WALL MANIFOLDS WITH INVOLUTION

BY

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ABSTRACT. Consider smooth manifolds W with involution t and a Wall structure described by a map $f: W \rightarrow S^1$ such that $ft = f$. For such objects we define cobordism theories $\mathbb{W}_*^I$ (in case W is closed, t unrestricted), $\mathbb{W}_*^F$ (for W closed, t fixed-point free), and $\mathbb{W}_*^{\text{rel}}$ (W with boundary, t free on W). We prove that there is an exact sequence

$$0 \rightarrow \mathbb{W}_*^I \rightarrow \mathbb{W}_*^{\text{rel}} \rightarrow \mathbb{W}_*^F \rightarrow 0.$$

As a corollary, $\mathbb{W}_*^I$ imbeds in the cobordism of unoriented manifolds with involution. We also describe how $\mathbb{W}_*^I$ determines the 2-torsion in the cobordism of oriented manifolds with involution.

1. Introduction. C. T. C. Wall defined the class of manifolds now named for him in order to compute the 2-torsion of oriented cobordism [8]. In this paper we will show how a similar construction determines, in principle, the 2-torsion of the cobordism of oriented manifolds with involution.

Let W be a smooth, unoriented manifold. A *Wall structure* on W is a homotopy class of maps $f: W \rightarrow S^1$ such that $f^*(\iota)$ is the Stiefel-Whitney class $w_1(W)$, where $\iota \in H^1(S^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is the generator. Equivalently, a Wall structure is given by a class $x \in H^1(W; \mathbb{Z})$ whose reduction mod 2 is $w_1(W)$. If $t: W \rightarrow W$ is a smooth involution, we say that (W, t, f) is a *Wall manifold with involution* provided $f(w) = f(t(w))$ for all $w \in W$. We define a cobordism theory $\mathbb{W}_*^I$ for such objects, a similar theory $\mathbb{W}_*^F$ for Wall manifolds with free involution, and a theory $\mathbb{W}_*^{\text{rel}}$ for Wall manifolds with boundary and an involution free on the boundary. The major result is that there is an exact sequence

$$0 \rightarrow \mathbb{W}_*^I \rightarrow \mathbb{W}_*^{\text{rel}} \rightarrow \mathbb{W}_*^F \rightarrow 0.$$

By comparison with a theorem of Conner and Floyd [3, p. 73] this proves that $\mathbb{W}_*^I$ imbeds in the cobordism $I_*(Z_2)$ of manifolds with involution.

In the last section we will define a cobordism theory $\mathbb{O}_*(Z_2)$ of oriented manifolds with involution, which contains as a summand the theory $\mathbb{C}_*(Z_2)$ studied by

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Rosenzweig [5] and Conner [2]. There is a Rohlin sequence

$$\begin{array}{ccc} W^I & \longrightarrow & O_*(Z_2) \\ & \nwarrow^* & \searrow_2 \\ & O_*(Z_2) & \end{array}$$

which is exact at the upper corners (but not at the lower one). This outlines a possible calculation of the 2-torsion of $O_*(Z_2)$, although the practical difficulties remain considerable.

2. Involutions and Wall structures. Let $(W, f: W \rightarrow S^1)$ be a Wall manifold, and let t be an involution on W , with $\pi: W \rightarrow W/t$ as quotient map. If $x \in H^1(W; Z)$ is the class whose mod 2 reduction is $w_1(W)$, we have this equivalent to our definition of a structure-preserving map:

Proposition 2.1. *(W, t, f) is a Wall manifold with involution if and only if there is a class $y \in H^1(W/t; Z)$ such that $\pi^*(y) = x$.*

The proof is trivial, since both conditions are equivalent to the existence of a map $b: W/t \rightarrow S^1$ such that $f \cong b\pi$.

