TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 169, July 1972

LATTICE-ORDERED INJECTIVE HULLS

BY

STUART A. STEINBERG

ABSTRACT. It is well known that the injective hull of a lattice-ordered group (l-group) M can be given a lattice order in a unique way so that it becomes an l-group extension of M. This is not the case for an arbitrary f-module over a partially ordered ring (po-ring). The fact that it is the case for any l-group is used extensively to get deep theorems in the theory of l-groups. For instance, it is used in the proof of the Hahn-embedding theorem and in the characterization of $\aleph_\alpha$-injective l-groups.

In this paper we give a necessary and sufficient condition on the injective hull of a torsion-free f-module M (over a directed essentially positive po-ring) for it to be made into an f-module extension of M (in a unique way). An f-module is called an i-f-module if its injective hull can be made into an f-module extension. The class of torsion-free i-f-modules is closed under the formation of products, sums, and Hahn products of strict f-modules. Also, an l-submodule and a torsion-free homomorphic image of a torsion-free i-f-module are i-f-modules.

Let R be an f-ring with zero right singular ideal whose Boolean algebra of polars is atomic. We show that R is a qf-ring (i.e., R_R is an i-f-module) if and only if each torsion-free R-f-module is an i-f-module. There are no injectives in the category of torsion-free R-f-modules, but there are $\aleph_\alpha$-injectives. These may be characterized as the f-modules that are injective R-modules and $\aleph_\alpha$-injective l-groups. In addition, each torsion-free f-module over R can be embedded in a Hahn product of l-simple $Q(R)$-f-modules. We note, too, that a totally ordered domain has an i-f-module if and only if it is a right Ore domain.

1. **Introduction.** Our methods and characterization of torsion-free i-f-modules are modelled after Anderson's work on the maximal right quotient ring of an f-ring [1]. Throughout this paper $\mathbf{Z}$ and $\mathbf{Q}$ will denote the totally ordered rings of integers and rational numbers, respectively. If R is a po-ring, then R_* will denote the po-ring obtained by freely adjoining $\mathbf{Z}$ to R.

We begin by recalling the requisite module theory. All modules will be right modules. An R-module E is *injective* if for every pair of R-homomorphisms $f: K \longrightarrow E$ and $g: K \longrightarrow L$, where g is monic, there exists an R-homomorphism $h: L \longrightarrow E$ such that $hg = f$. A submodule N of the R-module M is an *essential submodule* (and M is an *essential extension of* N) if $N \cap K \neq 0$ for every nonzero submodule K of M. Every module M has a maximal essential extension $E = E(M_R)$ which is unique up to an isomorphism over M. $E(M)$ is the smallest injective module containing M, and is called the *injective hull of* M. If F is an essential extension of M and

Presented to the Society, February 11, 1971; received by the editors February 10, 1971.
AMS 1970 *subject classifications*. Primary 06A70; Secondary 16A52, 06A60.
Key words and phrases. f-module, lattice-ordered injective hull, relative injectives, torsion-free f-module, Hahn embedding theorem.

it is injective, then $F = E(M)$ ([9] and [10]).

Let M^∇ be the set of essential submodules of M. Then M^∇ is a dual ideal in the lattice of submodules of M. If N is a submodule of M and T is a subset of M, then $(N:T) = \{r \in R\colon Tr \subseteq N\}$ is a right ideal of R. If $N \in M^\nabla$ and $x \in M$, then $(N:x) \in R^\nabla$. For a subset T of M we will sometimes write $r(T)$ for $(0\colon T)$. In [17] Johnson has defined the *singular submodule of* M by $Z(M) = \{x \in M\colon r(x) \in R^\nabla\}$. M is called a *torsion-free R-module* if $Z(M) = 0$. Note that when R is a commutative integral domain, $Z(M)$ is just the torsion submodule of M.

More generally, if N is a submodule of M, then the *closure of* N *in* M is defined by $\mathrm{Cl}_M N = \{x \in M\colon xD \subseteq N$ for some $D \in R^\nabla\}$. When no confusion is likely we will write $\mathrm{Cl}\, N$ for $\mathrm{Cl}_M N$. $\mathrm{Cl}\, N$ is, of course, a submodule of M containing N. In fact, $\mathrm{Cl}\, N/N = Z(M/N)$. N is said to be *closed in* M if $\mathrm{Cl}\, N = N$. In general, $\mathrm{Cl}\,\mathrm{Cl}\, N$ is the smallest closed submodule of M containing N [13]. The intersection of a family of closed submodules of M is clearly closed. Thus $C_r(M)$, the set of closed submodules of M, is a complete lattice with greatest lower bound being intersection.

If $Z(M) = 0$, then $\mathrm{Cl}\, N$ is the largest essential extension of N contained in M. In particular, when $Z(M) = 0$, every submodule of M has a unique injective hull contained in $E(M)$. If K is any essential extension of M (and $Z(M) = 0$), then the map $C_r(K) \longrightarrow C_r(M)$, given by $N \longrightarrow N \cap M$, is a lattice isomorphism. The inverse map sends N to its closure in K. For proof of these facts see [10].

An *f-module over the po-ring* R is a lattice-ordered R-module (l-module) that is embeddable in a product of a family of totally ordered R-modules. For the basic properties of f-modules see [20].

This paper is based on a portion of the author's dissertation written at the University of Illinois at Urbana-Champaign under the direction of Professor Elliot Weinberg.

2. **Torsion-free f-modules.** A po-ring R is called (*right*) *essentially positive* if $D \in R^\nabla$ implies $D^+R_* \in R^\nabla$. Since $D^+R_* = D^+ - D^+ + (D^+ - D^+)R$ is the right ideal of R generated by D^+, a directed po-ring is essentially positive if and only if each of its essential right ideals contains a directed essential right ideal. In order to characterize the torsion-free i-f-modules over R it will be necessary to assume that R is directed and essentially positive. Some examples of essentially positive po-rings are

(1) any totally ordered ring;

(2) any right quotient ring of a torsion-free essentially positive po-ring, in its canonical order;

(3) the n-by-n matrix ring over a totally ordered right Ore domain, ordered coordinatewise.

A word about (2) is in order. If R is a ring with $Z(R_R) = 0$, then a ring S is a right quotient ring of R if S_R is an essential extension of R_R. It is known that $E(R_R)$ can be made into a ring extension of R, so $Q(R) = E(R_R)$ is the maximal right quotient ring of R ([17], [22]). Now suppose that R is an essentially positive po-ring, and let S be a right quotient ring of R. Then

$$S^+ = \{s \in S: sD^+ \subseteq R^+ \text{ for some } D \in R^\nabla\}$$

is a partial order of the ring S (Lemma 3.1 implies that S_R is a po-module extension of R_R; but if $a, b \in S^+$, $aD^+ \subseteq R^+$, and $C = \{r \in R: br \in D\}$, then $abC^+ \subseteq aD^+ \subseteq R^+$; also see [1, Theorem 2.3]), called the canonical order of S. Note that $S^+ \cap R$ can contain R^+ properly, and, in fact, (S, S^+) is a po-ring extension of (R, R^+) if and only if R is *essentially semiclosed*, i.e. $rD^+ \subseteq R^+$ for $D \in R^\nabla$ implies $r \in R^+$. Let D be an essential right ideal of S. Then D_R is essential in S_R. (Suppose that $D \cap X = 0$ for some $X_R \subseteq S_R$. If $0 \neq d \in XS$, $d = \Sigma_{i=1}^n x_i s_i$ with $x_i s_i \neq 0$, then there is an element $t \in R$ such that $dt \neq 0$ and $s_i t \in R$. Thus $0 \neq dt \in D \cap X = 0$; so $D \cap XS = 0$ and $X = 0$.) Since $D \cap R \in R^\nabla$, $(D \cap R)^+ R_* \in R^\nabla$. But then $(D \cap R)^+ R_* S_* \in S^\nabla$, and since $D^+ S_* \supseteq (D \cap R)^+ R_* S_*$, $D^+ S_* \in S^\nabla$. Thus S is essentially positive.

Note, by Proposition 3.2, that a torsion-free right f-ring R (i.e., R_R is an f-module) is essentially semiclosed. Note also that any torsion-free f-ring is essentially positive [1, Lemma 2.1]. That (3) is true follows easily from (2). For if D is a totally ordered right Ore domain, then (R, R^+) is an essentially positive right f-ring with $Z(R_R) = 0$ and right quotient ring (D_n, D_n^+), where $R = \{[a_{ij}] \in D_n: a_{ij} = 0 \text{ for } j > 1\}$, $R^+ = \{[a_{ij}]: a_{ij} \geq 0\}$, and $D_n^+ = \{[a_{ij}] \in D_n: a_{ij} \geq 0\}$. That (D_n, D_n^+) is essentially positive is actually useless for our purposes, since it has no nontrivial f-modules [20].

An l-module M is called *distributive* if the map induced on M by each $r \in R^+$ is a lattice homomorphism. If M_R is distributive, and $x \in M$ and $xR = 0$ implies $x = 0$, then M is an f-module [20].

Lemma 2.1. *Let K be a po-module over the essentially positive po-ring R, and let M be an essential submodule of K.*

(a) *If N is a convex submodule of M, then $\mathrm{Cl}_K N$ is a convex submodule of K.*

(b) *If M is a distributive l-module and N is an l-submodule of M, then $\mathrm{Cl}_M N$ is an l-submodule of M.*

(c) *Suppose that K is a distributive l-module, and M is an l-submodule of K. If N is a prime submodule of M, then $\mathrm{Cl}_K N$ is a prime submodule of K.*

(d) *If M is archimedean and x and y are elements of K such that $nx \leq y$ for all $n \in \mathbf{Z}$, then $x \in Z(K)$. Thus K is archimedean if M is torsion-free and archimedean.*

Proof. Suppose that N is a convex submodule of M and $0 \leq x \leq y$ where $x \in K$

and $y \in \mathrm{Cl}_K N$. Then there exists $D \in R^\nabla$ such that $yD \subseteq N$ and $xD \subseteq M$. If $d \in D^+$, then $0 \leq xd \leq yd$; so $xD^+ \subseteq N$. Thus $x \in \mathrm{Cl}_K N$, and we have (a).

