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# ON ANTIFLEXIBLE ALGEBRAS

BY

DAVID J. RODABAUGH

ABSTRACT. In this paper we begin a classification of simple and semisimple totally antiflexible algebras (finite-dimensional) over splitting fields of char. $\neq 2, 3$. For such an algebra $A$, let $P$ be the largest associative ideal in $A^+$ and let $N^+$ be the radical of $P$. We determine all simple and semisimple totally antiflexible algebras in which $N \cdot N = 0$. Defining $A$ to be of type $(m, n)$ if $N^+$ is nilpotent of class $m$ with $\dim A = n$, we then characterize all simple nodal totally antiflexible algebras (over fields of char. $\neq 2, 3$) of types $(n, n)$ and $(n-1, n)$ and give preliminary results for certain other types.

1. **Introduction.** A totally antiflexible algebra is a nonassociative algebra (finite-dimensional) satisfying

$$(x, y, z) = (z, y, x) \quad \text{(the antiflexible law)} \tag{1}$$

and

$$(x, x, x) = 0, \tag{2}$$

where $(x, y, z) = (xy)z - x(yz)$. Throughout this paper we assume char. $\neq 2, 3$ and we define $x \cdot y = (xy + yx)/2$. The algebra $A^+$ is that formed from $A$ with multiplication $x \cdot y$. Define $(x, y) = xy - yx$.

Define $x^1 = x$, $x^{k+1} = x^k x$ and $x^{\cdot 1} = x$, $x^{\cdot k+1} = x^{\cdot k} \cdot x$. It is known that a totally antiflexible algebra $A$ with char. $\neq 0$ need not be power-associative [6]. However $A^+$ is known to be power-associative so $x^{\cdot m} \cdot x^{\cdot n} = x^{\cdot (m+n)}$ for all positive integers $m, n$. We will call $y$ nilpotent or nil if, for some $n$, $y^{\cdot n} = 0$. If $x$ in $A$ implies $x = \alpha 1 + z$ for $\alpha$ in the base field and $z$ nil and if the set of nil elements is not a subalgebra, we say that $A$ is *nodal.*

2. **Preliminaries.** We will state some known results on the structure of simple and semisimple totally antiflexible algebras. We also need (see [1], [7])

**Definition** 2.1. A field $K$ is said to be a splitting field for an algebra $A$ if every primitive idempotent $e$ of $A_K$ is absolutely primitive and if every element in $(A_K)_e(1)$ for $e$ primitive can be written as $ke + y$ with $k$ in $K$ and $y$ nilpotent or $y = 0$.

**Definition** 2.2. Let $A$ be an algebra over a field $F$ of char. $\neq 2, 3$. The mapping $\phi: A \times A \to B$ for $B \subset A$ will be called an antiflexible map provided $B \subseteq \{x: xy = yx \text{ for all } y \text{ in } A\}$ and

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(3) $\phi$ is bilinear over $F$,

(4) $\phi(x, x) = 0$,

(5) $\phi(x^2, x) = 0$,

(6) $\phi(x, y) = 0$ if $y$ is in $B$,

(7) $\phi((x, y), z) = 0$.

This $\phi$ in our definition is similar to maps used in [1], [4], [7]. For char. $\neq 2$, (4) is equivalent to

(8) $\phi(x, y) = -\phi(y, x)$.

Also, for char. $\neq 3$, (5) is equivalent to

(9) $\phi(x \cdot y, z) + \phi(y \cdot z, x) + \phi(z \cdot x, y) = 0$.

For $\alpha, \beta$ in $F$ and antiflexible maps $\phi_1, \phi_2$ define $\alpha\phi_1 + \beta\phi_2$ by

$$(\alpha\phi_1 + \beta\phi_2)(x, y) = \alpha\phi_1(x, y) + \beta\phi_2(x, y).$$

For char. $\neq 2, 3$, it is clear that $\alpha\phi_1 + \beta\phi_2$ is an antiflexible map.

**Definition 2.3.** Let $A$ be an algebra over a field of char. $\neq 2, 3$ and let $\phi$ be an antiflexible map. Define $A(\phi)$ as the algebra formed from $A$ with multiplication replaced by $x * y = xy + \phi(x, y)$.

It is known [4] that $A$ is antiflexible if and only if $A(\phi)$ is. From this, the following lemma is obvious.

**Lemma 2.1.** *Let $A$ be an algebra over a field of char. $\neq 2, 3$ and let $\phi$ be an antiflexible map. Then $A$ is totally antiflexible if and only if $A(\phi)$ is totally antiflexible. Also, if $\psi$ is an antiflexible map on $A(\phi)$ then $A(\phi)(\psi) = A(\phi + \psi)$.*

We now summarize certain results in [1], [4], [7] by the following two theorems.

**Theorem 2.1.** *If $A$ is a simple not associative totally antiflexible algebra over a field $F$ of char. $\neq 2, 3$ then $A^+$ is associative and $A = A_1 + \cdots + A_n$ where $A_i = A_{11}(e_i)$ for $e_i$ primitive. Furthermore, $\phi(x, y) = \frac{1}{2}(x, y)$ is an antiflexible map and $A = A^+(\phi)$.*

**Theorem 2.2.** *If $A$ is a semisimple totally antiflexible algebra over a field $F$ of char. $\neq 2, 3$ then $A = C + D$ where $C = 0$ or $C$ is an associative semisimple ideal with identity $e$ and $D^+$ is associative. If $D \neq 0$ then $D = A_1 + \cdots + A_n$ where $A_i = A_{11}(e_i)$ for $e_i$ primitive, $i \neq n$, and either $A_n = A_{11}(e_n)$ for $e_n$ primitive or $A_n$ is nil and $A_n = A_{00}(e + e_1 + \cdots + e_{n-1})$. Furthermore, if $w, x$ in $C$ and $y, z$ in $D$ define $\phi$ by $\phi(w + y, x + z) = \frac{1}{2}(y, z)$. Then $\phi$ is antiflexible and $A = (C \oplus D^+)(\phi)$.*

We will thus be interested in those algebras from which simple or semisimple algebras can be constructed.

**Definition 2. 4.** A totally antiflexible algebra will be called nearly simple (nearly semisimple) if there is an antiflexible map $\phi$ such that $A(\phi)$ is simple (semisimple).

We will now state some preliminary results on nearly simple and nearly semisimple algebras. Obviously, an associative semisimple algebra $C$ is nearly semisimple.

**Theorem 2.3.** *Let $A$ be a totally antiflexible algebra over a field $F$ of char. $\neq 2, 3$ and let $A = C + D$ where $C$ is a semisimple associative ideal with identity $e$ and $D = A_1 + \cdots + A_n$ with $A_i = A_{11}(e_i)$ for $e_i$ primitive, $i \neq n$, and either $A_n = A_{11}(e_n)$ for $e_n$ primitive or $A_n$ nil and $A_n = A_{00}(e + e_1 + \cdots + e_{n-1})$. Also, assume $D^+$ is associative. Then $A$ is nearly semisimple if and only if $C \oplus D^+$ is nearly semisimple.*

**Proof.** To begin with, let $w$, $x$ be in $C$ and $y$, $z$ in $D$. Define

$$\phi(w + y, x + z) = \tfrac{1}{2}(y, z).$$

We claim that $\phi$ is an antiflexible map. The proof is a routine verification of the conditions of Definition 2.2. Recall also that, in a totally antiflexible algebra, $A_{11}(f)A_{00}(f) = A_{00}(f)A_{11}(f) = 0$ for $f$ an idempotent. Also, $A = (C \oplus D^+)(\phi)$. Now suppose $A$ is nearly semisimple so $A(\psi)$ is semisimple for some $\psi$. Now $A(\psi) = (C \oplus D^+)(\phi + \psi)$ so $C \oplus D^+$ is nearly semisimple. Also, $C \oplus D^+ = A(-\phi)$. Now if $(C \oplus D^+)(\psi)$ is semisimple then $A(\psi - \phi)$ is semisimple.