Next, suppose that t is fixed point free. Then W/t is a smooth manifold with boundary $(\partial W)/t$, and π is a principal Z_2 -bundle. If $y \in H(W/t; Z)$ meets the conditions of (2.1), then

$$\begin{aligned} \pi^*(w_1(W/t)) &= w_1(W), \\ \pi^*(\rho(y)) &= \rho(\pi^*(y)) = w_1(W), \quad \rho = \text{mod } 2 \text{ reduction.} \end{aligned}$$

It does not follow that $w_1(W/t) = \rho(y)$, but recall that in the Gysin sequence of π there is an exact sequence

$$H^0(W/t; Z_2) \rightarrow H^1(W/t; Z_2) \xrightarrow{\pi^*} H^1(W; Z_2),$$

where the left map is multiplication by the characteristic class $w_1(\pi)$. This proves

Proposition 2.2. *If W/t is connected, then either $\rho(y) = w_1(W/t)$ or else $\rho(y) = w_1(W/t) + w_1(\pi)$.*

This provides a tool for studying W/t , which need not be a Wall manifold.

Our cobordism theories are established by standard arguments. Let $i: \partial W \subset W$, and recall that $w_1(\partial W) = i^*w_1(W)$. The boundary of a Wall manifold with involution (W, t, g) is the Wall manifold with involution (W, ti, gi) . Two closed, n -dimensional Wall manifolds with involution are *cobordant* if their disjoint union bounds a Wall manifold with involution. It is easy to check that this defines an equivalence relation, and that the set W_n^I of equivalence classes is a Z_2 -vector space

with operation induced by disjoint union. Further, the direct sum $\mathbb{W}_*^I$ of the $\mathbb{W}_n^I$ is a module over the Wall cobordism ring $\mathbb{W}_*$, with multiplication induced by Cartesian product, and by use of the multiplication $S^1 \times S^1 \rightarrow S^1$ of complex numbers of norm 1. A similar argument, with all involutions required to be free, produces the cobordism $\mathbb{W}_*^F$ of Wall manifolds with free involution.

Finally, if (W_1, t_1, g_1) and (W_2, t_2, g_2) are n -dimensional Wall manifolds with involutions t_i free on ∂W_i , then they are cobordant if there is a Wall manifold with involution (V, u, b) , having $W_1 \cup_{\partial} W_2$ as a submanifold of ∂V , u free on $\partial V - \text{int}(W_1 \cup W_2)$, $u|_{W_i} = t_i$ and $b|_{W_i} = g_i$. This is also an equivalence relation, and the equivalence classes form a $\mathbb{W}_*$ -module $\mathbb{W}_*^{\text{rel}}$.

The assignment $\partial(W, t, g) = (\partial W, ti, gi)$ defines a homomorphism $\mathbb{W}_n^{\text{rel}} \rightarrow \mathbb{W}_{n-1}^F$, also called ∂ . Let $r: \mathbb{W}_n^F \rightarrow \mathbb{W}_n^I$ and $s: \mathbb{W}_n^I \rightarrow \mathbb{W}_n^{\text{rel}}$ be the obvious forgetful homomorphisms. It is straightforward, if tedious, to prove the cobordism exact sequence.

Proposition 2.3. *The sequence*

$$\begin{array}{ccccc} \mathbb{W}_*^I & \xrightarrow{s} & \mathbb{W}_*^{\text{rel}} & \xrightarrow{\partial} & \mathbb{W}_*^F \\ & & \searrow & \swarrow & \\ & & & r & \end{array}$$

is exact.

