty} M|\alpha_i|\,\|h_i\|_1\,\|k\|_1 < Mn\epsilon + \tfrac{1}{2} = 1.$$

From this contradiction we conclude $Z(J)$ is open-closed.

Corollary 3.2. *If Γ is connected, no proper nonzero closed ideal of $L^1(G)$ can be countably generated.*

Corollary 3.3. *If $E \subset \Gamma$ is a set of nonsynthesis, then no closed ideal of $L^1(G)$ whose zero set is E can be countably generated.*

Proof. If J were such an ideal, $Z(J) = E$ would be open-closed; hence $F_E = I_E$, and by definition, E would be a set of synthesis.

Malliavin's theorem [17, 7.6.1] yields closed sets E of nonsysthesis in every nondiscrete LCA group, and Helson's result [17, 7.7.2] then yields ascending and descending chains of closed ideals in $L^1(G)$ whose zero sets are E. Whereas 3.3 implies that none of these can be generated algebraically by even an infinite set, they may be the *closure* of even principal ideals (cf. [17, 7.6.3, p. 174]).

The method of 3.1 can be used to yield more explicit information about the countably generated ideals in group algebras. We begin with

Theorem 3.4. *$L^1(G)$ is a countably generated ideal of itself iff G is discrete.*

Proof. Certainly if G is discrete, $L^1(G) = M(G)$ is generated by its identity. Suppose on the other hand that G is nondiscrete, but $L^1(G)$ is the smallest ideal of itself containing some sequence $\{w_n\}$. Since $L^1(G)$ has approximate identity, we may choose $h_i \in L^1(G)$ so that $\|h_i\|_1 \leq 1/2^i$ and $w_i \in h_i * L^1(G)$ (1.5). Thus $L^1(G)$ is the vector space sum of the $h_i * L^1(G)$'s, and, as before, the Dieudonné-Schwartz theorem yields a constant $M > 0$ such that given $g \in L^1(G)$ there are $g_1, \cdots, g_p \in L^1(G)$ satisfying (*). Choose n so that $\Sigma_{i=n+1}^{\infty} M\|h_i\|_1 < \frac{1}{2}$, set $\epsilon = 1/2Mn$ and choose a compact set $C \subset \Gamma$ so large that, off C, $|\hat{h}_i| < \epsilon/2$ for each $i = 1, 2, \cdots, n$. Since Γ is not compact, there is some γ_0 in the complement of C. Because the h_i's generate $L^1(G)$, $\hat{h}_i(\gamma_0) \neq 0$ for at least one i.

Letting $\alpha_i = |\hat{h}_i(\gamma_0)|/\hat{h}_i(\gamma_0)$ when $\hat{h}_i(\gamma_0) \neq 0$, 1 otherwise, and setting $s_i(x) = \overline{\alpha_i \gamma_0(x)}\, h_i(x)$, we observe that for each $g \in L^1(G)$ there are $g_1, \cdots, g_p \in L^1(G)$ such that $g = \Sigma_{i=1}^{p} s_i * g_i$ and $\|g_i\|_1 \leq M\|g\|_1$. For if $g_1', \ldots, g_p' \in L^1(G)$ are chosen to satisfy (*) for $\gamma_0 g$, then

$$g = \sum_{i=1}^{p} \overline{\gamma}_0(\alpha_i h_i * g_i'/\alpha_i) = \sum_{i=1}^{p} s_i * \overline{\gamma}_0 g_i'/\alpha_i,$$

and, setting $g_i = \overline{\gamma}_0 g_i'/\alpha_i$, we have $\|g_i\|_1 = \|\overline{\gamma}_0 g_i'\|_1/|\alpha_i| \leq M\|g\|_1$.