For (b), suppose $x \in \mathrm{Cl}_M N$ and $D \in R^\nabla$ such that $xD \subseteq N$. Then $x^+d = (xd)^+ \in N$ for all $d \in D^+$. Thus $x^+D^+ \subseteq N$, so $x^+ \in \mathrm{Cl}_M N$.

For (c), suppose that N is a prime submodule of M. By (a) and (b), $\mathrm{Cl}_K N$ is a convex l-submodule of K. Suppose that x and y are disjoint elements of K, and $x \notin \mathrm{Cl}_K N$. There exists $D \in R^\nabla$ such that $xD \subseteq M$ and $yD \subseteq M$. Since $x \notin \mathrm{Cl}_K N$, there exists $d \in D^+$ such that $xd \notin N$. If $e \in D^+$, then $xd \wedge ye = 0$; so $ye \in N$ since N is prime. Thus $yD^+ \subseteq N$, i.e. $y \in \mathrm{Cl}_K N$.

Finally, if $D = (M: x) \cap (M: y)$ where $nx \leq y$ for all $n \in \mathbf{Z}$, then $nxd \leq yd$ for all $d \in D^+$. Since M is archimedean, $xd = 0$. Therefore, $xD^+ = 0$, and $x \in Z(K)$.

Corollary 2.2. *Let M be a distributive l-module over an essentially positive po-ring. Then the closed convex l-submodules of M form a complete sublattice of $C_r(M)$.*

Proof. If $\{N_\alpha: \alpha \in A\}$ is a family of closed convex l-submodules of M, then clearly $\bigcap\{N_\alpha: \alpha \in A\}$ is a closed convex l-submodule. By 2.1, so is $\mathrm{Cl}\,\mathrm{Cl}\,\Sigma\, N_\alpha$.

If M is an l-module over a directed po-ring, then whether or not M is an f-module depends only on $P(M)$, the Boolean algebra of polars of M [20]. In light of this fact, the following proposition is not surprising. $X^\perp$ (or $X^{\perp M}$) will denote the polar of a subset X of M.

Proposition 2.3. *An f-module M over an essentially positive directed po-ring R is torsion-free if and only if each of its (principal) polars is a closed submodule.*

Proof. Suppose that M is torsion-free, and let N be a polar of M. If $xD \subseteq N$ for some $D \in R^\nabla$, then $|xd| \wedge |y| = 0$ for all $d \in D^+$ and for all $y \in N^\perp$. So $(|x| \wedge |y|)D^+ = 0$, and hence $x \in N^{\perp\perp} = N$.

The converse is obvious.

Because of 2.2 and 2.3 one might suspect that $P(M)$ is a sublattice of $C_r(M)$ when M is torsion-free. However, if $R = \mathbf{Q}$, then each submodule of an l-module is closed, but the sum of two polars need not be a polar.

An f-module whose Boolean algebra of polars is atomic will be called *irredundant*. An irredundant f-module M over a directed po-ring is an irredundant subdirect product of totally ordered modules, i.e. $M \subseteq \Pi M_\alpha$, where each M_α is totally ordered, $M \cap M_\alpha \neq 0$, and the set $\{M_\alpha\}$ consists of those homomorphic images of M whose kernels are the maximal polars of M ([14, p. 40] and [20]).

Corollary 2.4. *An irredundant f-module over an essentially positive directed po-ring is torsion-free if and only if it is a subdirect product of totally ordered torsion-free modules.*

Without the irredundancy hypothesis, 2.4 is false (see 2.7). By an *essential l-submodule* of an l-module we shall mean an l-submodule which is also an essential R-submodule.

Proposition 2.5. *Let K be a torsion-free f-module over the essentially positive directed po-ring R, and suppose that M is an essential l-submodule of K.*

(a) *If N is a convex l-submodule of M, then* $\mathrm{Cl}_K(N^{\perp M}) = [\mathrm{Cl}_K(N)]^{\perp K}$.

(b) *If N is a convex l-submodule of K, then* $N^{\perp K} \cap M = (N \cap M)^{\perp M}$.

Proof. (a) Let $K_1 = \mathrm{Cl}_K(N^{\perp M})$ and $K_2 = \mathrm{Cl}_K N$. Recall that N is an essential submodule of K_2, since M is torsion-free. Thus $K_1 \cap K_2 = 0$ since $N \cap N^{\perp M} = 0$. By 2.1, K_1 and K_2 are convex l-submodules of K. Hence $K_1 \subseteq K_2^{\perp K}$. Let $x \in K_2^{\perp K}$ and let $D = (M : x)$. Then $xD \subseteq N^{\perp M} \subseteq K_1$. Thus, since D is an essential right ideal and K_1 is closed in K, $x \in K_1$. So $K_1 = K_2^{\perp K}$.

(b) It is clear that $N^{\perp K} \cap M \subseteq (N \cap M)^{\perp M}$. Let x be a positive element of $(N \cap M)^{\perp M}$, and let y be a positive element of N. Since N is an essential extension of $N \cap M$, $yD \subseteq N \cap M$ for some essential right ideal D. If $d \in D^+$, then $x \wedge yd = 0$, so $(x \wedge y)D^+ = 0$. Since M is torsion-free, $x \in N^{\perp K}$. Thus $N^{\perp K} \cap M = (N \cap M)^{\perp M}$.

Corollary 2.6. *Let K be a torsion-free f-module over the essentially positive directed po-ring R, and suppose that M is an essential l-submodule of K. Then the map $N \longrightarrow \mathrm{Cl}_K N$ is an isomorphism between the Boolean algebras of polars of M and K. Its inverse is the map $N \longrightarrow N \cap M$.*

Proof. We already know that these correspondences are one-to-one between $C_r(M)$ and $C_r(K)$. By 2.5 they take polars to polars. In fact, if $N \in P(K)$, then $N = \mathrm{Cl}_K(N \cap M) = \mathrm{Cl}_K(N^{\perp K \perp K} \cap M) = \mathrm{Cl}_K[(N^{\perp K} \cap M)^{\perp M}]$. Thus the map $N \longrightarrow \mathrm{Cl}_K N$ between $P(M)$ and $P(K)$ is an order isomorphism that is onto. Hence it is an isomorphism of Boolean algebras.

A *qf-ring* is an f-ring R whose maximal right quotient ring Q is an f-ring extension [1]. If R is a qf-ring with $Z(R) = 0$, then $Q = E(R_R)$ is a (strongly) regular self-injective ring.

Proposition 2.7. *Let R be a semiprime (right) qf-ring with maximal right quotient ring Q. The following are equivalent:*

(a) *R_R is a subdirect product of totally ordered torsion-free modules.*

(b) *Q_R is a subdirect product of totally ordered torsion-free modules.*

(c) *Q_Q is a subdirect product of totally ordered torsion-free modules.*

(d) *The Boolean algebra of polars of Q is atomic.*

(e) *The Boolean algebra of polars of R is atomic.*

(f) *Q is the direct product of a family of totally ordered division rings.*

Proof. (a) implies (b). Let $\{N_\alpha: \alpha \in A\}$ be a collection of closed prime submodules of R_R whose intersection is zero, and let $E_\alpha = E((N_\alpha)_R) \subseteq Q$. By 2.1 each E_α is a closed prime submodule of Q_R. Since $C_r(R)$ and $C_r(Q)$ are isomorphic via the correspondence $N \longrightarrow E(N_R)$, $\bigcap\{E_\alpha: \alpha \in A\} = 0$.

That (b) implies (c) follows from the fact that every closed submodule of Q_R is a right ideal of Q [10, p. 70].

(c) implies (d). Let $\{E_\alpha: \alpha \in A\}$ be a collection of closed prime submodules of Q_Q whose intersection is zero. Since Q_Q is injective, $Q = E_\alpha \oplus F_\alpha$ as Q-modules. Since Q is a regular f-ring, each right ideal is an l-ideal, so the direct sum is a sum of f-rings. Thus each F_α is a totally ordered division ring. The projections onto the F_α induce an isomorphism of Q into the product of the F_α whose image contains the direct sum. Thus $P(Q)$ is atomic.

Finally, (d) and (e) are equivalent by 2.6, (e) implies (a) by 2.4, and (d) is equivalent to (f) by [10, p. 117].

It is, of course, not always the case that $P(R)$ is atomic. For instance, it is known that $C([0, 1])$, the f-ring of real-valued continuous functions defined on the unit interval, has no maximal polars. We do, however, have the following positive results:

Proposition 2.8. *Let R be a directed po-ring which has the property that a right ideal is essential if and only if it contains a positive regular element. If M is a torsion-free f-module over R, then every minimal prime submodule of M is closed. Thus M is a subdirect product of totally ordered torsion-free f-modules.*

Proof. First note that R is essentially positive. Suppose that N is a minimal prime submodule of M. By 2.1, $\mathrm{Cl}\, N$ is a convex l-submodule of M. If $N \subsetneq \mathrm{Cl}\, N$, there exists a positive element $x \in (\mathrm{Cl}\, N)\backslash N$. By hypothesis $(N : x)$ contains a positive regular element d. Since N is a minimal prime subgroup [20], there exists $y \in M^+\backslash N$ such that $y \wedge xd = 0$ [16]. Therefore $(y \wedge x)d = yd \wedge xd = 0$. Since M is torsion-free and dR is an essential right ideal of R, $x \wedge y = 0$. Thus $x \in N$, since N is prime.

A semiprime right Goldie ring can be characterized as a ring R that has a classical right quotient ring which is semisimple and artinian ([12] and [10]). R satisfies and, in fact, is characterized by the following condition: A right ideal I of R is essential if and only if it contains a regular element. If R is also an f-ring, then, since it has no nilpotent elements, it is of the following form: There is a family of totally ordered right Ore domains $\{R_i: i = 1, \cdots, n\}$ with totally ordered quotient division rings $\{D_i: i = 1, \cdots, n\}$ such that R is (isomorphic to) an l-subring of $D_1 \oplus \cdots \oplus D_n$ containing $R_1 \oplus \cdots \oplus R_n$ (see [10] and [1]).

The ring R of 2.8 is a semiprime right Goldie ring. It need not be an f-ring, however, but, for example, need only be directed and have the property that the square of every element is positive. We suspect that R cannot have any nilpotent elements.