In the next section we will need to know something about the characteristic classes of a projective space bundle. Suppose $x^k \rightarrow B$ is a vector bundle of dimension k , and let $Sx \rightarrow B$ be its $(k-1)$ -sphere bundle. The following calculation follows from well-known results of Borel and Hirzebruch, and is proved in [3, Theorem 23.3]:

Proposition 2.4. *Let $Px = Sx/t$, where $t: Sx \rightarrow Sx$ is the involution antipodal on each fiber, and let π and p be the obvious projections:*

$$\begin{array}{ccc} Sx & \xrightarrow{\pi} & Px \\ & \searrow & \swarrow p \\ & B & \end{array}$$

Then $w_1(Px) = p^(w_1(B) + w_1(x)) + kw_1(\pi)$.*

Corollary 2.5. *If the disk bundle Dx is a Wall manifold and k is even, then Px is also a Wall manifold.*

Proof. If $r: Dx \rightarrow B$ is the projection, note that $w_1(Dx) = r^*(w_1(B) + w_1(x))$ is an integral class, by hypothesis. Since B is homotopy equivalent to Dx it follows that $w_1(Px) = p^*(w_1(B) + w_1(x))$ is also integral. ***

3. Imbedding $\mathbb{W}_*^I$ in $I_*(Z_2)$. We begin by calculating $\mathbb{W}_*^F$ from Proposition 2.2. Suppose (W, f) is a Wall manifold and $g: W \rightarrow RP^\infty$ is any smooth map. The pullback of the universal Z_2 -bundle $S^\infty \rightarrow RP^\infty$ is a cover $q: W' \rightarrow W$, and $w_1(W') = q^*w_1(W) = (fq)^*(\iota)$. Therefore, if $t_q: W' \rightarrow W$ reverses the sheets of the cover, (W', t_q, fq) is a Wall manifold with free involution. This construction defines a homomorphism

$$k: \mathbb{W}_*(RP^\infty) \rightarrow \mathbb{W}_*^F.$$

Now let V be any smooth manifold, and $f: V \rightarrow S^1$ any smooth map. Corresponding to the cohomology class $f^*(\iota) + w_1(V) \in H^1(V; Z_2)$ is a map $V \rightarrow RP^\infty$ and hence a Z_2 -bundle $q: W' \rightarrow V$. By construction,

$$(fq)^*(\iota) + w_1(W') = q^*(f^*(\iota) + w_1(V)) = q^*(w_1(q)) = 0.$$

Therefore, (W', t_q, fq) is again a Wall manifold with free involution. This defines a homomorphism

$$j: \mathfrak{N}_*(S^1) \rightarrow \mathbb{W}_*^F.$$

There are monomorphisms $a: \mathbb{W}_* \rightarrow \mathfrak{N}_*(S^1)$ and $b: \mathbb{W}_* \rightarrow \mathbb{W}_*(RP^\infty)$ given by

$$a(W, f) = (W \rightarrow * \in S^1) \quad \text{and} \quad b(W, f) = (W, f; W \rightarrow * \in RP^\infty).$$

(See [6] for a discussion of these and other results from the theory of Wall cobordism.) Since $ja(W, f) = kb(W, f) = (W \times Z_2, -1, *)$, we have the commutative diagram:

$$\begin{array}{ccc} \mathbb{W}_*(RP^\infty) & \xrightarrow{k} & \mathbb{W}_*^F \\ b \uparrow & & \uparrow j \\ \mathbb{W}_* & \xrightarrow{a} & \mathfrak{N}_*(S^1) \end{array}$$

The computation of $\mathbb{W}_*^F$ is now easy:

Theorem 3.1. *k and j are monic, and*

$$k + j: \mathbb{W}_*(RP^\infty) \oplus \mathfrak{N}_*(S^1) \rightarrow \mathbb{W}_*^F$$

is epic, with kernel $a\mathbb{W}_ \oplus b\mathbb{W}_*$.*

Proof. k must be a monomorphism, for there is a commutative diagram

$$\begin{array}{ccc} \mathbb{W}_*(RP^\infty) & \xrightarrow{\phi} & \mathfrak{N}_*(RP^\infty) \\ k \downarrow & & \downarrow \cong \\ \mathbb{W}_*^F & \longrightarrow & \mathfrak{N}_*(Z_2) \end{array}$$

in which the horizontal maps forget Wall structure, and ϕ is known to be monic. Define an inverse for j , assigning to the class of (W, t, g) the class of the induced map $g_{\#}: W/t \rightarrow S^1$.