Let $\delta = \epsilon/(4 \max_{1 \leq i \leq n} \{1 + \|s_i\|_1\})$ and find a compact set $K \subset G$ so that the integral of each $|s_i|$ $(1 \leq i \leq n)$ over K^c is less than δ. Let $W = \{\gamma \in \Gamma: |1 - (x, \gamma)| < \delta,\ x \in K\}$ and choose a compact neighborhood V of 0 in Γ such that $V - V \subset W$. Let $s, t \in L^2(G)$ be functions whose Plancherel transforms are χ_V and χ_{-V} respectively, and set $k(x) = s(x)t(x)/(m(V))$. $\hat{k}(0) = 1 = \|k\|_1$ [17, 2.6.1], and we have $\|s_i * k\|_1 < \epsilon$ for $i = 1, \cdots, n$. For

$$s_i * k(x) = \int_G s_i(y)\{k(x-y) - k(x)\}\,dy + k(x)\hat{s}_i(0),$$

so that

$$\|s_i * k\|_1 \leq \int_G |s_i(y)|\, \|k_y - k\|_1\, dy + |\hat{s}_i(0)|\, \|k\|_1.$$

Since $-V$, $V \subset W$ and $\|k\|_1 = 1$, the computations of [17, p. 50] show that the right-hand side is less that $\epsilon/2 + |\hat{s}_i(0)| = \epsilon/2 + |\hat{h}_i(y_0)| < \epsilon$.

Let $g = \Sigma_{i=1}^{\infty} s_i * k$ and choose $g_1, \cdots, g_p \in L^1(G)$ so that $\|g_i\|_1 \leq M\|g\|_1$ and $g = \Sigma_{i=1}^{p} s_i * g_i$. Since

$$\sum_{i=1}^{\infty} |\hat{h}_i(y_0)| = \sum_{i=1}^{\infty} \hat{s}_i(0)\hat{k}(0)$$

$$= \sum_{i=1}^{p} |\hat{h}_i(y_0)|\,\hat{g}_i(0) \leq \sum_{i=1}^{p} |\hat{h}_i(y_0)||\hat{g}_i(0)|,$$

and not all the $\hat{h}_i(y_0)$'s are zero, $|\hat{g}_j(0)| \geq 1$ for at least one j. Hence

$$1 \leq \|g_j\|_1 \leq \sum_{i=1}^{n} M\|s_i * k\|_1 + \sum_{i=n+1}^{\infty} M|\alpha_i|\,\|\bar{y}_0 h_i\|_1\,\|k\|_1 < Mn\epsilon + \tfrac{1}{2} = 1.$$

This contradiction completes the proof.

Corollary 3.5. *If Γ is a noncompact connected group,* 0 *is the only closed, countably generated ideal of $L^1(G)$.*

However, nontrivial countably generated ideals in $L^1(G)$ can exist.

Proposition 3.6. *Let G and H be LCA groups with duals Γ and Ω respectively. Suppose G is compact and Ω is connected. Then* (i) *if H is nondiscrete,* 0 *is the only closed, countably generated ideal of $L^1(G \times H)$, whereas* (ii) *if H is discrete, a closed ideal J in $L^1(G \times H)$ is finitely generated iff $Z(J) = E \times \Omega$ where E is a cofinite subset of Γ.*

Proof. Let J be a closed, countably generated ideal of $L^1(G \times H)$. Then $Z(J)$ is an open-closed S-set in $\Gamma \times \Omega$ (3.1), so $Z(J) = E \times \Omega$ and $J = I_{E\times\Omega}$, $E \subset \Gamma$. As before there is a sequence $\{h_n\}$ on $L^1(G \times H)$ with $\|h_n\|_1 \leq 1/2^n$, so that J is the sum of its subspaces $\{h_n * J\}$; in fact, there is a constant $M > 0$ so that given $g \in J$, there are $g_1, \cdots, g_p \in J$ satisfying $(*)$. Find $n > 0$ so that $\Sigma_{i=n+1}^{\infty} M\|h_i\|_1 < \tfrac{1}{2}$, let $\epsilon = 1/2Mn$ and choose $C \subset \Gamma \times \Omega$ compact so that $|\hat{h}_i(y, \eta)| < \epsilon/2$ off C for each $i = 1, 2, \cdots, n$.