A module M over the semiprime ring R is called *l-torsion-free* [18] if $NJ = 0$ for $0 \neq N_R \subseteq M$ and some ideal J of R implies $JK = 0$ for some nonzero ideal K. Note that if M is torsion-free, then it is l-torsion-free. For if $NJ = 0$, then J_R essential in R implies $J = 0$, while $J \cap K = 0$ for $0 \neq K_R \subseteq R$ implies $KJ \subseteq J \cap K = 0$, and hence $JK = 0$.

Theorem 2.9. *Let R be a torsion-free right qf-ring. Then every torsion-free f-module over R is a subdirect product of totally ordered torsion-free modules if and only if the Boolean algebra of polars of R is atomic.*

Proof. Suppose that $P(R)$ is atomic. By 2.7, $Q(R) = \Pi D_\alpha$, where D_α $(\alpha \in A)$ is a totally ordered division ring. Let $R_\alpha = \text{image}(R \longrightarrow D_\alpha)$. Then R is an irredundant subdirect product of the R_α: $R \subseteq \Pi R_\alpha \subseteq \Pi D_\alpha$. Since $Q(\Pi R_\alpha) = \Pi D_\alpha$, $D_\alpha = Q(R_\alpha)$ [22, 2.2]. But then R_α is a right Ore domain [1, 5.2].

Let M be a torsion-free f-module over R, let $N_\alpha = \{x \in M\colon x(R \cap R_\alpha) = 0\}$, and let $P_\alpha = \text{kernel}(R \longrightarrow R_\alpha)$. By [18, 3.7], $MP_\alpha \subseteq N_\alpha$, $M_\alpha = M/N_\alpha$ is an l-torsion-free R-module (R_α-module), and M is a subdirect product of the M_α(as R-modules). It is easily seen that N_α is a convex l-submodule of M, so the subdirect product is one of f-modules. Clearly, M_α is an f-module over R_α.

Let E_α be the R_α-injective hull of M_α. Then E_α is the R-injective hull of M_α and $E(M_R) = E = \Pi E_\alpha$[18, 4.1]. Thus each E_α is a torsion-free R-module. But $R_\alpha = R/P_\alpha$ and $M_\alpha P_\alpha = 0$, so M_α is a torsion-free R_α-module [13, Lemma 3.4].

Let N_α be a minimal prime R-submodule of M_α. Then N_α is a minimal prime R_α-submodule of M_α, and so, by 2.8, is a closed R_α-submodule. Suppose that $xT \subseteq N_\alpha$ for some $x \in M_\alpha$ and some essential right ideal T of R. If $T \subseteq P_\alpha$, then $P_\alpha \in R^\nabla$. This contradicts the fact that R_α is a torsion-free R-module (2.4). Thus $T \subsetneq P_\alpha$, and, since every nonzero right ideal of R_α is essential, $x \in N_\alpha$. So N_α is a closed R-module. Therefore M_α, and hence M, is a subdirect product of totally ordered torsion-free R-modules.

The converse is given by the equivalence of (a) and (e) of 2.7.

Proposition 2.10. *Let M be an essential l-submodule of the f-module K_R. If R is directed, or if it is essentially positive and M is torsion-free, then any weak order unit of M is a weak order unit of K. Thus, K is totally ordered if and only if M is.*

Proof. Suppose that $x \in K$ with $x^\perp \neq 0$. If R is directed, then $x^\perp$ is a submodule of K, so $x^\perp \cap M \neq 0$. Suppose M is torsion-free and R is essentially positive, and let $0 \neq y \in x^\perp$. There exists $d \in (M:y)^+$ such that $yd \neq 0$. So $yd \in x^\perp \cap M$. Thus, in either case, $x^\perp \cap M \neq 0$, and a weak order unit of M is a weak order unit of K. If M is totally ordered and x is a nonzero element of K, then any positive nonzero element of $x^{\perp} \cap M$ is a weak order unit of M. So $x^{\perp} \cap M = 0$. If R is directed this says that $x^{\perp} = 0$, since $x^{\perp}$ is a submodule of K. If R is essentially positive, M is torsion-free, and $0 \neq y \in x^{\perp}$, then $0 \neq yd \in x^{\perp} \cap M$ for some $d \in (M:y)^{+}$. Thus $x^{\perp} = 0$. So K is totally ordered in both cases.

An element g in the f-module M (over the directed po-ring R) is called *basic* [4] if $C_R(g)$, the convex l-submodule generated by g, is totally ordered. This is equivalent to saying that $C(g)$ $(= C_{\mathbf{Z}}(g))$ is totally ordered. A subset X of M is a *basis* of M if X is a maximal set of disjoint elements and each element of X is basic.

Proposition 2.11. *Let R be a directed po-ring, and let M be an essential l-submodule of the f-module K_R. Then M has a basis of cardinality u if and only if K has a basis of cardinality u.*

Proof. If x and y are two elements of K such that $0 \neq x \in y^{\perp}$, and if $r \in R_*$ is such that $0 \neq xr \in M$, then $xr \in y^{\perp}$, since every polar of K is a submodule. Let $X = \{x_\alpha : \alpha \in A\}$ be a maximal set of disjoint elements of K. There is a subset $\{r_\alpha : \alpha \in A\}$ of R_* such that $Y = \{|x_\alpha r_\alpha| : \alpha \in A\}$ is a set of disjoint elements of M. If X is now a basis of K, then, clearly, each element of Y is basic. Suppose that y is a nonzero element of M such that $y \wedge |x_\alpha r_\alpha| = 0$ for all $\alpha \in A$. Let α be an element in A such that $y \wedge x_\alpha > 0$. Then $y \wedge x_\alpha \wedge |x_\alpha r_\alpha| = 0$, contradicting the fact that x_α is basic. So Y is a basis of M.

On the other hand, if $Y = \{y_\alpha : \alpha \in A\}$ is a maximal set of disjoint elements of M, and if $0 \neq x \in K$ is such that $x \wedge y_\alpha = 0$ for all $\alpha \in A$, then $Y \cup \{|xr|\}$ is a disjoint set in M for some $r \in R_*$. Thus x does not exist. So Y is a maximal set of disjoint elements of K. Since the convex l-submodule of K generated by an element of M is totally ordered exactly when the convex l-submodule of M generated by it is totally ordered, Y is a basis of M if and only if it is a basis of K.

For an f-module M_R, $\Gamma_R(M)$ will denote the rooted po-set of R-values of M [20]. It is known [7] that an f-module M has a finite basis with n elements if and only if $\Gamma_{\mathbf{Z}}(M)$ has exactly n roots. Since there is a one-to-one correspondence between the roots of Γ_R and the minimal prime submodules of M, and since the sets of minimal prime subgroups and submodules coincide, M has a basis containing n elements exactly when Γ_R has n roots. This fact gives the following corollary.

Corollary 2.12. *Let M be an i-f-module over the directed po-ring R. Then $\Gamma_R(M)$ has exactly n roots if and only if $\Gamma_R(E)$ has exactly n roots.*

Let M be an l-group, and let $d(M)$ be its $\mathbf{Z}$-injective hull. There is a lattice isomorphism between the lattice of convex l-subgroups of M, $\mathfrak{L}(M)$, and the lattice of convex l-subgroups of $d(M)$ given by: $N \in \mathfrak{L}(M)$ corresponds to the convex

l-subgroup of $d(M)$ generated by N, and $N \in \mathfrak{L}(d(M))$ corresponds to $N \cap M$. With respect to this correspondence, the value sets of M and $d(M)$ are isomorphic. That the R-value sets of M and $E(M_R)$ are not always isomorphic for an (totally ordered torsion-free) i-f-module M_R is shown by the following example.

Example 2.13. Let $R = \mathbf{Q}[[x]]$ be the formal power series ring with coefficients in $\mathbf{Q}$ and exponents in $\mathbf{Z}^+$. Order R lexicographically with the constant term dominating. Thus $R = \{\Sigma_{i=0}^{\infty} a_i x^i: a_i \in \mathbf{Q}\}$ and $R^+ = \{\Sigma_{i=n}^{\infty} a_i x^i: a_n > 0\} \cup \{0\}$. The units of R are the elements with nonzero constant term. Every element of R is of the form $x^k u$, u a unit or zero. The quotient field F of R is $\{\Sigma_{i=-n}^{\infty} a_i x^i: n \geq 0, a_i \in \mathbf{Q}\}$. It is a totally ordered field if its positive cone is defined by $F^+ = \{\Sigma_{i=-n}^{\infty} a_i x^i: a_{-n} > 0\}$.

Let $M = R_R$. Then $E(M_R) = F_R$, so M is an i-f-module. The convex l-submodules of M are

$$R \supseteq xR \supseteq x^2 R \supseteq \cdots,$$

and the convex l-submodules of E are

$$F \supseteq \cdots \supseteq x^{-2}R \supseteq x^{-1}R \supseteq R \supseteq xR \supseteq x^2 R \supseteq \cdots,$$

Thus $\Gamma_R(M) \cong \mathbf{Z}^+$ and $\Gamma_R(E) \cong \mathbf{Z}$.

3. i-f-**modules**. In this section we show that part of Anderson's characterization of a unital qf-ring [1] characterizes the torsion-free i-f-modules over an essentially positive directed po-ring. Using this characterization we show that a totally ordered domain has torsion-free i-f-modules if and only if it is a right Ore domain. We also show that every torsion-free f-module over a torsion-free irredundant f-ring R is an i-f-module if and only if R is a qf-ring. In addition, we examine some properties of the class of torsion-free i-f-modules, and we show that the i-f-property is a local property over a right noetherean ring.

Let M be a po-module over the po-ring R, and let N be an R-module containing M. Define

$$N^+ = \{x \in N: xD^+ \subseteq M^+ \text{ for some } D \in R^\nabla\}.$$

Notice that if R contains an essential right ideal D for which $D^+ = 0$, then $N^+ = N$. When $R = \mathbf{Z}$, N^+ consists of those elements x of N such that $nx \in M^+$ for some positive integer n.

Lemma 3.1.

(a) $N^+ + N^+ \subseteq N^+$.

(b) $N^+ R^+ \subseteq N^+$.

(c) *If R is essentially positive, then* $N^+ \cap -N^+ = Z(N_R)$.

(d) *If R is essentially positive and if M is a distributive l-module, then* $M^+ \subseteq N^+ \cap M = \{x \in M: x^- \in Z(M_R)\}$.

Proof. The first statement is an immediate consequence of the fact that R^∇ is closed under intersection.