In a similar way, we can prove

**Theorem 2.4.** *Let $A$ be a totally antiflexible algebra over a field of char. $\neq$ 2, 3 and assume $A^+$ is associative. Then $A$ is nearly simple (nearly semisimple) if and only if $A^+$ is nearly simple (nearly semisimple).*

**Proof.** The only additional fact we need is the fact that $\phi(x, y) = \tfrac{1}{2}(x, y)$ is an antiflexible map on $A$. It is known [3, p. 474] that if $A^+$ is associative and $A$ is antiflexible then $((w, x), y) = 0$. It is then easy to verify the fact that $\phi$ is an antiflexible map.

**Theorem 2.5.** *Let $A$ satisfy the hypotheses of Theorem 2.3 and let $Z$ = center of $C$. Then $A$ is nearly semisimple if and only if $Z \oplus D^+$ is nearly semisimple.*

**Proof.** For any antiflexible map $\phi$ on $A$, $\{\phi(x, y)\} \subseteq \{x: xy = yx$ for all $y$ in $A\}$. Hence, $\{\phi(x, y)\} \cap C \subseteq Z$. The proof is then routine.

We remark that $Z \oplus D^+$ is the largest associative ideal in $A^+$.

The above results reduce the problem of finding all simple (semisimple) algebras to the following two problems:

I. Find all nearly simple (nearly semisimple) associative commutative algebras.

II. Given a nearly simple (nearly semisimple) associative commutative algebra $A$, find all simple (semisimple) algebras that can be constructed, using antiflexible maps, from $A$.

A nearly simple algebra possesses an identity element and the adjunction of an identity element to a nearly semisimple algebra does not destroy its being nearly semisimple. Hence, throughout the rest of this paper, we will assume that each algebra considered has an identity element.

## 3. Conditions on $\phi(x, y)$.

**Theorem 3.1.** *Let $P$ be an associative commutative algebra over a field of char. $\neq 2, 3$ and let $\phi$ be a bilinear map from $P \times P \to B \subset P$ such that $\phi(P, B) = 0$. Then $\phi$ is an antiflexible map if and only if, for every $n$; $y_1, \cdots, y_n$,*

$$\sum_{j=1}^{n} \phi\left(\prod_{i \neq j} y_i, y_j\right) = 0. \tag{10}$$

**Proof.** If $\phi$ satisfies (10) then it must satisfy (4) and (5). Also $(x, y) = 0$ in $P$ so (7) is satisfied and $\phi$ is an antiflexible map. Conversely, let $\phi$ be an antiflexible map. Then, for $n = 1, 2$; $\phi$ satisfies (10). Assume (10) for $n \leq k$ and let $y_1, \cdots, y_{k+1}$ be given. For $z = \Pi_{i=1}^{k-1} y_i$, we have from (9) (since $P$ is commutative)

$$\phi(y_{k+1}y_k, z) + \phi(y_k z, y_{k+1}) + \phi(z y_{k+1}, y_k) = 0. \tag{11}$$

But, using (10) with $n = k$ yields

$$\phi(y_{k+1}y_k, z) = -\phi(z, y_{k+1}y_k) = \sum_{j=1}^{k-1} \phi\left(\prod_{i \neq j} y_i, y_j\right). \tag{12}$$

Putting (12) in (11) yields (10) with $n = k + 1$ and we are done.

Except where otherwise stated, we will assume that $A$ is a totally antiflexible algebra with identity element over a splitting field $K$ of char. $\neq 2, 3$ and that $A^+$ is associative. Hence, $A = A_1 + \cdots + A_n$ with $A_i = A_{11}(e_i)$ for $e_i$ primitive and $A_i A_j = 0$ if $i \neq j$. For, since $A^+$ is associative then $A_{10}(e) + A_{01}(e) = 0$ for any idempotent $e$ (see also [5], [7]). In addition, since $K$ is a splitting field, each element in $A_i$ has the form $\alpha e_i + z$ for $\alpha$ in $K$ and $z$ nil. Thus, $A$ has a basis consisting of primitive idempotents and nil elements. We define the following sets:

(13) $N = \{x: x \text{ is nil}\}$,

(14) $N_i = N_{i-1} \cdot N$ with $N_1 = N$,

(15) $N'_i = N_i - N_{i+1}$ (quotient or difference algebra),

(16) $M_i = \{x: x \cdot N \subseteq M_{i-1}\}$ with $M_0 = 0$.

Define $T_x: y \to y \cdot x$ and note that, since there is an identity element 1 in $A$ and $A^+$ is associative, $x \to T_x$ is an isomorphism of $A^+$ with $\{T_x\}$. Thus, if

$\dim A = n$, we can think of $A^+$ or of one of its subalgebras as an algebra of commutative $n \times n$ matrices.

For some $m$, $N_m = 0$ with $N_{m-1} \neq 0$. We say that $A$ (or $N$) is of type $(m, n)$ if $A^+$ (or $N^+$) is isomorphic to an algebra of commutative $n \times n$ matrices for $n = \dim A$ with $N_m = 0 \neq N_{m-1}$. The algebra $A$ (or $N$) is said to be of class $m$.

**Definition** 3.1. The algebra $A$ (or $N$, the radical of $A^+$) is of type $(m, n, d_1, \cdots, d_q)$ if $A$ (or $N$) is of type $(m, n)$, $\dim N'_i = d_i$ for $1 \leq i \leq q$ and $\dim N'_i = 1$ for $q < i \leq m - 1$.

Note that if $N_i = N_{i+1}$ then $N_i = N_j$ for all $j \geq i$. Hence, either $N_i = 0$ or $\dim N'_i \geq 1$.

**Lemma** 3.1. *The following hold for $x$ in $M_i$, $y$ in $N_j$ and $z$ in $N_{j+1}$ with $j \geq i \geq 1$:*

(a) $x \cdot y = 0$.

(b) *If $\phi$ is an antiflexible map,* $\phi(x, z) = 0$.

**Proof.** The proof of (a) is by induction on $i$. By definition, $M_1 \cdot N_j = 0$. Suppose $M_{i-1} \cdot N_k = 0$ for $k \geq i - 1$ and choose $x$ in $M_i$, $y$ in $N_{j-1}$ and $z$ in $N$ where $j \geq i \geq 1$. Then $x \cdot (y \cdot z) = (x \cdot z) \cdot y = 0$ for $x \cdot z$ is in $M_{i-1}$ and $M_{i-1} \cdot N_{j-1} = 0$. Therefore, $M_i \cdot N_j = 0$. If $\phi$ is an antiflexible map on $A$, we can regard $\phi$ as an antiflexible map on $P = A^+$. Hence, (a) and Theorem 3.1 imply (b).

The results of the following theorem are found in [1], [7].