Proposition 2.2 shows that $k + j$ is epic. Now suppose $\pi: W' \rightarrow W$ is a double cover, and $b: W \rightarrow S^1$ is such that $(W', t_{\pi}, b\pi)$ is a cobordism between $(M'_1, t_{\pi}i_1, b\pi i_1)$, a representative of $m_1 \in \text{Im } k$, and $(M'_2, t_{\pi}i_2, b\pi i_2)$, a representative of $m_2 \in \text{Im } j$, where $i_x: M'_x \subset W'$. Then W is a cobordism between $M_1 = \pi M'_1$ and $M_2 = \pi M'_2$, and by hypothesis, $b^*\iota|_{M_1} = w_1(M_1)$, while $b^*\iota|_{M_2} = w_1(M_2) + w_1(\pi|M_2)$. By (2.2), either $b^*\iota = w_1(W)$, in which case $w_1(\pi|M_2) = 0$ and $m_2 \in \text{Im } ja$, or else $b^*\iota = w_1(W) + w_1(\pi)$, in which case $w_1(\pi|M_1) = 0$ and $m_1 \in \text{Im } kb$. This completes the proof. ***

The next task is to describe the structure of $\mathbb{W}_*^{\text{rel}}$.

Theorem 3.2. *There is an isomorphism*

$$\beta: \mathbb{W}_*^{\text{rel}} \cong \tilde{\mathbb{W}}_* \left(\bigvee_k TBO_k \right)$$

given by a standard "normal bundle to the fixed set" construction (as in [3, p. 73]).

Proof. To review the construction, let (M, t, g) represent an element of $\mathbb{W}_*^{\text{rel}}$, let F^k be the k -dimensional component of the fixed set of t , and let $\nu^{n-k} \rightarrow F^k$ be its normal bundle in M . There is a tubular neighborhood of F^k equivariantly diffeomorphic to the disk bundle $D\nu^{n-k}$, where $D\nu^{n-k}$ is given an involution antipodal in each fiber. $D\nu^{n-k}$ thus acquires a Wall structure by imbedding in M , so classifying $D\nu^{n-k}$ produces an element of $\tilde{\mathbb{W}}_n(TBO_{n-k})$.

Conversely, suppose $b: (M, \partial M) \rightarrow (TBO_j, \infty)$ represents some element of $\tilde{\mathbb{W}}_n(TBO_j)$, with $f: M \rightarrow S^1$ defining the Wall structure. Set $BO_j \subset TBO_j$ as the zero section of the universal bundle γ^j , and make b transverse regular on BO_j . Then $F = b^{-1}BO_j$ has a tubular neighborhood D diffeomorphic to $Dh^*\gamma^j$, so that, if we give D the Wall structure induced by that of M , and the involution s antipodal in the fibers of $h^*\gamma^j$, we obtain a Wall manifold together with an involution whose fixed set is F . This construction produces an inverse for β , as in [3], but we must verify that $(D, s, f|_D)$ is a Wall manifold with involution. Recall that there is a retraction $r: D \rightarrow F$ factoring through $\pi: D \rightarrow D/s$. Since r^* is an isomorphism, π^* is epic, and Proposition 2.1 applies. ***

Theorem 3.3. *The map $\partial: \mathbb{W}_*^{\text{rel}} \rightarrow \mathbb{W}_*^F$ of (2.3) is an epimorphism. Thus there is an exact sequence*

$$0 \rightarrow \mathbb{W}_*^I \xrightarrow{s} \mathbb{W}_*^{\text{rel}} \xrightarrow{\partial} \mathbb{W}_*^F \rightarrow 0.$$

Proof. There are two steps; we show that $\partial\tilde{W}_*(\bigvee_{k \text{ odd}} TBO_k)$ covers $\text{Im } j$, and that $\partial\tilde{W}_*(\bigvee_{k \text{ even}} TBO_k)$ covers $\text{Im } k$.