(i) If $E = \Gamma$, we are done. If $y_0 \in E^c$, then since Ω is not compact, $(y_0, \eta_0) \notin C$ for some $\eta_0 \in \Omega$. Let $\alpha_i = |\hat{h}_i(y_0, \eta_0)|/\overline{\hat{h}_i(y_0, \eta_0)}$ if $\hat{h}_i(y_0, \eta_0) \neq 0$, 1 otherwise. Define $s_i \in L^1(G \times H)$ by $s_i(x, y) = \alpha_i \overline{\eta_0(y)}\, h_i(x, y)$ and observe that, as before, given $g \in J$ there are $g_1, \cdots, g_p \in J = I_{E\times\Omega}$ such that $g = \Sigma_{i=1}^{p} s_i * g_i$ and $\|g_i\|_1 \leq M\|g\|_1$. If we define $A_i \in L^1(H)$ by

$$A_i(w) = \int_G s_i(x, w)\overline{y_0(x)}\, dx,$$

then since $|\hat{A}_i(0)| = |\hat{s}_i(\gamma_0, 0)| = |\hat{b}_i(\gamma_0, \eta_0)| < \epsilon/2$ for $i = 1, 2, \cdots, n$, we may construct $k \in L^1(H)$ so that $\hat{k}(0) = 1 = \|k\|_1$ and $\|A_i * k\|_1 < \epsilon$ for $1 \leq i \leq n$ (cf. 3.4).

Since the transform of $\gamma_0 k: (x, y) \to \gamma_0(x) k(y)$ vanishes on $E \times \Omega$, $g = \sum_{i=1}^{\infty} s_i * \gamma_0 k \in J$, and hence for some $g_i \in J$ with $\|g_i\|_1 \leq M\|g\|_1$, we have $g = \sum_{i=1}^{p} s_i * g_i$. But then

$$\sum_{i=1}^{\infty} |\hat{b}_i(\gamma_0, \eta_0)| = \sum_{i=1}^{\infty} \hat{s}_i(\gamma_0, 0)\widehat{\gamma_0 k}(\gamma_0, 0) \leq \sum_{i=1}^{p} |\hat{b}_i(\gamma_0, \eta_0)|\, |\hat{g}_i(\gamma_0, 0)|,$$

and because at least one $\hat{b}_i(\gamma_0, \eta_0)$ is nonzero, this means $|\hat{g}_j(\gamma_0, 0)| \geq 1$ for some j. Since $s_i * \gamma_0 k(x, y) = \gamma_0(x) A_i * k(y)$, $\|s_i * \gamma_0 k\|_1 = \|A_i * k\|_1$, and we reach the following absurdity:

$$\begin{aligned} 1 &\leq \|g_j\|_1 \leq M\|g\|_1 \\ &\leq \sum_{i=1}^{n} M\|A_i * k\|_1 + \sum_{i=n+1}^{\infty} M|\alpha_i|\,\|b_i\|_1\|\gamma_0 k\|_1 < Mn\epsilon + \tfrac{1}{2} = 1. \end{aligned}$$

(ii) Since Γ is discrete, the compact set C is contained in a product $F \times \Omega$, $F \subset \Gamma$ finite. But then if the complement of E in Γ is infinite, there is some $\gamma_0 \notin F \cup E$. Taking any $\eta_0 \in \Omega$, we have $(\gamma_0, \eta_0) \notin C \cup Z(J)$, and with the argument used above we obtain the required contradiction.

Of course, if the complement of E is a finite set $\{\gamma_1, \cdots, \gamma_n\}$, then since $L^1(H)$ has identity δ_0, $g(x, y) = \sum_{i=1}^{n} \gamma_i(x)\delta_0(y)$ is an $L^1(G \times H)$ function whose transform is the characteristic function of the complement of $E \times \Omega$. Since $\partial E \times \Omega = \emptyset$, $J = I_{E\times\Omega}$ is the only closed ideal of $L^1(G \times H)$ with zero set $E \times \Omega$, and clearly g generates this ideal.