**Theorem** 3.2. *Let $A$ be a totally antiflexible algebra over a field of char. $\neq$ 2, 3. Then $A$ is simple (semisimple) if and only if $(I, A) \not\subseteq I$ where $I$ is any ideal (nil ideal) of $A^+$.*

**Theorem** 3.3. *Let $A$ be a totally antiflexible algebra over a splitting field of char. $\neq$ 2, 3 with $A^+$ associative. Then $A$ is semisimple if and only if*

(17) *for every nonzero $x$ in $M_1$ there is a $y$ in $N$ with $(x, y) \neq 0$,*

(18) *no nil element in $\{(x, y)\}$ generates a proper nil ideal.*

**Proof.** First, suppose $A$ is semisimple and note that (18) is trivially satisfied. Now, let $J = \{x$ in $M_1$: $(x, y) = 0$ for all $y$ in $N\}$. The algebra $A$ has a basis of idempotents and nil elements. We have $JN = NJ = J \cdot N = 0$. Since $A^+$ is associative, if $e$ is an idempotent, $A = A_{11}(e) + A_{00}(e)$. If $x$ is in $J$ and $y$ is in $N$ then $x = x_1 + x_0$ and $y = y_1 + y_0$ for $x_1$, $y_1$ in $A_{11}(e)$ and $x_0$, $y_0$ in $A_{00}(e)$. The product $xy = 0$ so $0 = xy = x_1y_1 + x_0y_0$ and $x_1y_1 = x_0y_0 = 0$. Hence, $(ex)y = (xe)y = x_1y = x_1y_1 = 0$. Similarly, $y(ex) = y(xe) = 0$ so $(ex, y) = (xe, y) = 0$. Since $J^2 = 0$ then $(ex)^2 = (e \cdot x)^2 = (e \cdot e) \cdot (x \cdot x) = 0$. Thus, $ex = xe$ in $J$ and $J$ is a nil ideal of $A$. We conclude that $J = 0$ so (17) is satisfied.

Conversely, suppose (17) and (18) are satisfied in $A$ and let $J$ be a proper nil ideal of $A$ with $x \neq 0$, $x$ in $J$. We first show $J \cap M_1 \neq 0$. For, if $x$ is not in $M_1$ then, since $M_0 \subset M_1 \subset \cdots \subset M_{n-1} = N$ where $N^{\cdot n} = 0$, we have an integer $i$ with $x$ in $M_i$ but not in $M_{i-1}$. There must be an element $y$ in $N$ such that $x \cdot y$ is in $M_1$ and since $x \cdot y$ is in $J$ we have $M_1 \cap J \neq 0$. Now let $u$ be nonzero in $M_1 \cap J$. By (17) there is a $v$ in $N$ with $z = (u, v) \neq 0$. Clearly, $z$ is in $J$. If $z$ is not nil we contradict the assumption that $J$ is nil. If $z$ is nil and $I$ is the ideal generated by $z$ then $I$ is not nil by (18). However $I \subseteq J$ so $J$ is not nil. We have proved $A$ semisimple.

Define $H(A)$ by

$$H(A) = \{x: (x, y) = 0 \text{ for all } y \text{ in } A\}. \tag{19}$$

In all known examples of semisimple totally antiflexible algebras (see [3], [4], [6]), $H(A) \cap N = 0$. In many of these, $N \cdot N = 0$.

**Corollary 3.1.** *Let $A$ be a totally antiflexible algebra over a field of char. $\neq$ 2, 3 with $A^+$ associative. If either $H(A) \cap N = 0$ or $N \cdot N = 0$ then $A$ is semisimple if and only if $A$ satisfies* (17).

**Proof.** Since $((x, y), z) = 0$ for all $x, y, z$ then $\{(x, y)\} \subseteq H(A)$ and the condition $H(A) \cap N = 0$ implies (18). Suppose $N \cdot N = 0$ and $A$ satisfies (17). Observe that $N = M_1$ so (17) implies that if $x$ is nil then $x$ is not in $H(A)$. Hence, $H(A) \cap N = 0$ and we are done.

**Lemma 3.2.** *If $R$ is a nodal algebra over a field $F$ with $R^+$ power-associative and if $J \neq R$ is an ideal of $R$ then $J$ is nil or zero. Thus $R$ is simple if and only if $R$ is semisimple.*

**Proof.** Let $x$ be a nonnil member of $J$ and write $x = \alpha \cdot 1 + z$ with $z$ nil and $\alpha \neq 0$, $\alpha$ in $F$. Define $u = -(1/\alpha)z$ and define $n$ as an integer with $u^{\cdot n} = 0$. Then $1 = (1 - u) \cdot (1 + u + \cdots + u^{\cdot n-1}) = (1/\alpha)x \cdot (1 + u + \cdots + u^{\cdot n-1})$ is in $J$ so $J = R$.

We shall construct two nodal algebras $A$, $B$ with $A^+ = B^+$ in which $H(A) = H(B)$ contains nil elements. The algebra $B$ satisfies (17) but not (18) and $A$ is simple but $H(A) \cap N \neq 0$. Let $P$ be the associative commutative algebra generated by $1, w, x, y, z$ subject only to the conditions that $w^2 = x^2 = y^2 = z^2 = 0$ and $N \cdot N \cdot N = 0$ where $N$ is generated by $w, x, y, z$ and 1 is the identity element of $P$. Thus, $P$ has a basis $1, w, x, y, z, w \cdot x, w \cdot y, w \cdot z, x \cdot y, x \cdot z, y \cdot z$.

Let $\phi(x, y)$ be defined on this basis by $\phi(z \cdot y, x) = -\phi(x, z \cdot y) = \phi(z \cdot w, y) = -\phi(y, z \cdot w) = -\phi(z \cdot x, y) = \phi(y, z \cdot x) = -\phi(z \cdot y, w) = \phi(w, z \cdot y) = 1$, $\phi(x \cdot y, w) = -\phi(w, x \cdot y) = \phi(y \cdot w, x) = -\phi(x, y \cdot w) = z$, $\phi(w \cdot x, y) = -\phi(y, w \cdot x) = -2z$ and $\phi(u, v) = 0$ where $(u, v)$ is any other pair of basis elements. Extend $\phi$ bilinearly to all of $P \times P$. Now, define $\psi(u, v)$ by $\psi(u, v) = \alpha \cdot 1 + \beta z$ if $\phi(u, v) = \beta \cdot 1 + \alpha z$. Assume char. $\neq 2, 3$ and let $A = P(\phi)$, $B = P(\psi)$. Since $\phi(N, N) = \psi(N, N) \not\subseteq N$, $A$ and $B$ are nodal.

It is verified that $\phi$ and $\psi$ are antiflexible maps by routinely checking (8) and (9). In addition, $H(A) = H(B) = \{\alpha 1 + \beta z \colon \alpha, \beta \text{ in } F\}$ so $H(A) \cap N = H(B) \cap N = \{\beta z \colon \beta \text{ in } F\}$. In both $A$ and $B$, (17) holds. Routinely, we can show that $A$ is simple while $z$, $z \cdot x$, $z \cdot y$ and $z \cdot w$ span a nil ideal of $B$.

**Theorem** 3.4. *Let $A$ be a totally antiflexible algebra over a splitting field $F$ of char. $\neq 2, 3$ with $A^+$ associative. Then $A$ is simple if and only if*

(20) *for every $x$ in $M_1$ there is a $y$ in $N$ with $(x, y) \neq 0$,*

(21) *no element of $\{e(x, y)\}$ generates a proper ideal where $e$ is a primitive idempotent,*

(22) *for each primitive idempotent $e$ in $A$, $\{e(x, y)\}$ is not nil.*

**Proof.** If $A$ is simple then (20) is true from Theorem 3.3 and (21) is obvious. If $e$ is primitive with $\{e(x, y)\}$ nil, recall the fact that $A = A_{11}(e) + A_{00}(e)$ and write $C = (N \cap A_{11}(e)) + A_{00}(e)$. It is routine to check $C \cdot A \subseteq C$. Also, $e$ is in $H(A)$. If $u$ is in $(C, A)$ then $u = eu + u_0$ with $u_0$ in $A_{00}(e)$ and $eu$ in $A_{11}(e)$. Since $\{e(x, y)\}$ is nil, $eu$ is in $N$ so $(C, A) \subseteq C$ and $C$ is a proper ideal of $A$.