For (b), take $x \in N^+$, $a \in R^+$, and $D \in R^\nabla$ such that $xD^+ \subseteq M^+$. Let $I = (D : a)$. Then $I \in R^\nabla$, and $(xa)I^+ = x(aI^+) \subseteq xD^+ \subseteq M^+$. Hence $xa \in N^+$.

If $x \in Z(N)$, then $xD = 0$ for some $D \in R^\nabla$; so, clearly, $x, -x \in N^+$. Conversely, if $x, -x \in N^+$, then $xD^+, -xF^+ \subseteq M^+$ for some $D, F \in R^\nabla$. Therefore, $x(D \cap F)^+ \subseteq M^+ \cap -M^+ = 0$, $x \in Z(N)$, and $N^+ \cap -N^+ = Z(N)$.

Finally, if $x \in M$ with $x^- \in Z(M)$, then $xD^+ = (x^+ - x^-)D^+ = x^+D^+ \subseteq M^+$ for some $D \in R^\nabla$. Thus $x \in N^+ \cap M$. On the other hand, if $x \in N^+ \cap M$ and $xD^+ \subseteq M^+$ for some $D \in R^\nabla$, then $x^-d = (xd)^- = 0$ for all $d \in D^+$. So $x^- \in Z(M)$.

Proposition 3.2. *Let M be a distributive l-module over the essentially positive po-ring R, and let N be a torsion-free R-module containing M. Then $N^+ = \{x \in N : xD^+ \subseteq M^+$ for some $D \in R^\nabla\}$ is a partial order on N, and (N, N^+) is a po-module extension of (M, M^+). If N is an essential extension of M, then*

(a) *N is semiclosed.*

(b) *The greatest lower bound (least upper bound) of two elements of M is also their greatest lower bound (least upper bound) in N.*

(c) *N^+ is the largest partial order P of N for which (N, P) is a po-module extension of (M, M^+).*

(d) $N^+ = \{x \in N : x(N : x)^+ \subseteq M^+\}$.

Proof. By 3.1, (N, N^+) is a po-module extension of (M, M^+). Suppose that N is an essential extension of M. Let $x \in M$ and $y \in N$ with $y \geq x, 0$. Since N is an essential extension of M, $(M : y) \in R^\nabla$. If $d \in (M : y)^+$, then $yd \geq (xd)^+ = x^+d$. Therefore, $(y - x^+)(M : y)^+ \subseteq M^+$, so $y - x^+ \in N^+$. Thus x^+ is the least upper bound of x and 0 in N, and we have (b). A similar argument gives (a).

Suppose that P is a partial order on N for which $P \cap M = M^+$ and $PR^+ \subseteq P$. If $y \in P$, then $y(M : y)^+ \subseteq PR^+ \cap M \subseteq P \cap M = M^+$. Thus $y \in N^+$ and $P \subseteq N^+$. Therefore (c) is true.

The last statement is obvious.

Proposition 3.3. *Let M be a distributive torsion-free l-module over the essentially positive po-ring R, and let N_R be an essential extension of M_R. If (N, P) is a distributive l-module extension of (M, M^+), then $P = N^+$.*

Proof. By 3.2(c) we only have to show that $N^+ \subseteq P$. Take $x \in N^+$ and let $D = (M : x) \cap (M : x^+)$. (All lattice operations are with respect to P.) If $x \notin P$, then $x^- \neq 0$, and $(xd)^- = x^-d \neq 0$ for some $d \in D^+$, i.e. $xd \in M \cap N^+ = M^+$ and $(xd)^- \neq 0$. Thus $x \in P$, and $N^+ \subseteq P$.

Notice that 3.3 says that N^+ is the only partial order P of N for which (N, P) can be a distributive l-module extension of (M, M^+).

Theorem 3.4. *Let M be a torsion-free right f-module over the essentially positive directed po-ring R. Then the injective hull E of M is an f-module extension of M if and only if for all $x \in E$ and, for all $d_1, d_2 \in R^+$ for which $xd_i \in M$,*

$$(xd_1)^+ \wedge (xd_2)^- = 0.$$

When this is the case the lattice order of E is uniquely determined by that of M.

Proof. First note that this condition is equivalent to: If ϕ is a homomorphism of M onto a totally ordered R-module and $x \in E$, then $\phi(x(M:x)^+) \subseteq \phi(M)^+$ or $-\phi(x(M:x)^+) \subseteq \phi(M)^+$. For if this latter condition is satisfied and $d_1, d_2 \in (M:x)^+$, then

$$\phi[(xd_1)^+ \wedge (xd_2)^-] = \phi(xd_1)^+ \wedge \phi(xd_2)^- = 0,$$

for arbitrary ϕ. Therefore, $(xd_1)^+ \wedge (xd_2)^- = 0$, since M is a subdirect product of totally ordered R-modules. Conversely, suppose that the condition in the theorem is satisfied and $\phi(xd_1) > 0$, $\phi(xd_2) < 0$. Then $\phi[(xd_1)^+ \wedge (xd_2)^-] = \phi(xd_1)^+ \wedge \phi(xd_2)^- = \phi(xd_1) \wedge -\phi(xd_2) > 0$, which contradicts the fact that $(xd_1)^+ \wedge (xd_2)^- = 0$.

If E is an f-module extension of M, then clearly $(xd_1)^+ \wedge (xd_2)^- = x^+d_1 \wedge x^-d_2 = 0$. Conversely, suppose that the condition holds. By 3.2, (E, E^+) is a po-module extension of (M, M^+). Let $x \in E$, and consider the correspondence h: $(M:x)^+R \to E$ given by $\Sigma_{i=1}^n d_ir_i \to \Sigma_{i=1}^n (xd_i)^+r_i$. Suppose that $\Sigma\, d_ir_i = 0$. Let $\phi : M \to \phi(M)$ be any homomorphism onto a totally ordered R-module. If $\phi(x(M:r)^+) \subseteq \phi(M)^+$, then

$$0 = \phi\left(\sum xd_ir_i\right) = \sum \phi(xd_i)^+r_i = \phi\left[\sum (xd_i)^+r_i\right],$$

and similarly, if $-\phi(x(M:x)^+) \subseteq \phi(M)^+$, then

$$\phi\left[\sum (xd_i)^+r_i\right] = \sum \phi(xd_i)^+r_i = 0.$$

Since M is a subdirect product of totally ordered R-modules, $\Sigma\,(xd_i)^+r_i = 0$, and thus h is a well-defined function. Clearly, it is an R-homomorphism. Since E is an injective R-module, h can be extended to an R-homomorphism g defined on R_*.

Let $y = g(1)$. Then $h(dr) = g(dr) = ydr$ for all $d \in (M:x)^+$ and $r \in R$, so $ydr = (xd)^+r$. Since $Z(E) = 0$, $yd = (xd)^+$ for all $d \in (M:x)^+$. We claim that $y = x^+$. Since $y(M:x)^+ \subseteq M^+$, y is in E^+. Also, $(y-x)d = (xd)^+ - xd \geq 0$ for all $d \in (M:x)^+$, so $y - x \in E^+$. Suppose that $z \in E$ with $z \geq \{0, x\}$. If $0 \leq d \in (M:x) \cap (M:z)$, then $zd \geq (xd)^+ = yd$, so $z - y \in E^+$. Therefore $y = x^+$, and E is an l-module. Note that $x^+d = (xd)^+$ for all $d \in (M:x)^+$.

All that remains is to show that E is an f-module. Take $x \in E$, $a \in R^+$, and let $C = \{b \in R : ab \in (M : x)\}$. Then $C \in R^{\nabla}$, and if $b \in C^+$, $(x^+a)b = (xab)^+ = (xa)^+b$. Thus $[x^+a - (xa)^+]C^+ = 0$, and hence $x^+a = (xa)^+$. Therefore E is a distributive l-module, and, hence it is an f-module.

Note that without the assumption that R is directed, Theorem 3.4 becomes

E is a distributive l-module extension of M if and only if $(xd_1)^+ \wedge (xd_2)^- = 0$ for all $x \in E$ and for all $d_1, d_2 \in R^+$ such that $xd_i \in M$.

Corollary 3.5. *Let M be a torsion-free f-module over the po-ring R. Then M is an i-f-module if either*

(a) *R is commutative, essentially positive, and directed, or*

(b) *R is an f-ring and a semiprime right Goldie ring.*

Proof. (a) If $x \in E(M)$ and $d_1, d_2, d \in (M : x)^+$, then

$$[(xd_1)^+ \wedge (xd_2)^-]d = (xd_1d)^+ \wedge (xd_2d)^- = (xd)^+d_1 \wedge (xd)^-d_2 = 0.$$

Thus $[(xd_1)^+ \wedge (xd_2)^-](M : x)^+ = 0$, so $(xd_1)^+ \wedge (xd_2)^- = 0$.

(b) Let $x \in E(M)$, $d_1, d_2 \in (M : x)^+$, and $d \in (M : x)^+$ with d regular. There are elements $a, b, c, e \in R^+$, with a and c regular, such that $d_1da = db$ and $d_2dc = de$. Therefore,

$$(xd_1da)^+ \wedge (xd_2dc)^- = (xdb)^+ \wedge (xde)^- = (xd)^+b \wedge (xd)^-e = 0;$$

so $(xd_1)^+da \wedge (xd_2)^-dc = 0$. Let s and t be positive regular elements of R such that $as = ct$. Then

$$[(xd_1)^+ \wedge (xd_2)^-]das = (xd_1)^+das \wedge (xd_2)^-dct = 0.$$

But das is regular. Therefore $dasR$ is an essential right ideal of R, and $(xd_1)^+ \wedge (xd_2)^- = 0$.

Note that any commutative, semiprime, directed po-ring, in which the square of every element is positive, is an example of a po-ring satisfying the conditions of 3.5(a). An archimedean semiprime f-ring is, of course, such a po-ring.

The equivalence of (1) and (2) in the next corollary is a generalization of the fact that a semiprime f-ring R with the maximum condition on polars (i.e., $P(R)$ is finite) is a right qf-ring if and only if it is a right Goldie ring [1, Theorem 6.1].

Corollary 3.6. *The following statements are equivalent for an irredundant torsion-free f-ring R.*

(1) *R is a qf-ring.*

(2) *Each component of the irredundant representation of R is a totally ordered right Ore domain.*

(3) *Every torsion-free f-module over R is an i-f-module.*

Proof. That (1) implies (2) has already been observed in the proof of 2.9, and that (3) implies (1) follows from the fact that Q $(= E(R))$ is an f-module extension of R_R if and only if it is an f-ring extension of R.