Lemma 3.4. *The composite*

$$\tilde{W}_*(TBO_1) \xrightarrow{\beta^{-1}} W_{*}^{\text{rel}} \xrightarrow{\partial} W_*^F \xrightarrow{j^{-1}} \mathfrak{N}_*(S^1)$$

is well defined and an isomorphism.

Proof. Let $(W, t, f\pi)$, where $\pi: W \rightarrow W/t$, represent an element of $\text{Im } j \in W_*^F$. Let $d: D\xi \rightarrow W/t$ be the disk bundle of the line bundle associated with π . By definition of j ,

$$\begin{aligned} w_1(\xi) &= w_1(\pi) = f^*(t) + w_1(W/t), \\ w_1(D\xi) &= d^*(w_1(\xi) + w_1(W/t)) = (f/d)^*(t), \end{aligned}$$

so $D\xi$ is a Wall manifold with boundary, and $(W, t, f\pi)$ represents the image of $[D\xi] \in \tilde{W}_*(TBO_1)$. Thus $j^{-1}\partial\beta^{-1}$ is defined and epic. A simple calculation shows that $\mathfrak{N}_n(S^1)$ has the same rank as $\tilde{W}_{n+1}(TBO_1) \cong W_n(RP^\infty)$, so $j^{-1}\partial\beta^{-1}$ is an isomorphism.

Lemma 3.5. *The composite*

$$\tilde{W}_*(TBO_2) \xrightarrow{\beta^{-1}} W_{*}^{\text{rel}} \xrightarrow{\partial} W_*^F \xrightarrow{k^{-1}} \tilde{W}_*(RP^\infty)$$

is well defined and an epimorphism.

Proof. If $a \in \tilde{W}_*(TBO_2)$, $\partial\beta^{-1}(a)$ is represented as the sphere bundle Sx of some 2-dimensional bundle x , together with the antipodal involution. By Corollary 2.5, the quotient space Px is also a Wall manifold, so $\partial\beta^{-1}(a) \in \text{Im } k$.

We will exhibit a W_* -basis of $\tilde{W}_*(RP^\infty)$ which lies in the image of $k^{-1}\partial\beta^{-1}$. To be specific, there are manifolds W^j , in each dimension $j \geq 1$, with these properties:

- (1) There is an $\alpha \in H^1(W^j; Z_2)$ so that α^j is the generator of $H^j(W^j; Z_2) \cong Z_2$.
- (2) Each W^j is a Wall manifold.
- (3) Each $W^j = Px_j$ for some two-dimensional bundle x_j .
- (4) $w_1(W^j)$ is the image of some integral class in the cohomology of the base

of x_j .

If $b_j: W^j \rightarrow RP^\infty$ classifies α , then the first and second properties assure that (W^j, b_j) represents a set of generators of $\tilde{W}_*(RP^\infty)$ over W_* . The third and fourth properties, together with Proposition 2.4, assure that the Dx_j are Wall manifolds, so that the W^j lie in the image of $k^{-1}\partial\beta^{-1}$. Since all three maps are clearly W_* -homomorphisms, the lemma will follow.