Conversely, suppose $A$ satisfies (20), (21) and (22). Let $J$ be a proper ideal and suppose $x \neq 0$, $x$ in $J$. Since $A = A_{11}(e_1) + \cdots + A_{11}(e_n)$ for $e_i$ primitive then $x = x_1 + \cdots + x_n$ for $x_i$ in $A_{11}(e_i)$. We have some $x_i \neq 0$ so $y = x_i = e_i x$ is in $J$. Either $y$ is nil or $y$ is not nil. If $y$ is not nil then $y = \alpha e_i + z$ with $\alpha$ in $F$, $\alpha \neq 0$ and $z$ nil. Write $u = -(1/\alpha)z$ and note that for some $n$, $e_i = (1/\alpha)(e_i - u) \cdot (e_i + u + \cdots + u^{\cdot n})$ is in $J$. Now, for arbitrary $u$, $v$, $e_i(u, v)$ is in $J$. Since $\{e_i(u, v)\}$ is not nil, some $z = e_i(u, v) \neq 0$, and by (21), $z$ in $J$ must generate $A$ so $A = J$.

Now, suppose $y$ is nil. If $y$ is in $M_1$ let $u = y$; if not there is a $z$ with $u = y \cdot z$ in $M_1$: In either case, $u$ is in $J \cap M_1$. Note also that $u$ is in $A_{11}(e_i)$. There is a $v$ such that $(u, v) \neq 0$. From [4], we know that $(u, v)$ is in some $A_{11}(e_j)$ so $e_j(u, v) \neq 0$ and $e_j(u, v)$ is in $J$ so $e_j(u, v)$ generates $A$. Hence $J = A$ and we have proved $A$ simple.

## 4. Algebras with $N \cdot N = 0$.

**Lemma** 4.1. *If $A$ is a semisimple algebra over a splitting field of char. $\neq 2, 3$ with $A^+$ associative and $N \cdot N = 0$ then $\{(x, y)\} \cap N \subseteq H(A) \cap N = 0$.*

**Proof.** We need only note that if $z$ is in $H(A) \cap N$ then $\{\alpha z\}$ is a nil ideal.

We will first be interested in those associative commutative algebras which give rise to nodal simple algebras. The following definition is thus convenient.

**Definition** 4.1. An algebra $A$ will be called nearly nodal if $A = F \cdot 1 + N$ where $F$ is the base field, 1 is the identity of $A$ and $N$ is the set of nil elements of $A$.

Note that a nearly nodal algebra is nodal if and only if $N^2 \not\subseteq N$.

**Theorem** 4.1. *Let $P$ be a nearly nodal associative commutative algebra over a field of char. $\neq 2, 3$ with $N \cdot N = 0$. Let $\{x_i\}_{i=1}^n$ be a basis for $N$. If $\phi$ is an antiflexible map then $P(\phi)$ is simple if and only if there is a nonsingular matrix $X = ((x_{ij}))$ with $\phi(x_i, x_j) = x_{ij} \cdot 1$.*

**Proof.** Suppose $P(\phi)$ is simple. Then $H(P) \cap N = 0$ so $\{\phi(x, y)\} = \{\alpha \cdot 1\} = H(P)$. Hence $\phi(x_i, x_j) = x_{ij} \cdot 1$. Now $y$ in $N$ can be written $y = \sum_{i=1}^n \alpha_i x_i$. By the bilinearity of $\phi$, $\phi(y, x_j) = \sum_{i=1}^n \alpha_i x_{ij} \cdot 1$. Hence, $X$ can be regarded as a linear mapping from $N$ into $V_n(F)$ (space of $n$-tuples over $F$). By Lemma 3.2 and Corollary 3.1, $P(\phi)$ is simple if and only if $y \neq 0$ implies $X(y) \neq 0$. This says that $X$ is nonsingular. Conversely, if $X$ is nonsingular then for each $y$ ther is an $x_j$ with $\phi(y, x_j) = \alpha \cdot 1$, $\alpha \neq 0$. Hence, there can be no ideals in $P(\phi)$.

**Definition** 4.2. If $\phi$ is an antiflexible map from $A \times A \to F \cdot 1$ and if $X = ((x_{ij}))$ such that, for a basis $\{x_i\}_{i=1}^n$ of $N$, $\phi(x_i, x_j) = x_{ij} \cdot 1$ then $X$ is said to represent $\phi$ relative to the basis $\{x_i\}_{i=1}^n$.

**Lemma** 4.2. *Let $\phi$ be an antiflexible map from $A \times A \to F \cdot 1$. Two matrices $X$ and $Y$ represent $\phi$ relative to different bases if and only if they are congruent.*

**Proof.** The proof follows from observing that $\phi$ can be regarded as a bilinear form and then using standard linear algebra results (see [2, pp. 177–180]).

**Theorem** 4.2. *A nodal antiflexible algebra over a field of char. $\neq 2, 3$ and $N \cdot N = 0$ is simple if and only if, for some basis $\{x_i\}_{i=1}^n$ of $N$, $\phi(x, y) = \frac{1}{2}(x, y)$ is represented by the matrix*

$$x = \begin{vmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_k \end{vmatrix}$$

*where* $x_i = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$.

**Proof.** We know that $\phi(x, y)$ is skew-symmetric. It is a well known fact (see Exercise 9, p. 186 in [2]) that any skew-symmetric matrix $C$ is congruent to a matrix having the following diagonal block form:

$$\begin{vmatrix} C_1 & & & 0 \\ & \ddots & & \\ & & C_t & \\ 0 & & & 0 \end{vmatrix}, \quad C_i = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}.$$

Our result then follows from Theorem 4.1.

**Theorem 4.3.** *If $P$ is a nearly nodal associative commutative with $N \cdot N = 0$ algebra over a field of char. $\neq 2, 3$ then $P$ is nearly simple if and only if* $\dim P$ *is odd.*

**Proof.** If $P$ is nearly simple then there is a $\phi$ with $P(\phi)$ simple. Relative to some basis, $\phi$ is represented by the matrix $X$ of Theorem 4.2. Thus, $\dim N$ is even so $\dim P$ is odd. Conversely, the $\phi$ represented by $X$ in Theorem 4.2 yields a simple algebra $P(\phi)$.

**Theorem 4.4.** *Let $P$ be an associative commutative algebra over a splitting field $F$ of char. $\neq 2, 3$ with $N \cdot N = 0$. Then $P$ is nearly simple if and only if*

(23) *there is an identity element in $P$,*

(24) *for every primitive idempotent $e$,* $\dim P_{11}(e) \geq 3$,

(25) *either 1 is not primitive or* $\dim P$ *is odd.*

**Proof.** If $P$ is nearly simple then (23) is satisfied. If 1 is a primitive idempotent, $P$ is nearly nodal and Theorem 4.3 tells us that $\dim P$ is odd. Thus (25) is satisfied. We will now prove (24).