Let M be a torsion-free f-module over R. Assuming (2), we have (see the proof of 2.9): $R \subseteq \Pi R_\alpha$ and $M \subseteq \Pi M_\alpha \subseteq \Pi E_\alpha = E(M)$, where R_α is a totally ordered right Ore domain, and E_α is the R_α- (R-)injective hull of the torsion-free R_α- (R-)f-module M_α. By 3.5, each E_α is an f-module extension of M_α(over R_α, hence over R). Thus $E(M)$ is an f-module extension of M, and (2) implies (3).

An interesting example of a torsion-free i-f-module may be obtained as follows. Let R be a semiprime right qf-ring with maximal right quotient ring Q. Suppose that I is an l-submodule of Q_R, and let $I' = E(I_R) \subseteq Q$. Then $I' = eQ$ for some idempotent e of Q. Consequently, $S = \mathrm{Hom}_R(I_R, I_R) \cong \{q \in Qe: qI \subseteq I\}$, and $T = \mathrm{Hom}_R(I'_R, I'_R) \cong eQ$. Thus S and T can be made into f-rings in a natural way. Now S_S is an essential submodule of T_S [10, p. 97], and ${}_SI$ is a torsion-free f-module [21, Theorem 3.25]. If ${}_SS$ is essential in ${}_ST$, then ${}_SI$ is an i-f-module.

There are i-f-modules that are not torsion-free. For instance, over a quasi-Frobenius f-ring R each unital module can be made into an f-module. If R is totally ordered, but not a division ring, then it has no torsion-free modules [21, p. 114]. It is not hard to see that if M is any i-f-module over a directed essentially positive po-ring, then $M/Z_2(M)$ is a torsion-free i-f-module, where $Z_2(M) = \mathrm{Cl}\,\mathrm{Cl}\,0$ is the torsion submodule of M.

We now present an example of a totally ordered, archimedean, torsion-free f-module, over a unital essentially positive l-ring, that is not an i-f-module, but whose injective hull is an l-module extension.

Example 3.7. Let $R = \{\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right): a, b, c \in \mathbf{Q}\}$, and let $R^+ = \{\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right): a, b, c \in \mathbf{Q}^+\}$. Then R is an essentially positive unital l-ring. Let $M = \{\left(\begin{smallmatrix} 0 & 0 \\ a & 0 \end{smallmatrix}\right): a \in \mathbf{Q}\}$, and let $M^+ = \{\left(\begin{smallmatrix} 0 & 0 \\ a & 0 \end{smallmatrix}\right): a \in \mathbf{Q}^+\}$. Then M_R is a simple R-module and a totally ordered torsion-free f-module. But $E(M_R) = \{\left(\begin{smallmatrix} 0 & 0 \\ a & b \end{smallmatrix}\right): a, b \in \mathbf{Q}\}$ cannot be made into an f-module extension of M (use 3.4).

More generally, if R is any essentially positive sub-po-ring of the canonically ordered matrix ring D_n $(n > 1$, D a totally ordered division ring), such that $Q(R) = D_n$, then 3.12 implies that R has no torsion-free i-f-module (D_n has no nontrivial f-modules). Any l-subring R of D_n with $Q(R) = D_n$ is essentially positive.

Proposition 3.8. *Let M be a torsion-free i-f-module over an essentially positive po-ring R.*

(a) *Every l-submodule of M is an i-f-module.*

(b) *If N is a closed convex l-submodule of M, then M/N is an i-f-module.*

Proof. Let $E = E(M_R)$. If N is an l-submodule of M, then $E(N) = \mathrm{Cl}_E N$ is an l-submodule of E by 2.1(b). Thus N is an i-f-module. If N is a closed convex

l-submodule of M, then $E_1 = E(N)$ is a convex l-submodule of E by 2.1, and $N = E_1 \cap M$. Therefore, $M/N \to E/E_1$ is a mononomorphism of f-modules. Since $E = E_1 \oplus E_2$ as R-modules, E/E_1 is an injective R-module. But M/N is an l-submodule of E/E_1, so M/N is an i-f-module by (a). In fact, $E/E_1 = E(M/N)$.

Using the notation of (b), note that E_2 has two partial orders. One comes from E/E_1, and one is inherited from E. (E_2 is not uniquely determined by N, though E_1 is.) Let P be the partial order of E_2 coming from E/E_1. Then $P = \{x \in E_2 : x + E_1 \geq 0\}$ contains $E_2 \cap E^+$. The following example shows that, in general, this containment is proper. Let $R = \mathbf{Z}$, $M = \mathbf{Z} \oplus \mathbf{Z}$, $E = \mathbf{Q} \oplus \mathbf{Q}$, $N = \mathbf{Z} \oplus 0$, $E_1 = \mathbf{Q} \oplus 0$, and $E_2 = (1, -1)\mathbf{Q}$. Then $P = (1, -1)\mathbf{Q}^+$ and $E_2 \cap E^+ = \{0\}$. Thus (E_2, P) is not a sublattice of E. If, however, $E = \operatorname{lex} E_1$, i.e. $M = \operatorname{lex} N$ (see 3.14), then $P = E_2 \cap E^+$. For then, $x \in P$ and $x + E_1 \geq 0$ implies $x \geq 0$. This is the case if M is totally ordered.

It is easy to see that a finite direct sum of i-f-modules (over any po-ring) is an i-f-module.

Corollary 3.9. *Let R be an essentially positive po-ring, and let $\{M_\alpha : \alpha \in A\}$ be a collection of torsion-free f-modules. The following are equivalent.*

(a) *M_α is an i-f-module for each $\alpha \in A$.*

(b) *ΠM_α is an i-f-module.*

(c) *$\Sigma \bigoplus M_\alpha$ is an i-f-module.*

Proof. (b) implies (c) and (c) implies (a) by 3.8(a). Let $E_\alpha = E(M_\alpha)$. Then (a) implies that ΠE_α is an i-f-module, so ΠM_α is an i-f-module, again by 3.8(a).

Corollary 3.10. *The inverse limit of torsion-free i-f-modules over an essentially positive po-ring is an i-f-module.*

Proof. Let $(\{M_\alpha : \alpha \in A\}, \{f_{\alpha\beta} : \alpha \geq \beta\})$ be an inverse limit system of f-modules. Then $M = \{(m_\alpha) \in \Pi M_\alpha : f_{\alpha\beta} m_\alpha = m_\beta$ whenever $\alpha \geq \beta\}$ is the inverse limit of this system, and M is an l-submodule of ΠM_α. Thus M is an i-f-module if each M_α is, by 3.9 and 3.8.

Corollary 3.11. *Let M be a torsion-free f-module over an essentially positive directed po-ring R. Then M is an i-f-module if and only if every torsion-free homomorphic image of M is an i-f-module.*

Proof. This follows immediately from 3.8(b).

Proposition 3.12. *Let R be an essentially positive po-ring with $Z(R) = 0$, and let S be a directed (in its canonical order) right quotient ring of R. Suppose that M is a torsion-free f-module over R. Then M is an i-f-module if and only if it is contained in a torsion-free i-f-module over S. In fact, $E(M_R)$ is an f-module over S and an injective S-module.*

Proof. Let $E = E(M_R)$ be an f-module extension of M. By [23, Theorem 3 and Proposition 10], E is a Q-module, where Q is the maximal right quotient ring of R. If $x \wedge y = 0$ in E, $q \in Q^+$, and $D \in R^\nabla$ with $qD \subseteq R$, then $xqD^+ \subseteq E^+$, so $xq \in E^+$. Also, if $d \in D^+$, then $(xq \wedge y)d = xqd \wedge yd = 0$, so $xq \wedge y = 0$. Thus E is a distributive l-module over Q. Since S is directed (and E_S is torsion-free) E_S is an f-module. But any S-essential extension of E is also an R-essential extension of E, so E_S is injective.

Now suppose that M is an R-submodule of a torsion-free injective f-module K_S. Since $E(K_R)$ is an S-module, it is an S-essential extension of K, so K is R-injective. By 3.8, M is an i-f-module.

Theorem 3.13. *A totally ordered domain has a nonzero torsion-free i-f-module if and only if it is a right Ore domain.*

Proof. Let M be a torsion-free i-f-module over the totally ordered domain R, and let $0 < x \in M$. Then $R \longrightarrow xR$ is an f-module homomorphism, so $R/r(x) \cong xR$. Since M is torsion-free, $r(x)$ is not an essential right ideal of R. Let $0 \neq J$ be a right ideal such that $J \cap r(x) = 0$, and take $0 < y \in J$. Then yR is isomorphic to an f-submodule of M, so it is an i-f-module. But $yR \cong R$ as f-modules, so R_R is an i-f-module. Therefore, $E(R_R)$ satisfies the condition of Theorem 3.4, and so R is a right Ore domain, by [1, 3.1 and 5.2].

Theorem 3.13 is not true for an arbitrary f-ring with $Z(R) = 0$, i.e. a torsion-free f-ring can have a torsion-free i-f-module without being a qf-ring. In particular, let R be an irredundant semiprime f-ring, and let M be a torsion-free f-module over R. Let $R \subseteq \Pi R_\alpha$ and $M \subseteq \Pi M_\alpha$ be the decompositions of R and M. Then M is an i-f-module if and only if $M_\alpha = 0$ for each α for which R_α is not a right Ore domain.