Theorem 3.7. *Let G be a compactly generated LCA group. A nonzero closed, ideal J of $L^1(G)$ is countably generated iff G is $Z^n \times K$, K a compact group, and $J = I_{T^n\times E}$ where E is a cofinite subset of the dual of K.*

Proof. G is a group of the form $R^m \times Z^n \times K$, K compact [12, II 9.8, p. 90]. If $L^1(G)$ has a nonzero closed, countably generated ideal J, then 3.6(i) applied to $H = R^m \times Z^n$ shows that $m = 0$. So G is $Z^n \times K$, and 3.6(ii) applied to $H = Z^n$ yields $Z(J) = T^n \times E$, a set of synthesis; whence $J = I_{T^n\times E}$. The converse also follows from 3.6(ii).

In fact, the proof of 3.6 reveals that for a compactly generated group G, a closed ideal in $L^1(G)$ will either be principal or else require uncountably many generators. Thus, in $L^1(T)$, say, no maximal ideal I_{γ_0} can be generated by the characters different from γ_0. More generally, we have

Corollary 3.8. *For any LCA group G, $L^1(G)$ has a countably generated maximal ideal if and only if G is finite.*

Proof. If M is such an ideal, M is regular (cf. 1.6) so that $Z(M)$ is an isolated point in Γ (3.1). By duality G is compact, so that 3.7 implies that the complement of the singleton $Z(M)$ is finite. Thus Γ, and hence G, is finite. Of course if G is finite, every ideal of $L^1(G)$ is principal.

Simplification of these methods yields an improvement of 2.6 for the class of function algebras.

Theorem 3.9. *If J is a closed, countably generated ideal in a function algebra A, then $Z(J) \cap \Gamma_A$ is open-closed in Γ_A.*

Proof. J is the sum of spaces $\{h_i A\}$ where $h_i \in J$ has norm less than $1/2^i$. As before, the Dieudonné-Schwartz theorem applied to the map $\bigoplus_1^\infty A \to J$ given by $\{g_i\} \to \Sigma_i h_i g_i$ yields a constant $M > 0$ so that, given $g \in J$, there are $g_1, \cdots, g_p \in A$ with $g = \Sigma_{i=1}^p g_i h_i$ and $\|g_i\|_\infty \leq M\|g\|_\infty$. Choose n so that

$$\sum_{i=n+1}^{\infty} M\|\hat{h}_i\|_\infty < ½.$$

If $\epsilon = 1/2Mn$, $U = \bigcap_{i=1}^n \hat{h}_i^{-1}(B(0,\epsilon))$ is an open neighborhood of $Z(J)$ in $\mathfrak{M}_A$. If $Z(J) \cap \Gamma_A$ is not open in Γ_A, $V = \Gamma_A \cap U - Z(J) = U \cap \Gamma_A - Z(J)$ is a nonempty open set in Γ_A. Since ∂A is dense in Γ_A, we may choose some $x_0 \in \partial A \cap V$. x_0 is a strong boundary point for $\hat{A}$ [2, 2.3.4]: there is a $k \in A$ with $k(x_0) = 1 = \|\hat{k}\|_\infty$ and $|\hat{k}| < 1$ off U. If k assumes its maximum modulus on the compact set $M_A - U$ at a, choose m so large that $|k(a)|^m < \epsilon/(1 + \max_{1 \leq i \leq n}\|\hat{h}_i\|_\infty)$. Notice $\|\hat{k}^m\hat{h}_i\|_\infty < \epsilon$ for $i = 1, \cdots, n$. For if $b \in U$, $|\hat{k}(b)^m\hat{h}_i(b)| \leq \|\hat{k}\|_\infty^m |\hat{h}_i(b)| < \epsilon$, and if $b \notin U$, $|\hat{k}(b)^m\hat{h}_i(b)| \leq \|\hat{h}_i\|_\infty|\hat{k}(a)|^m < \epsilon$.