If $e$ is primitive with $\dim P_{11}(e) = 1$ then $P_{11}(e) = \{\alpha e: \alpha \text{ in } F\}$ is an ideal in any algebra $P(\phi)$. If $e = 1$, Theorem 4.3 implies $\dim P_{11}(e) \neq 2$. Suppose $\dim P_{11}(e) = 2$ and $e \neq 1$ and let $A = P(\phi)$. Then $A = A_{11}(e) + A_{00}(e)$ and $A_{11}(e) \cap N = \{\alpha x: \alpha \text{ in } F\}$ for some $x$ with $x^2 = 0$. If $y$ is in $A_{00}(e)$ then $xy = yx = 0$. Hence $A_{11}(e) \cap N$ is a nil ideal in $A$ and $A$ is not simple.

Suppose $P$ satisfies (23), (24), and (25) with 1 not primitive and write $P = P_{11}(e_1) \oplus \cdots \oplus P_{11}(e_q)$ with each $e_i$ primitive. We will define two antiflexible maps $\phi$ and $\psi$ on each $P_{11}(e_m)$ and then extend them bilinearly to all of $P$. Let $\{x_i\}_{i=1}^n$ be a basis for $P_{11}(e_m) \cap N$. If $n$ is even let $n = 2k$ while if $n$ is odd let $n = 2k + 1$. Define

$$\phi(x_i, x_j) = \begin{cases} x_{ij} \cdot 1, & \text{if } i \leq 2k \text{ and } j \leq 2k, \\ 0, & \text{otherwise,} \end{cases}$$

where $X = ((x_{ij}))$ is the matrix of Theorem 4.2. If $n$ is even define $\psi(x_i, x_j) = 0$, while if $n$ is odd define

$$\psi(x_i, x_j) = \begin{cases} e_m, & i = 1,\ j = n, \\ -e_m, & i = n,\ j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Extend $\phi$ and $\psi$ to all of $P_{11}(e_m)$ bilinearly with $\phi(e, N) = \psi(e, N) = \phi(N, e) = \psi(N, e) = 0$.

After extending $\phi, \psi$ to all of $P$ define $A = P(\phi + \psi)$. We claim that $A$ is simple and totally antiflexible. It is clear that $\phi$ and $\psi$ are antiflexible so we need only show that $A$ is simple. Let $J$ be an ideal of $A$, $J \neq 0$. Since $e_i J \subseteq J$ then, for some $m$, $J \cap P_{11}(e_m) \neq 0$. It is easy to show that $J \cap (P_{11}(e_m) \cap N) \neq 0$ so choose $x \neq 0$ in $J \cap P_{11}(e_m) \cap N$. Also, let $\{x_i\}_{i=1}^n$ be a basis for $P_{11}(e_m) \cap N$ with $k$ as previously defined. Now, $x = \sum_{i=1}^n \alpha_i x_i$. If $n = 2k$ it is clear that there is a $y$ with $\phi(x, y) = 1$ so $J = A$ so we will assume that $n + 2k + 1$. If $\phi(x, x_j) = 0$ for $j \leq 2k$ then, since $X$ is nonsingular, $\alpha_i = 0$ for $i \leq 2k$. Hence $x = \alpha_n x_n$ with $\alpha_n \neq 0$ and $e_m = \psi(x_1/\alpha_n, x)$ is in $J$. However, $x_1 = e_m x_1$ is then in $J$ so $1 = \phi(x_2, x_1)$ is in $J$. Consequently $J = A$ and we have proved $A$ simple.

**Theorem 4.5.** *Let $P$ be an associative commutative algebra over a splitting field $F$ of char. $\neq 2, 3$ with $N \cdot N = 0$. Then $P$ is nearly semisimple if and only if*

(26) *$P$ is not nil,*

(27) *for every primitive idempotent $e$,* $\dim P_{11}(e) \neq 2$,

(28) *$e$ principal implies* $\dim P_{00}(e) \neq 1$,

(29) *$e$ principal and primitive implies* $\dim P_{11}(e)$ *is odd and* $\dim P_{00}(e)$ *is even.*

**Proof.** If $e$ is primitive then $\dim(P_{11}(e) \cap N) = \dim P_{11}(e) - 1$. Let $P$ be nearly semisimple so that some $P(\phi)$ is semisimple. Clearly, (26) is satisfied. If dim $P_{11}(e) = 2$ then, as above, $P_{11}(e) \cap N$ is a nil ideal of $P(\phi)$. Thus (27) holds. If $e$ is principal and not the identity element then adjoin an identity element 1 to $P(\phi)$. It is routine to show $1 - e$ primitive and the algebra formed is semisimple. Hence, (28) is true. Now, if $e$ is primitive and principal with $P_{00}(e) = 0$ then $P(\phi)$ is nodal and simple so $\dim P_{11}(e)$ is odd. Suppose $e$ is primitive and principal with $P_{00}(e) \neq 0$ and let $A = P(\phi)$ be semisimple. We know $A_{11}(e)A_{00}(e) = A_{00}(e)A_{11}(e) = 0$ so $\phi(A_{11}(e), A_{00}(e)) = \phi(A_{00}(e), A_{11}(e)) = 0$. By Lemma 4.1, $\{\phi(x, y)\} \subseteq \{\alpha e\}$. Thus for $x$, $y$ in $A_{11}(e)$, $\phi(x, y) = \alpha_{xy} e$ and the restriction of $\phi$ to $A_{11}(e) \cap N$ yields a mapping $S$ from $(A_{11}(e) \cap N) \times (A_{11}(e) \cap N)$ to $F$. By Theorem 3.3 and the fact that $\phi(A_{11}(e), A_{00}(e)) = \phi(A_{00}(e), A_{11}(e)) = 0$, $S$ is nonsingular and $\dim(P_{11}(e) \cap N)$ is even. Similarly, $\dim P_{00}(e) = \dim(P_{00}(e) \cap N)$ is even. This establishes (29).

Conversely, let $P$ satisfy (26), (27), (28) and (29). Since $P$ is associative and commutative,

$$P = P_{11}(e_1) \oplus \cdots \oplus P_{11}(e_q) \oplus P_{00}(e)$$

where each $e_i$ is primitive and $e = e_1 + \cdots + e_q$ is principal. Of course, $P_{00}(e)$ may be zero. As before, we will define two antiflexible maps $\phi$ and $\psi$ on each $P_{11}(e_m)$ and on $P_{00}(e)$ and then extend them bilinearly to all of $P$. Let $\{x_i\}_{i=1}^n$

be a basis for $P_{11}(e_m) \cap N$ or $P_{00}(e)$. If $n$ is even let $n = 2k$ and if $n$ is odd let $n = 2k + 1$. Define

$$\phi(x_i, x_j) = \begin{cases} x_{ij}e & \text{if } i \leq 2k \text{ and } j \leq 2k, \\ 0, & \text{otherwise,} \end{cases}$$

where $X = ((x_{ij}))$ is the matrix of Theorem 4.2. If $n$ is even, define $\psi(x_i, x_j) = 0$; otherwise

$$\psi(x_i, x_j) = \begin{cases} e, & i = 1, j = n, \\ -e, & i = n, j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Extend $\phi$ and $\psi$ to all of $P_{11}(e_m)$ or $P_{00}(e)$ bilinearly.

After extending $\phi, \psi$ to all of $P$, define $A = P(\phi + \psi)$. We claim that $A$ is semisimple and totally antiflexible. Clearly, $A$ is totally antiflexible. If $x$ is nonzero in $N$, $x = x_1 + \cdots + x_q + x_0$; $x_i$ in $P_{11}(e_i) \cap N$ for $i > 0$, $x_0$ in $P_{00}(e)$. If $x_j \neq 0$ then there is a $y$ in $P_{11}(e_j)$ if $j > 0$ or $P_{00}(e)$ if $j = 0$ such that $\phi(x_j, y) \neq 0$. Since $\phi(x_i, y) = 0$, $i \neq j$, we have $\phi(x, y) \neq 0$. Since $\{\phi(x, y)\}$ contains no nil elements, Corollary 3.1 implies $A$ is semisimple.