Proposition 3.14 (see [4, p. 239]). *Let K be a torsion-free f-module over an essentially positive po-ring, and let M be an essential l-submodule of K. Suppose that N is a convex l-submodule of M, and let $K_1 = \mathrm{Cl}_K N$.*

(a) *If $M = \mathrm{lex}\ N$, then $K = \mathrm{lex}\ K_1$.*

(b) *If $K = \mathrm{lex}\ K_1$, then $M = \mathrm{lex}\ \mathrm{Cl}_M N$.*

(c) *Suppose that $K = E(M)$ and $K = \mathrm{lex}\ K_1$. If K_2 is any R-complement of K_1 in K, then K_2 is a totally ordered submodule of K, and*

$$K^+ = \{y + z \in K_1 \oplus K_2 \colon z > 0, \text{ or } z = 0 \text{ and } y \geq 0\}.$$

Proof. (a) Suppose that $y \in K^+ \backslash K_1$ and $x \in K_1$. Then $(y-x)^+ \neq 0$. Assume that $(y-x)^- \neq 0$, also. There exists $0 \leq d \in (M : y) \cap (M : x) = D$ such that $(y-x)^- d \neq 0$. So $(y-x)^+ D^+ \subseteq N$, since N contains all the nonunits of M [5, p. 111]. Therefore $(y-x)^+ \in K_1$. Similarly, $(y-x)^- \in K_1$. Thus $y - x \in K_1$ and $y \in K_1$. So $(y-x)^- = 0$ and $y \geq x$. Since K_1 is prime in K by 2.1(c), $K = \mathrm{lex}\ K_1$.

(b) If $x \in M^{+}\backslash \mathrm{Cl}_M N$, then $x \in K^{+}\backslash K_1$, since $K_1 \cap M = \mathrm{Cl}_M N$. Therefore $x > \mathrm{Cl}_M N$. Clearly, $\mathrm{Cl}_M N$ is prime in M.

(c) The remarks following 3.8 show that K_2 is totally ordered. Since $K = \operatorname{lex} K_1$, we clearly have $K^{+} = \{y + z \in K_1 \oplus K_2 : z > 0 \text{ or } z = 0 \text{ and } y \geq 0\}$.

We show next that the i-f-property is a local property.

Theorem 3.15. *Suppose that R is a right noetherian, essentially positive, directed po-ring, and let M be a subdirect product of totally ordered torsion-free f-modules. Then M is an i-f-module if and only if $C_R(g)$ is an i-f-module for each $g \in M$.*

Proof. Suppose that M is a subdirect product of the family $\{M_\alpha : \alpha \in A\}$ of totally ordered torsion-free f-modules. We may assume that $M_\alpha = M/N_\alpha$ for each $\alpha \in A$. Therefore, if $x + N_\alpha \in M_\alpha$, then $C_R(x + N_\alpha) = (C_R(x) + N_\alpha)/N_\alpha$. If $C_R(x)$ is an i-f-module, then so is $(C_R(x) + N_\alpha)/N_\alpha$, by 3.11. Now suppose that M is an i-f-module locally, i.e. $C_R(g)$ is an i-f-module for each $g \in M$. The preceding remarks show that each M_α is also an i-f-module locally. If each M_α is an i-f-module, then M is an i-f-module by 3.9 and 3.8. So, without loss of generality, we may assume that M is totally ordered.

Let $x \in E(M) = E$ and let $\{N_\alpha : \alpha \in A\} = \{C_R(xd) : d \in (M:x)^{+}\}$. If $E_\alpha = E(N_\alpha) \subseteq E$, then E_α is an f-module extension of N_α by hypothesis. Since M is totally ordered, $\{N_\alpha : \alpha \in A\}$ is totally ordered by inclusion, and so is the family $\{E_\alpha : \alpha \in A\}$. Thus $N = \bigcup N_\alpha$ is a submodule of $E_1 = \bigcup E_\alpha$. Since R is noetherian, and since E_1 is the direct limit of the E_α, E_1 is an injective R-module ([3, p. 17] and [10, p. 53]). Clearly, E_1 is an f-module extension of N with positive cone $\bigcup E_\alpha^{+}$, and it is the injective hull of N.

Now $x(M:x)^{+} \subseteq N \subseteq E_1$; so $x \in E_1$, since E_1 is closed in E. If $d_1, d_2 \in R^{+}$, then $(xd_1)^{+} \wedge (xd_2)^{-} = x^{+}d_1 \wedge x^{-}d_2 = 0$. But then M is an i-f-module, by 3.4.

The converse follows from 3.8.

Corollary 3.16. *Let R be a right noetherian, essentially positive, directed po-ring, and let M be a subdirect product of totally ordered torsion-free R-modules. If M is a finitely-valued f-module, then M is an i-f-module if and only if $C_R(g)$ is an i-f-module for every special element g of M.*

Proof. If $g \in M$, then $C_R(g) = C_R(g_1) \oplus \cdots \oplus C_R(g_n)$, where each g_i is special [20]. Now use 3.14.

Note that the proof of 3.15 is valid if, instead of assuming that R is noetherian, one assumes that M has the maximum condition on convex l-submodules.

We consider next the Hahn product of strict i-f-modules. Let Γ be a rooted po-set, and suppose that for each $p \in \Gamma$, M_p is an f-module over the po-ring R. Suppose further that M_p is totally ordered if p is not a minimal element of Γ.

For $v \in \Pi M_p$ (R-module product) define the *support* (*supp*) *of* $v = \{p \in \Gamma : v(p) \neq 0\}$. It is well known ([7] or [24]) that

$$V(\Gamma, M_p) = \{v \in \Pi M_p : \text{ supp } v \text{ has the maximum condition}\}$$

is an l-group when provided with the positive cone

$$V^+ = \{v \in V: v(p) > 0 \text{ whenever } p \text{ is a maximal element of supp } v\} \cup \{0\}.$$

An f-module M is called *strict* if it satisfies the following condition: $(x, r) \in M^+ \times R^+$ and $xr = 0$ implies $x = 0$ or $r = 0$. Note that if R has a strict f-module, then R is a po-domain, i.e. $a, b \in R^+$ and $ab = 0$ implies $a = 0$ or $b = 0$.

Proposition 3.17. *Suppose that R is directed and each M_p is strict. Then $V = V(\Gamma, M_p)$ is an f-module over R.*

Proof. Since $(vr)(p) = v(p)r$ for all $v \in \Pi M_p$, $r \in R$, and $p \in \Gamma$, supp $vr \subseteq$ supp v. Thus V is an R-module. If $v \in V^+$, p is a maximal element of supp v, and $0 < r \in R^+$, then $(vr)(p) = v(p)r > 0$, since M_p is strict. So V is an l-module over R. If P is a maximal chain of Γ, then $V_P = \{v \in V : \text{supp } v \subseteq P\} \cong V(P, M_p)$ as l-modules, and V is a subdirect product of the V_P. Thus V is an f-module provided each V_P is an f-module. So we may assume that Γ is totally ordered.

Let $0 \neq v \in V$, $0 < r \in R^+$, and let q be the maximal element of supp v. Note that supp v = supp vr, and $v(q) < 0$ if and only if $(vr)(q) < 0$, since M_q is strict. Therefore, $v^+r = (vr)^+$. So V is a distributive l-module over R. Hence it is an f-module.

Proposition 3.18. *Let R be an essentially positive directed po-ring, and let M_p be an i-f-module for each p in the rooted po-set Γ. Suppose further that M_p is totally ordered if p is not a minimal element of Γ. If each $E_p = E(M_p)$ is strict, then $V(\Gamma, E_p) = E(V(\Gamma, E_p))$ and $V(\Gamma, M_p)$ is an i-f-module.*

Proof. Since ΠE_p is an injective module, $V(\Gamma, E_p) \subseteq \Pi E_p$ has an injective hull $E(V(\Gamma, E_p)) = E$ contained in ΠE_p. Let $w \in E$ and $0 \neq d \in R^+$ such that $wd \in V(\Gamma, E_p)$. Suppose that $p_1 < p_2 < \cdots$ is a chain in supp w. Since p_i is not a minimal element of Γ for $i > 1$, $w(p_i) > 0$ or $w(p_i) < 0$. Thus $p_i \in$ supp wd for $i > 1$. This is impossible, since supp wd has the maximum condition. So $w \in V(\Gamma, E_p)$ and $E(V(\Gamma, E_p)) = V(\Gamma, E_p)$. Since $V(\Gamma, M_p)$ is an l-submodule of $V(\Gamma, E_p)$, $V(\Gamma, M_p)$ is an i-f-module.

If R is commutative, then E is strict provided M is, but we do not know if this is true for any R. The po-ring R in 3.18 need not be a domain. Diem [8] has given the following example of a commutative l-domain that is not a domain: $R = \mathbf{Q}a \oplus \mathbf{Q}b$ as l-groups and $a^2 = b^2 = ab = ba = a$. We remark that the essential ideals of R are those ideals that contain $\mathbf{Q}a + \mathbf{Z}qb$ for some $0 \neq q \in \mathbf{Q}$. Thus R

is essentially positive. As a ring $R = \mathbf{Q}a \oplus \mathbf{Q}(a - b)$, so $\mathbf{Q}a$ is a strict i-f-module over R.

We close this section with a remark about extending homomorphisms.

Proposition 3.19. *Let N and K be distributive l-modules over the essentially positive po-ring R, and suppose that M is an essential l-submodule of K. If $f \in \operatorname{Hom}_R(M, N)$ is an l-homomorphism and if N is torsion-free, then any $g \in \operatorname{Hom}_R(K, N)$ extending f is an l-homomorphism, and g is unique.*

Proof. Since N is torsion-free and M is essential in K, g is unique. If $x \in K$ and $d \in (M : x)^+$, then

$$[g(x^+)]d = g(x^+d) = g[(xd)^+] = f[(xd)^+] = [f(xd)]^+ = [g(xd)]^+ = [(gx)^+]d.$$

So $g(x^+) = (gx)^+$.

Corollary 3.20. *Let R be an essentially positive po-ring, and let M be an i-f-module over R. Then any l-homomorphism from M into a torsion-free injective f-module N has a unique extension to an l-homomorphism from $E(M)$ to N.*

4. **Relative injective f-modules.** Ribenboim [19] has observed that there are no injectives in the category of unital po-modules over a po-domain. Consequently, he defined and studied a certain type of relative injectivity. Let $\mathcal{C}$ be a category whose objects are sets, and let $\aleph_\alpha$ be an infinite cardinal number. An object E of $\mathcal{C}$ is called *$\aleph_\alpha$-injective* if whenever C is a subobject of A in $\mathcal{C}$ and $\operatorname{card}(A) < \aleph_\alpha$, then every morphism $C \longrightarrow E$ can be extended to a morphism $A \longrightarrow E$.

Weinberg [25] has recently given a characterization of the $\aleph_\alpha$-injectives in the category of l-groups. In this section we show that his characterization (and his methods) holds in the category of torsion-free f-modules over an irredundant semiprime right qf-ring.