Let $s_i = \alpha_i h_i$ where $\alpha_i = |h_i(x_0)|/h_i(x_0)$ if $h_i(x_0) \neq 0$, 1 otherwise. Since α_i has modulus 1, we may choose $g_1, \cdots, g_p \in J$ so that $\Sigma_{i=1}^p g_i s_i = \Sigma_{i=1}^\infty k^m s_i = g \in J$ and $\|g_i\|_\infty \leq M\|g\|_\infty$. Then

$$\sum_{i=1}^{\infty} |h_i(x_0)| = \sum_{i=1}^{\infty} k(x_0)^m s_i(x_0) = \sum_{i=1}^{p} g_i(x_0)\, s_i(x_0) \leq \sum_{i=1}^{p} |g_i(x_0)|\, |h_i(x_0)|.$$

Since $x_0 \notin Z(J)$, we conclude that $|g_j(x_0)| \geq 1$ for at least one j, and finally that

$$\begin{aligned} 1 &\leq \|g_j\|_\infty \leq M\|\hat{g}\|_\infty \\ &\leq \sum_{i=1}^{n} M|\alpha_i|\, \|\hat{k}^m\hat{h}_i\|_\infty + \sum_{i=n+1}^{\infty} M|\alpha_i|\, \|\hat{h}_i\|_\infty \|\hat{k}\|_\infty^m < Mn\epsilon + ½ = 1. \end{aligned}$$

This contradiction completes the proof.

3.9 has an obvious corollary when Γ_A is connected. In particular we have

Corollary 3.10. *In the disc algebra, a nonzero closed ideal is countably generated iff its zero set lies in the open unit disc.*

Proof. Plainly 3.9 makes the zero set condition necessary. On the other hand, if the zero set of a nonzero ideal J is contained in the open disc D, then $Z(J)$ is finite. For otherwise $Z(J)$ would have a cluster point $p \in Z(J) \subset D$, and the identity principle for analytic functions would force $J = 0$. If $Z(J)$ is empty, J is generated by 1. If $Z(J) = \{p_1, \cdots, p_n\}$, a compactness argument (similar to [9, p. 235]) shows that J is the principal ideal generated by $\Pi_{i=1}^n (z - p_i)^{k_i}$ where

$$k_i = \inf \{m\colon \text{ some } f \in J \text{ has a zero of order } m \text{ at } p_i\}.$$

3.9 has application to other familar examples. Consider for instance a Swiss cheese space K formed by deleting from the closed unit disc Δ countably many open discs $\{\Delta_j\}$, with centers $\{b_j\}$ and radii $\{r_j\}$ whose closures are pairwise disjoint, in such a way that K has no interior in the plane. As the intersection of a descending chain of compact, connected sets in the plane, K is connected [10, 16.14, p. 246]. The closure $R(K)$ in $C(K)$ of the rational functions with poles off K is a function algebra on its maximal ideal space K whose Šilov boundary is $\partial K = K$ [8, p. 27]. Thus 3.9 implies

Corollary 3.11. *If K is a Swiss cheese,* 0 *and $R(K)$ are the only countably generated, closed ideals of $R(K)$.*

If the radii $\{r_j\}$ are choosen so that $\Sigma r_j < \infty$, $R(K) \neq C(K)$ and the result is nontrivial (cf. [5, Corollary 3, p. 177]). Further, Sidney observes [19, p. 148] that if $0 \in K$ and $\Sigma r_j/(|b_j| - r_j)^m < \infty$ for $m = 1, 2, \cdots$, then the closed powers $\overline{I_0^n}$ form a strictly decreasing chain of closed, primary ideals in $R(K)$. In view of 3.11, none of them can be countably generated.

We can also use 3.9 to obtain purely algebraic information.

Corollary 3.12. *If A is a regular function algebra, then a closed ideal of A which is countably generated is actually principal.*

Proof. If J is such an ideal, $Z(J)$ is open-closed in $\mathfrak{M}_A = \Gamma_A = X$. By Šilov's idempotent theorem [8, p. 88], there is some $e \in A$ vanishing on $Z(J)$ which is identically 1 elsewhere. By regularity, $e \in J$ and hence $eA = J$.

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