These two theorems characterize those associative commutative algebras that are either nearly simple or nearly semisimple when $N \cdot N = 0$. Also, Theorems 4.1 and 4.3 characterize the nodal simple antiflexible algebras in which $N \cdot N = 0$.

5. **Nodal algebras of type $(n, n)$ and $(n-1, n)$.** We now focus attention on nodal algebras. If $A$ is such an algebra then $\dim A = 1 + \dim N$. The following is immediate from Theorem 3.1.

**Lemma 5.1.** *If $\phi$ is an antiflexible map on an associative commutative algebra $P$ of char. $\neq 2, 3$ then for $x$ in $P$ and integers $n$, $\alpha$ with $n \geq \alpha \geq 1$, $\phi(x^{n-\alpha}, x^{\alpha}) = \alpha\phi(x^{n-1}, x)$ and $n\phi(x^{n-1}, x) = 0$.*

**Theorem 5.1.** *Suppose $N$ is an associative commutative nilpotent algebra over a field $F$. If $\dim N_i' = 1$ then there is an $x$ in $N_{i-1}$ (if $i = 1$, set $x = 1$ in $F$) and an $a$ in $N$ such that $xa$ is not in $N_{i+1}$. If $x$ in $N_{i-1}$ (if $i = 1$, $x$ in $F$) and $a$ in $N$ are such that $xa$ is not in $N_{i+1}$ and if $c$ is in $N_j$ for $j \geq i$ then $c = \alpha x a^{j-i+1} + n$ for $\alpha$ in $F$ and $n$ in $N_{j+1}$. Furthernore, if $j \geq i$ then $\dim N_j' = 1$ or $N_j = 0$.*

**Proof.** Since $\dim N_i' = 1$ then there is a $y$ in $N_i$ such that if $c$ is in $N_i$ then $c = \alpha y + n$ for $\alpha$ in $F$ and $n$ in $N_{i+1}$. By definition, $N_i = N_{i-1}N$ so there is an $x$ in $N_{i-1}$ and $a$ in $N$ with $xa$ not in $N_{i+1}$. That is, $xa = \beta y + n_1$ with $\beta \neq 0$, $\beta$ in $F$ and $n_1$ in $N_{i+1}$. Clearly, $c = (\alpha/\beta)xa + n - (\alpha/\beta)n_1$ and $n - (\alpha/\beta)n_1$ is in $N_{i+1}$. Such a formula holds for any $x$ in $N_{i-1}$, $a$ in $N$ with $xa$ not in $N_{i+1}$.

We fix $x$ and $a$ and note that the general result holds for $j = i$. Suppose it holds for $j = k$. If $d$ is in $N_{k+1}$ then $d = \sum_{m=1}^{s} \beta_m c_m b_m$ for $c_m$ in $N_k$, $b_m$ in $N$ and $\beta_m$ in $F$. However, $c_m = \gamma_m xa^{k-i+1} + n_m$ for $\gamma_m$ in $F$ and $n_m$ in $N_{k+1}$. We now observe that $xb_m$ is in $N_i$ so $xb_m = \delta_m xa + n'_m$ with $\delta_m$ in $F$ and $n'_m$ in $N_{i+1}$. Thus

$$
\begin{aligned}
d &= \sum_{m=1}^{s} \beta_m c_m b_m \\
&= \sum_{m=1}^{s} \beta_m \gamma_m x b_m a^{k-i+1} + \sum_{m=1}^{s} \beta_m n_m b_m \\
&= \sum_{m=1}^{s} \beta_m \gamma_m \delta_m x a^{k-i+2} + \sum_{m=1}^{s} (\beta_m \gamma_m n'_m a^{k-i+1} + \beta_m n_m b_m).
\end{aligned}
$$

However, $n'_m a^{k-i+1}$ and $n_m b_m$ are each in $N_{k+2}$ so $d = \alpha' x a^{k-i+2} + n'$ for $\alpha'$ in $F$ and $n'$ in $N_{k+2}$. Finally, if $j \geq i$, either $N_j = 0$ or $\dim N'_j = 1$.

The following follows from a footnote in [8, p. 10].

**Lemma 5.2.** *If $N$ is an associative commutative nilpotent algebra of class $k$ over a field $F$ of char. 0 or char. $\geq k$ then there is an $x$ in $N$ such that $x^{k-1} \neq 0$.*

**Lemma 5.3.** *If $N$ is an associative commutative nilpotent algebra of class $k$ with* $\dim N'_m = 1$ *over a field $F$ of char. $> m$ then there is an $x$ in $N$ with $x^{k-1} \neq 0$.*

**Proof.** Write $Q = N - N_{m+1}$ and note that $Q$ is of class $m + 1$. Let $[x]$ be the image of $x$ in $N$ in the natural map from $N \to N - N_{m+1}$. Since char $F \geq m + 1$, there is an element $[y]$ in $Q$ with $[y]^m \neq 0$. Thus, $y^m$ is not in $N_{m+1}$. If $m = 1$, set $x = 1$ while, if $m > 1$, set $x = y^{m-1}$. In either case define $a = y$. We have $x$ in $N_{m-1}$ and $a$ in $N$ with $xa$ not in $N_{m+1}$. Since $N_{k-1} \neq 0$ there is a nonzero element $c$ in $N_{k-1}$. From Theorem 5.1 and the fact that $N_k = 0$, $c = \alpha x a^{k-m} = \alpha y^{k-1}$. Therefore, $y^{k-1} \neq 0$.

**Theorem 5.2.** *Suppose $N$ is an associative commutative nilpotent algebra of dimension $n - 1$ over a field $F$. If $N$ is of class $k$ and if* char $F = 0$ *or* char $F$ $\geq k$ *or* char $F \geq n - k + 2$ *then there is an $x$ in $N$ with $x^{k-1} \neq 0$.*

**Proof.** By Lemma 5.2, we need only consider the case $k >$ char $F \geq n - k + 2$. By Lemma 5.3, it is sufficient to have $\dim N'_m = 1$ where $m = n - k + 1$. Assume $\dim N'_i > 1$ for $i \leq m$ so $\dim N'_i \geq 2$ for $i \leq m$. Now, $n - 1 = \dim N'_1 + \cdots + \dim N'_{k-1} \geq 2m + \dim N'_{m+1} + \cdots + \dim N'_{k-1} \geq 2m + (k - m - 1) = m + k - 1 = n$ which is impossible. Thus, $\dim N'_m = 1$.

**Theorem 5.3.** *If $N$ is of type $(n, n)$ then there is an element $a$ in $N$ such that $N$ is spanned by $a, a^2, \cdots, a^{n-1}$.*

**Proof.** When $k = n$, the hypotheses of Theorem 5.2 are always satisfied. Theorem 5.3 becomes a corollary to Theorem 5.2.