Definition 4.1. An l-group M is an almost-η_α-group if, for each pair of subsets X and Y of M such that $X < Y$ and $\operatorname{card}(X \cup Y) < \aleph_\alpha$, there exists an a in M such that $X \le a \le Y$.

Definition 4.2 (Weinberg [25]). An element y in an l-group M is said to *split b from a* if $y \ge a^+$, $y \wedge a^- = 0$, and $(b - y)^+ \wedge y = 0$. M is *self-splitting* if each ordered pair of elements of M is split by some element in M.

The cardinal number $\aleph_\alpha$ is called *regular* provided $\bigcup\{X_i : i \in I\}$ has cardinality less than $\aleph_\alpha$ whenever I and each X_i have cardinality less than $\aleph_\alpha$.

Theorem 4.3 (Weinberg [25]). *Let $\aleph_\alpha$ be a regular cardinal number, and let M be an l-group. The following are equivalent.*

(a) *M is $\aleph_\alpha$-injective.*

(b) *M is a divisible, self-splitting, almost-η_α-group in which any two pairwise disjoint subsets of cardinality less than $\aleph_\alpha$ have disjoint upper bounds.*

Now let M_R be an f-module. We will show, with not too many restrictions on R, that Weinberg's four conditions are necessary for M to be an $\aleph_\alpha$-injective f-module. The converse is much harder and we only have it for the case that R is an irredundant semiprime qf-ring.

Lemma 4.4. *Let M be a torsion-free i-f-module over the directed po-ring R, let $\mathcal{C}$ be the category of torsion-free f-modules over R. If M is $\aleph_\alpha$-injective in $\mathcal{C}$, then so is $E = E(M_R)$.*

Proof. Suppose that $N \in \mathcal{C}$ with $\operatorname{card}(N) < \aleph_\alpha$. Let N' be an l-submodule of N, and let f be a homomorphism from N' into E. Then $N'' = f^{-1}(M)$ is an essential l-submodule of N'. We have the following diagram:

$$\begin{tikzcd}
N'' \arrow[d, "f"'] & \subseteq & N' \arrow[d, "f"'] & \subseteq & N \arrow[dllll, dashed, "g"'] \\
M & \subseteq & E & &
\end{tikzcd}$$

where g comes from the fact that M is $\aleph_\alpha$-injective. Since $N'' \subseteq \ker(g - f)$, since N'' is essential in N', and since M is torsion-free, $N' \subseteq \ker(g - f)$. Thus $f(N') \subseteq M$, and g is an extension of f.

Corollary 4.5. *Let M be a torsion-free i-f-module over the directed po-ring R. If $\aleph_\alpha > \operatorname{card}(R)$ and if M is $\aleph_\alpha$-injective in the category of torsion-free f-modules over R, then $M = E(M_R)$.*

Proof. Take $x \in E$ and let $N_1 = xR_*$. It is known that the sublattice of E generated by N_1 is a subgroup, and is given by

$$L(N_1) = \left\{ \bigvee_{j=1}^{m} \bigwedge_{i=1}^{n} s_{ij} : s_{ij} \in N_1 \right\}.$$

If $r \in R^+$, then $L(N_1)r \subseteq L(N_1)$, since E is an f-module. Thus $L(N_1)$ is an l-submodule of E, since R is directed. Clearly, $\operatorname{card}(L(N_1)) < \aleph_\alpha$. In the proof of 4.4, let $N' = N = L(N_1)$, $N'' = L(N_1) \cap M$, and let f be the inclusion map from N' to E. Then $f(L(N_1)) \subseteq M$, so $x \in M$.

Lemma 4.6. *Let R be a directed po-domain which has a strict totally ordered module K. Then any f-module M can be embedded in an f-module which contains, for each a in M, an element y that splits b from a for every b in M.*

Proof. M can be embedded in a product of a family of totally ordered modules $\{M_i : i \in I\}$. Let $N_i = M \oplus K$ as R-modules with positive cone defined by $N_i^+ = \{(m, k) : k > 0, \text{ or } k = 0 \text{ and } m \geq 0\}$. Define $y \in \prod N_i$ by $y(i) = 0$ if $a^+(i) = 0$, and

$y(i) = (0, k_0)$ if $a^+(i) > 0$, where k_0 is a fixed nonzero positive element of K.

Proposition 4.7. *Let R be a directed po-domain which has a strict totally ordered module M, and let $\aleph_\alpha$ be a cardinal number greater than* card(R). *If M is an $\aleph_\alpha$-injective f-module, then M is self-splitting and each pair of pairwise disjoint subsets of M of cardinality less than $\aleph_\alpha$ have disjoint upper bounds.*

Proof. Let $a, b \in M$, and let A be the l-submodule of M generated by a and b. By 4.6, A is imbeddable in an f-module C containing an element y which splits b from a. Let B be the l-submodule of C generated by A and y. Then card$(B) < \aleph_\alpha$, and the injection of A into M can be extended to a homomorphism ϕ from B into M, since M is $\aleph_\alpha$-injective. Clearly, $\phi(y)$ splits b from a. Thus M is self-splitting.

Let A_1 and A_2 be pairwise disjoint subsets of M such that card(A_1) + card$(A_2) < \aleph_\alpha$. Embed M in a product of totally ordered R-modules $M \longrightarrow \Pi M_i$. As in the proof of 4.6, let N_i be the lexicographic extension of M_i by K. Define t_j in $N = \Pi N_i$, for $j = 1, 2$, by $t_j(i) = (0, k_0)$ if $\pi_i(A_j) \neq 0$ and $t_j(i) = 0$ if $\pi_i(A_j) = 0$. Then $t_1 \wedge t_2 = 0$ and $t_j \geq A_j$. Let L be the l-submodule of N generated by $A_1 \cup A_2$, and let P be the l-submodule of N generated by L and $\{t_1, t_2\}$. Then card$(P) < \aleph_\alpha$, and the injection of L into M can be extended to a homomorphism ϕ from P into M. Thus $\phi(t_1) \wedge \phi(t_2) = 0$, and $\phi(t_j) \geq A_j$.

A totally ordered set T is said to be *dense* if for all $a < b$ in T there exists $x \in T$ such that $a < x < b$. If M is a totally ordered module over a po-ring R, then $\mathbf{Q} \otimes_Z M$ is a dense totally ordered R-module containing M.

Lemma 4.8. *Let M be a totally ordered module over the po-ring R, and suppose that X and Y are subsets of M such that $X < Y$. Then M can be embedded in a totally ordered module N containing an element a satisfying $X < a < Y$. Moreover, if R is right noetherian and M is torsion-free, then such an N may be found which is torsion-free.*

Proof. By the preceding remark we may assume that M is dense. Suppose that no element of M lies between X and Y. Then either X is nonempty and has no last element or Y is nonempty and has no first element. Suppose, for example, that X is nonempty and has no last element. For $x \in X$, let $A_x = \{z \in X : z > x\}$. Since X has no last element, $\{A_x : x \in X\}$ has the finite intersection property. Therefore, there is an ultrafilter $\mathfrak{F}$ on X containing $\{A_x : x \in X\}$.

For each $h \in M^X$ let $Z(h) = \{x \in X : h(x) = 0\}$. Let $I = \{f \in M^X : Z(f) \in \mathfrak{F}\}$. Then I is a prime submodule of M^X. Embed M in M^X/I via the map $m \longrightarrow \langle m \rangle + I$, where $\langle m \rangle(x) = m$ for all $x \in X$. If $e \in M^X$ is the identity on X, i.e. $e(x) = x$ for all $x \in X$, then $\langle X \rangle + I < e < \langle Y \rangle + I$.

Suppose that M is torsion-free and R is right noetherian. Let $f \in M^X$ and $D \in R^\nabla$ such that $fd \subseteq I$. Since R is noetherian, $D = d_1R_* + \cdots + d_nR_*$. Clearly,

$Z(f) \subseteq \bigcap\{Z(fd_i): i = 1, \cdots, n\}$. Conversely, if $(fd_i)(x) = 0$ for $1 \le i \le n$, then $f(x)D = 0$. Thus $f(x) = 0$, since M is torsion-free. So $Z(f) = \bigcap\{Z(fd_i): i = 1, \cdots, n\}$. Hence $Z(f) \in \mathfrak{S}$ and $f \in I$.

Proposition 4.9. *Let M be an $\aleph_\alpha$-injective f-module over the directed po-ring R, and suppose $\aleph_\alpha >$ card(R). Then M is an almost-η_α-module.*

Proof. Suppose that X and Y are subsets of M, each of which has cardinality less than $\aleph_\alpha$, and $X < Y$. Assume M is embedded in the product $\prod M_j$, where each M_j is totally ordered. Let N_j be a totally ordered module containing M_j and an element t_j such that $\pi_j(X) \le t_j \le \pi_j(Y)$. Let $t \in \prod N_j = K$ be defined by $t(j) = t_j$. Finally, let A be the l-submodule of M generated by $X \cup Y$, and let B be the l-submodule of K generated by A and t. Then card$(B) < \aleph_\alpha$, and the injection of A into M can be extended to B. Thus the image of t in M lies between X and Y.

Suppose that R is right noetherian (and directed), and M is a subdirect product of totally ordered torsion-free R-modules. If M is $\aleph_\alpha$-injective in the category of torsion-free f-modules over R, then M is an almost-η_α-module. The proof is the same as that of 4.9. Of course, if M is a torsion-free f-module over an essentially positive po-ring R, then M is $\aleph_\alpha$-injective in the category of R-f-modules if and only if it is $\aleph_\alpha$-injective in the category of torsion-free R-f-modules. For then the torsion submodule, $\mathrm{Cl}\,\mathrm{Cl}_A\, 0 = Z_2(A)$, of each f-module A is a convex l-submodule; so an f-module homomorphism $A \longrightarrow M$ induces a torsion-free f-module homomorphism $A/Z_2(A) \longrightarrow M$.

For the remainder of this section $\aleph_\alpha$ will be a regular cardinal number.

Theorem 4.11. *Let R be a totally ordered right Ore domain, and let M be a torsion-free f-module over R. If $\aleph_\alpha >$ card(R), then M is $\aleph_\alpha$-injective in the category of torsion-free f-modules over R if and only if it is injective in the category of R-modules and $\aleph_\alpha$-injective in the category of l-groups.*

Proof. If M is $\aleph_\alpha$-injective, then 4.5, 4.7, 4.9, and 4.3 imply that M_R is injective and that M is an $\aleph_\alpha$-injective l-group.