As an immediate corollary to Lemma 5.1, we have

**Lemma 5.4.** *If $\phi$ is an antiflexible map on an associative commutative algebra $P$ of char. $\neq 2, 3$ then for $x$ in $P$ and integers $n$, $\alpha$ with $n \geq \alpha \geq 1$, $\phi(x^{n-\alpha}, x^\alpha) = 0$ if $n \not\equiv 0 \pmod{p}$.*

**Theorem 5.4.** *Let $P = F \cdot 1 \oplus N$ where $N$ is an associative commutative nilalgebra of type $(n, n)$ over a field $F$ of char. $\neq 2, 3$. The algebra $P$ is nearly simple if and only if* char *$F$ divides $n$.*

**Proof.** We first assume $P$ is nearly simple. By Theorem 5.3, there is an element $a$ in $N$ with $N$ spanned by $a, a^2, \cdots, a^{n-1}$. It is easy to verify each $M_i$ is spanned by $a^{n-i}$ and each $N_i$ is spanned by $a^i, \cdots, a^{n-1}$. Now, Lemma 3.1 implies $\phi(a^i, a^j) = 0$ whenever $i + j > n$. Theorem 3.3 (or 3.4) then states $\phi(a^{n-1}, y) \neq 0$ for some $y = \alpha_1 a + \cdots + \alpha_{n-1} a^{n-1}$. We conclude $\phi(a^{n-1}, a) \neq 0$. However $n\phi(a^{n-1}, a) = 0$ so $n \equiv 0 \pmod{p}$ for $p =$ char $F$.

Now, assume $n = kp$ for $p =$ char $F$. If we define

$$\phi(a^i, a^j) = \begin{cases} 0, & i + j \neq n, \\ j, & i + j = n, \end{cases}$$

then the proof in [6] for the case $k = 1$ will generalize and $P(\phi)$ is a simple nodal algebra.

We have determined all nearly simple associative commutative algebras of class 2. In classifying nearly simple associative commutative algebras of type $(m, n)$, we can assume $m \geq 3$.

Having determined nearly simple associative commutative algebras of type $(n, n)$, our next interest is those of type $(n-1, n)$. If $\dim N'_1 = 1$ then $\dim N'_i = 1$ for all $i \leq n-2$ so that $\dim N = n-2$. Since $\dim N = n-1$ we conclude $\dim N'_1 = 2$ and $\dim N'_i = 1$ for $2 \leq i \leq n-2$. We have proved the following lemma.

**Lemma 5.5.** *If $N$ is of type $(n-1, n)$ then $N$ is of type $(n-1, n, 2)$.*

**Theorem 5.5.** *Let $N$ be an associative commutative nilalgebra of type $(n-k, n, k+1)$ over a field $F$ with char. $\neq 2$. If $n \geq k+3$ then there are elements $a$, $b_i$, $i = 1, \cdots, k$, with $N$ spanned by $a, \cdots, a^{n-k-1}, b_1, \cdots, b_k$; $ab_i = 0$; $b_i^2 = \alpha_i a^{n-k-1}$; $b_i b_j = \lambda_{ij} a^{n-k-1}$.*

**Proof.** By Lemma 5.3, since char $F \neq 2$ and $\dim N'_2 = 1$, there is an element

$a$ in $N$ with $a^{n-k-1} \neq 0$. Let $c_1, \cdots, c_k$ be chosen so they are not in $N_2$ and $a, \cdots, a^{n-k-1}, c_1, \cdots, c_k$ are a basis for $N$. (This is possible since $\dim N'_1 = k+1$.) We know $a^2, \cdots, a^{n-k-1}$ form a basis for $N_2$,

$$ac_j = \sum_{i=2}^{n-k-1} \beta_{ij} a^i = a\left(\sum_{i=2}^{n-k-1} \beta_{ij} a^{i-1}\right), \quad j = 1, \cdots, k.$$

Define $b_j = c_j - \sum_{i=2}^{n-k-1} \beta_{ij} a^{i-1}$, $j = 1, \cdots, k$. Clearly $ab_j = 0$, $j = 1, \cdots, k$. Now, $b_j^2$ is in $N_2$ so $b_j^2 = \sum_{i=2}^{n-k-1} \gamma_{ij} a^i$ and

$$0 = (ab_j)b_j = ab_j^2 = \sum_{i=2}^{n-k-2} \gamma_{ij} a^{i+1}.$$

Hence, $\gamma_{ij} = 0$ for $j = 1, \cdots, k$ and $i = 2, \cdots, n-k-2$. Defining $\alpha_j = \gamma_{n-k-1,j}$ we have $b_j^2 = \alpha_j a^{n-k-1}$.

**Lemma** 5.6. *If $\phi$ is an antiflexible map on an associative commutative algebra $P$ of char. $\neq 2, 3$ in which $ab = 0$ then $\phi(a^r, b^s) = 0$ if $r > 1$ or $s > 1$.*

**Proof.** If $r > 1$ then

$$\phi(a^r, b^s) + \phi(b^s a, a^{r-1}) + \phi(b^s a^{r-1}, a) = 0.$$

Since $b^s a = b^s a^{r-1} = 0$, $\phi(a^r, b^s) = 0$. The proof when $s > 1$ is similar.

**Theorem** 5.6. *Let $P = F \cdot 1 \oplus N$ where $N$ is an associative commutative nilalgebra of type $(n-k, n, k+1)$ with $n - k > 2$ over a field $F$ of char. $\neq 2, 3$. The algebra $P$ is nearly simple if and only if the following hold:*

(a) *$N$ is spanned by $a, \cdots, a^{n-k-1}, b_1, \cdots, b_k$ where $ab_i = b_i^2 = b_i b_j = 0$, $i, j = 1, \cdots, k$.*

(b) *Either $n - k = \text{char } F$ with $k$ even or $n - k = m \text{ char } F$ for $m > 1$.*

**Proof.** By Theorem 5.5, there are elements $a, b_1, \cdots, b_k$ with $N$ spanned by $a, \cdots, a^{n-k-1}, b_1, \cdots, b_k$. Furthermore, $ab_i = 0$, $b_i^2 = \alpha_i a^{n-k-1}$, $b_i b_j = \lambda_{ij} a^{n-k-1}$ for all $i, j$ where each $\alpha_i, \lambda_{ij}$ is in $F$. From this, it is clear that $M$ is a subspace of the space spanned by $a^{n-k-1}, b_1, \cdots, b_k$.

Assume $P$ is nearly simple. Then there is a $\phi$ with $P(\phi)$ simple. We first show that each $b_i$ is in $M$. To do this, it is necessary and sufficient to prove that each $\alpha_i = 0$ and each $\lambda_{ij} = 0$. If $x \neq 0$ is in $M$, Theorem 3.4 assures the existence of a $y$ in $N$ with $\phi(x, y) \neq 0$. Thus, if $x$ in $M$ has the property that $\phi(x, y) = 0$ for all $y$ in $N$ then $x = 0$. Since $a^{n-k-1}$ is in $M$, each $b_i^2$ and each $b_i b_j$ are in $M$.

Lemma 5.6 implies $\phi(b_i^2, a^j) = 0$ for all $i, j$. Since $n - k - 1 > 1$, $\phi(b_i^2, b_j) = \alpha_i \phi(a^{n-k-1}, b_j) = 0$ for all $i, j$ (also by Lemma 5.6). Thus, for each $i$, $\phi(b_i^2, y) = 0$ for each $y$ in $N$ so $b_i^2 = 0$. If $p > 1$, $a^p$ is in $N_2$ so $\phi(b_i b_j, a^p) =$

$\lambda_{ij}\phi(a^{n-k-1}, a^p) = 0$ by Lemma 3.1. From Lemma 5.6, since $n-k-1 > 1$, we derive $\phi(b_ib_j, b_p) = \lambda_{ij}\phi(a^{n-k-1}, b_p) = 0$. Finally by Theorem 3.1, $\phi(b_ib_j, a) = -\phi(b_ia, b_j) - \phi(b_ja, b_i) = 0$. We conclude $b_ib_j = 0$ and have shown that $M$ is spanned by $a^{n-k-1}, b_1, \cdots, b_k$.

Since $a^{n-k-1}$ is in $M$ there is a $y$ in $N$ with $\phi(a^{n-k-1}, y) \neq 0$. If $p > 1$, $a^p$ is in $N_2$ so $\phi(M, a^p) = 0$. Also, as above, for each $i$, $\phi(a^{n-k-1}, b_i) = 0$. We conclude $\phi(a^{n-k-1}, a) \neq 0$. From Lemma 5.1, $(n-k)\phi(a^{n-k-1}, a) = 0$ so char $F$ divides $n-k$. We further note that $\phi(a^{n-k-\alpha}, a^\alpha) = \alpha\phi(a^{n-k-1}, a)$ so, for $\alpha$ not divisible by char $F$, $\phi(a^{n-k-\alpha}, a^\alpha) \neq 0$.

Define $q =$ char $F$ and assume $n-k = q$. Since $P$ is spanned by $1, a, \cdots, a^{n-k-1}, b_1, \cdots, b_k$, if $x$ and $y$ are arbitrary,

$$\phi(x, y) = \delta_0 + \sum_{i=1}^{q-1} \delta_i a^i + \sum_{i=1}^{k} \gamma_i b_i.$$

From (6), we know that, for any $z$, $\phi(\phi(x, y), z) = 0$. Lemma 5.1 implies $\phi(a^i, a^j) = j\phi(a^{i+j-1}, 1)$ and $(i+j)\phi(a^{i+j-1}, a) = 0$. Hence, $\phi(a^i, a^j) = 0$ unless $i + j = q$. For $s > 1$, Lemma 5.6 implies $\phi(\sum_{i=1}^k \gamma_i b_i, a^s) = 0$. Thus, for $s > 1$,

$$0 = \phi(\phi(x, y), a^s) = \sum_{i=1}^{q-1} \delta_i\phi(a^i, a^s)$$

$$= \delta_{q-s}\phi(a^{q-s}, a^s) = s\delta_{q-s}\phi(a^{q-1}, a).$$

Since $1 < s < q$ and $\phi(a^{q-1}, a) \neq 0$, $\delta_{q-s} = 0$. Letting $y_1 = \delta_{q-1}a^{q-1} + \sum_{i=1}^k \gamma_i b_i$, we have $\phi(x, y) = y_1 + \delta_0$ with $y_1$ in $M$. Now, for any $z$, $0 = \phi(\phi(x, y), z) = \phi(y_1, z) + \phi(\delta_0, z) = \phi(y_1, z)$. If $y_1 \neq 0$ there must be a $z$ in $N$ with $\phi(y_1, z) \neq 0$ so we conclude $y_1 = 0$ and $\phi(x, y)$ is in $F \cdot 1$.

We know that for $b = \sum_{i=1}^k \eta_i b_i \neq 0$ there is a $z$ in $N$ with $\phi(b, z) \neq 0$. Since $ab = 0$, $\phi(b, a^s) = 0$ when $s > 1$. Thus, if $b$ satisfies $\phi(b, b_j) = 0$ for all $j$ then $\phi(b, a) \neq 0$. If we write $\phi(b, a) = \beta_1 \neq 0$ and $\phi(a^{q-1}, a) = \beta_2 \neq 0$, we have shown $\beta_1$ and $\beta_2$ to be in $F \cdot 1$. Define $x = \beta_2 b - \beta_1 a^{q-1}$ and verify $\phi(x, z) = 0$ for all $z$ in $P$. However, the simplicity of $P(\phi)$ implies $\phi(x, z) \neq 0$ for some $z$. We have proved that for any $b$ there is a $j$ with $\phi(b, b_j) \neq 0$.

Define $\beta_{ij} = \phi(b_i, b_j)$, $i, j = 1, \cdots, k$. For any set $\eta_1, \cdots, \eta_k$ there is a $j$ with $\sum_{i=1}^k \eta_i\beta_{ij} = \sum_{i=1}^k \eta_i\phi(b_i, b_j) \neq 0$. Defining $B$ as the matrix $((\beta_{ij}))$ we conclude that $B$ is a nonsingular matrix. If we let $Q = F \cdot 1 \oplus F \cdot b_1 \oplus \cdots \oplus F \cdot b_k$ and let $\phi' = \phi$ restricted to $Q$, then $Q(\phi')$ is a nodal simple subalgebra of $P(\phi)$. Since $Q(\phi')$ is of class 2, Theorem 4.3 says that $\dim Q$ is odd so $k$ is even.

For the converse, first assume that $P$ satisfies (a) with $k$ even. Write $r = n - k$. Define $\phi$ on the basis as follows:

(30) $$\phi(a^i, a^j) = \begin{cases} 0, & i+j \neq r, \\ j, & i+j = r. \end{cases}$$

(31) $\phi(b_i, b_j) = x_{ij} \cdot 1$ where $X = ((x_{ij}))$ is the matrix of Theorem 4.2.

(32) $\phi(a^s, b_i) = \phi(b_i, a^s) = 0$ for all $i, s \geq 1$.

(33) $\phi(1, x) = \phi(x, 1) = 0$ for all $x$ in $P$.

Extend $\phi$ bilinearly to $P \times P$. Let $x = \delta_0 + \sum_{i=1}^{r-1} \delta_i a^i + \sum_{i=1}^{k} \gamma_i b_i$ be a nonzero element in an ideal $J$ of $P(\phi)$. If some $\gamma_i \neq 0$ then there is a $j$ with $\phi(x, b_j) = \pm\gamma_i$ in $J$. If each $\gamma_i = 0$ then, for some $j$, $\delta_j \neq 0$. If $\delta_j \neq 0$ and char $F$ divides $j$ (or $j = 0$) then $\phi(xa, a^{r-j-1}) = (-j-1)\delta_j = -\delta_j$ is in $J$. If $\delta_j \neq 0$ with $j$ and char $F$ relatively prime then $\phi(x, a^{r-j}) = -j\delta_j \neq 0$ in in $J$. In any case, $F \cdot 1 \subseteq J$ so $J = P(\phi)$.

Now, suppose $P$ satisfies (a) with $k = h + 1$ odd so $h$ is even. Write $r = n - k$. We know $r = m$ char $F$ with $m > 1$. Let $q =$ char $F$. Define $\phi$ on the basis as follows:

(34) $$\phi(a^i, a^j) = \begin{cases} 0, & i+j \neq r, \\ j, & i+j = r. \end{cases}$$

(35) $\phi(b_i, b_j) = x_{ij} \cdot 1$ for $i, j = 1, \cdots, h$ where $X = ((x_{ij}))$ is the matrix of Theorem 4.2.

(36) $\phi(b_k, a) = -\phi(a, b_k) = a^q$.

(37) $\phi(a^s, b_i) = \phi(b_i, a^s) = 0$ unless $s = 1$ and $i = k$.

(38) $\phi(1, x) = \phi(x, 1) = 0$ for all $x$ in $P$.

Extend $\phi$ bilinearly to $P \times P$. It is straightforward to verify $P(\phi)$ is simple.

As an immediate corollary, we have

**Corollary 5.** *Let $P = F \cdot 1 \oplus N$ where $N$ is an associative commutative nil-algebra of type $(n-1, n)$ with $n - 1 > 2$ over a field $F$ of char. $\neq 2, 3$. The algebra $P$ is nearly simple if and only if $N$ is spanned by $a, \cdots, a^{n-2}, b$ where $ab = b^2 = 0$ and $n - 1 = m$* char *$F$ with $m > 1$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65201