Conversely, if M_R is torsion-free and injective, then M is a vector lattice over D, where D is the totally ordered right quotient division ring of R (see 3.12). Thus we may assume that $R = D$. Now copy the proof of 4.3 given in [24] for the case that $R = \mathbf{Q}$.

Corollary 4.12. *Let R be an irredundant semiprime right qf-ring, and let M be a torsion-free f-module over R. Suppose that $\aleph_\alpha >$ card(R). Then M is $\aleph_\alpha$-injective in the category of (torsion-free) f-modules over R if and only if M is an injective R-module and an $\aleph_\alpha$-injective l-group.*

Proof. Let $R \subseteq \prod R_\lambda \subseteq \prod D_\lambda = Q$ be the decomposition of R (see 2.7). If M is either $\aleph_\alpha$-injective or R-injective, then $M = \prod E_\lambda$, where E_λ is an f-module over

$R(R_\lambda)$ and an injective R- (R_λ-)module (3.6, 4.5, and the proof of 2.9). By the usual argument, it is easily seen that M is $\aleph_\alpha$-injective if and only if each E_λ is $\aleph_\alpha$-injective.

Now E_λ is an $\aleph_\alpha$-injective f-module over R if and only if it is an $\aleph_\alpha$-injective f-module over R_λ. For suppose that E_λ is $\aleph_\alpha$-injective with respect to R. Let A be an f-submodule of the R_λ-f-module B ($\mathrm{card}(B) < \aleph_\alpha$), and let $f: A \to E$ be a map in the category of R_λ-f-modules. Since R_λ is a homomorphic image of R, A and B are naturally f-modules over R, and then f is an R-homomorphism. Let $g: B \to E_\lambda$ be an R-extension of f. Then g is clearly an R_λ-extension of f; so E_λ is $\aleph_\alpha$-injective over R_λ.

On the other hand, suppose that E_λ is $\aleph_\alpha$-injective over R_λ. Let A be an f-submodule of the torsion-free R-f-module B ($\mathrm{card}(B) < \aleph_\alpha$), and let $f: A \to E$ be a map in the category of R-f-modules. By 3.20, we may assume that A and B are injective R-modules. Thus A and B (and E_λ) are Q-modules (see 3.12), and f is a Q-map. Let $g: B \to E_\lambda$ be the R_λ-extension of f. Then g is, in fact, a Q-extension of f, hence an R-extension. So E_λ is $\aleph_\alpha$-injective over R.

In summary, we have M is R-$\aleph_\alpha$-injective if and only if $M = \prod E_\lambda$, where E_λ is R_λ-$\aleph_\alpha$-injective. But E_λ is R_λ-$\aleph_\alpha$-injective if and only if it is $\mathbf{Z}$-$\aleph_\alpha$-injective, by Theorem 4.11. Thus M is R-$\aleph_\alpha$-injective exactly when it is an injective R-module and an $\aleph_\alpha$-injective l-group.

Note that there are no injectives in the category of torsion-free R-f-modules. For any nonzero torsion-free injective R-f-module gives rise to a nonzero torsion-free injective R_λ-f-module for some λ, and there are none. There are quasi-injectives, however. In particular, if R is any semiprime qf-ring and f is a homomorphism from the l-submodule M of Q_R into Q_R, then f may be extended to $E(M) \subseteq Q$, by 3.20, and thus it can be extended to Q, since $E(M)$ is a summand of Q. So Q_R is a quasi-injective R-f-module.

The next corollary is an immediate consequence of Weinberg's theorem (Theorem 4.3) and 4.12.

Corollary 4.13. *The following statements are equivalent for a torsion-free f-module M over an irredundant semiprime right qf-ring R* ($\mathrm{card}(R) < \aleph_\alpha$).

(a) *M is $\aleph_\alpha$-injective.*

(b) *M is an injective R-module, and a self-splitting almost-η_α-module in which any two pairwise disjoint subsets of cardinality less than $\aleph_\alpha$ have disjoint upper bounds.*

Finally, we remark that there are enough $\aleph_\alpha$-injectives for embedding purposes when R is an irredundant semiprime right qf-ring. The proof is the same as that given by Weinberg in [25] for $R = \mathbf{Z}$. We can reduce to the case where R is a totally ordered division ring. Then the idea is to enlarge the given vector lattice M, successively, via vector lattices in which every pair of elements of M is split, every pair of disjoint subsets of M of small cardinality have disjoint upper bounds, and in which there is an element between every pair of subsets of M of small cardinality, one of which is smaller than the other. By repeating this procedure inductively one eventually gets a vector lattice having the properties of 4.3(b).

5. **Remarks on an Hahn embedding theorem for f-modules.** Let M be a torsion-free f-module over the irredundant semiprime right qf-ring R, and let $R \subseteq \Pi R_\lambda \subseteq \Pi D_\lambda = Q$ and $M \subseteq \Pi M_\lambda \subseteq \Pi E_\lambda = E$ be the representations of R and M, respectively. Let Γ_λ be the D_λ-value set of E_λ. For each lower submodule $M_\alpha \in \Gamma_\lambda$, let M^α be the convex l-submodule of E_λ that covers it. Since the proof of the Hahn embedding theorem for l-groups [7] is valid for a vector lattice over a totally ordered division ring, E_λ is D_λ-value embedded in the Hahn product $V_\lambda = V(\Gamma_\lambda, M^\alpha/M_\alpha)$. Thus the f-module M_R is embedded in the product of the V_λ. Let Γ be the cardinal sum of the Γ_λ. Then the map $(v_\lambda) \to \bar{v}$, $\bar{v}(\alpha) = v_\lambda(\alpha)$ embeds the Q-f-module ΠV_λ onto the Hahn product $V(\Gamma, M^\alpha/M_\alpha) = V$. ($V$ is a Hahn product as a Q-f-module, i.e. $V^+Q^+ \subseteq V^+$: Suppose $0 < v \in V$ and $0 < q = (q_\lambda) \in Q$. Let α be a maximal element in the support of vq, $\alpha \in \Gamma_\mu$. Then α is a maximal element in the support of v, and $(vq)(\alpha) = v(\alpha)(q_\lambda) = v(\alpha)q_\mu > 0$.)

If M has only a finite number of nonzero components, in particular, if R is a semiprime right Goldie f-ring, then Γ is the Q-value set of $E(M)$, but in general Γ is only contained in the latter. If a component R_λ of R is archimedean, in particular, if R is archimedean, then the M^α/M_α for $\alpha \in \Gamma_\lambda$ are D_λ-submodules of the reals. In this case Γ_λ is isomorphic to the value set of M_λ.

REFERENCES

1. F. W. Anderson, *Lattice-ordered rings of quotients*, Canad. J. Math. 17 (1965), 434–448. MR 30 #4801.

2. G. Birkhoff and R. S. Pierce, *Lattice-ordered rings*, An. Acad. Brasil. Ci. 28 (1956), 41–69. MR 18, 191.

3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956. MR 17, 1040.

4. P. F. Conrad, *Some structure theorems for lattice-ordered groups*, Trans. Amer. Math. Soc. 99 (1961), 212–240. MR 22 #12143.

5. ———, *The lattice of all convex l-subgroups of a lattice-ordered group*, Czechoslovak Math. J. 15 (90) (1965), 101–123. MR 30 #3926.

6. P. F. Conrad and J. E. Diem, *The ring of polar preserving endomorphisms of an abelian lattice-ordered group*, Illinois J. Math. 15 (1971), 222–240.

7. P. F. Conrad, J. Harvey and C. Holland, *The Hahn embedding theorem for abelian lattice-ordered groups*, Trans. Amer. Math. Soc. 108 (1963), 143–169. MR 27 #1519.

8. J. E. Diem, *A radical for lattice-ordered rings*, Pacific J. Math. 25 (1968), 71–82. MR 37 #2653.

9. B. Eckmann and A. Schopf, *Über injektive Moduln*, Arch. Math. (Hochschild) 4 (1953), 75–78. MR 15, 5.

10. C. Faith, *Lectures on injective modules and quotient rings*, Lecture Notes in Math., no. 49, Springer-Verlag, New York, 1967. MR 37 #2791.

11. L. Fuchs, *Teilweise geordnete algebraische Strukturen*, Vandenhoeck and Ruprecht in Gottingen, 1966. MR 34 #4386.

12. A. W. Goldie, *Rings with maximum condition*, Lecture Notes, Yale University, New Haven, Conn., 1964.

13. ———, *Torsion-free modules and rings*, J. Algebra 1 (1964), 268–287. MR 29 #2282.

14. P. Jaffard, *Les systèmes d'idéaux*, Travaux et Recherches Mathématiques, 4, Dunod, Paris, 1960. MR 22 #5628.

15. D. G. Johnson, *A structure theory for a class of lattice-ordered rings*, Acta Math. 104 (1960), 163–215. MR 23 #A2447.

16. D. G. Johnson and J. Kist, *Prime ideals in vector lattices*, Canad. J. Math. 14 (1962), 517–528. MR 25 #2010.

17. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. 2 (1951), 891–895. MR 13, 618.

18. L. Levy, *Unique subdirect sums of prime rings*, Trans. Amer. Math. Soc. 106 (1963), 64–76. MR 26 #136.

19. P. Ribenboim, *On ordered modules*, J. Reine Angew. Math. 225 (1967), 120–146. MR 34 #5877.

20. S. A. Steinberg, *Finitely-valued f-modules*, Pacific J. Math. (to appear).

21. ———, *Lattice-ordered rings and modules*, Dissertation, University of Illinois, Urbana, Ill., 1970.

22. Y. Utumi, *On quotient rings*, Osaka Math. J. 8 (1956), 1–18. MR 18, 7.

23. Diana Yun-dee Wei, *On the concept of torsion and divisibility for general rings*, Illinois J. Math. 13 (1969), 414–431. MR 9, 280.

24. E. C. Weinberg, *Lectures on ordered groups and rings*, Lecture Notes, University of Illinois, Urbana, Ill., 1968.

25. ———, *Relative injectives and universals for categories of ordered structures*, Trans. Amer. Math. Soc. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, ST. LOUIS, MISSOURI 63121

Current address: Department of Mathematics, University of Toledo, Toledo, Ohio 43606