The necessary examples were constructed by P. G. Anderson in [1]. If $\lambda \rightarrow RP^n$ is the canonical line bundle, define $(m+n)$ -manifolds

$$M(m, n) = P(\lambda \oplus m \rightarrow RP^n).$$

Then

$$H^*(M(m, n); Z_2) = Z_2[\alpha, \beta] / \langle \alpha^{m+1} + \alpha^m \beta, \beta^{n+1} \rangle, \quad \dim \alpha = \dim \beta = 1.$$

If m is odd and n is even, $M(m, n)$ is orientable. If m is odd and $n = 1$, $M(m, 1)$ is a Wall manifold, and $w_1(M(m, 1)) = \beta + (m+1)\alpha = \beta$ is induced from an integral class in the cohomology of the base S^1 . Next, let $\lambda' \rightarrow M(m, 1)$ be the canonical line bundle. Define an $(m+2)$ -manifold

$$M(1, m, 1) = P(\lambda' \oplus 1 \rightarrow M(m, 1)).$$

Then

$$H^*(M(1, m, 1); Z_2) = Z_2[\alpha, \beta, \gamma] / \langle \alpha^2 + \alpha\beta, \beta^{m+1} + \beta^m \gamma, \gamma^2 \rangle, \\ \dim \alpha = \dim \beta = \dim \gamma = 1.$$

For m even, $M(1, m, 1)$ is a Wall manifold, and $w_1(M(1, m, 1)) = \gamma + (m+2)\beta = \gamma$ is induced from an integral class in the cohomology of the base $M(m, 1)$. All these calculations are expanded in [1].

The required manifolds are

$$W^j = \begin{cases} M(1, 2k), & \text{if } j = 2k + 1 \geq 3, \\ M(1, 2k - 2, 1), & \text{if } j = 2k \geq 4. \end{cases}$$

In the low dimensions, $W^1 = S^1 = P(1 \oplus 1 \rightarrow pt)$ and $W^2 = M(1, 1) =$ the Klein bottle. Properties (2), (3), and (4) are clear from Anderson's construction; property (1) for the class $\alpha \in H^1(W^j; Z_2)$ is immediate from the indicated structure of $H^*(W^j; Z_2)$.

This completes the proof of Theorem 3.3. ***

Corollary 3.6. *There is a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_*^I & \xrightarrow{s} & W_*^{\text{rel}} & \xrightarrow{\partial} & W_*^F \longrightarrow 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ 0 & \longrightarrow & I_*(Z_2) & \longrightarrow & \mathfrak{N}_*\left(\bigvee_k TBO_k\right) & \longrightarrow & \mathfrak{N}_*(Z_2) \longrightarrow 0 \end{array}$$

the lower sequence being that of [3, Theorem 28.1], and a and b are monomorphisms.

This follows at once, since the map

$$W_*\left(\bigvee_k TBO_k\right) \rightarrow \mathfrak{N}_*\left(\bigvee_k TBO_k\right),$$

forgetting Wall structure, is known to be monic.

4. **Oriented manifolds with involution.** Rosenzweig [5] and Conner [2] have studied the cobordism group $\mathcal{O}_*(Z_2)$ of orientation-preserving involutions on closed oriented manifolds. In this section we consider this subject from a different point of view.

Let $t: M \rightarrow M$ be an involution on an oriented manifold with boundary. Clearly, t will be orientation-preserving on ∂M if and only if it is orientation-preserving on M . If ∂M is the disjoint union of N_1 and N_2 then we say that (M, t) is a cobordism between $(N_1, t|N_1)$ and $(N_2, t|N_2)$, and thus construct the cobordism group $\mathbf{O}_*(Z_2)$ of oriented manifolds with involution (no restriction on t 's action on the orientation). $\mathbf{O}_*(Z_2)$ is obviously a module over the oriented cobordism ring Ω_* , with multiplication induced by the Cartesian product. $\mathcal{O}_*(Z_2) \subset \mathbf{O}_*(Z_2)$, and indeed

$$\mathbf{O}_*(Z_2) = \mathcal{O}_*(Z_2) \oplus \mathcal{R}_*(Z_2),$$

where $\mathcal{R}_*(Z_2)$ can be interpreted as the cobordism of oriented manifolds with orientation-reversing involution.

In order to relate $\mathbf{O}_*(Z_2)$ to $\mathbf{W}_*(Z_2)$, one would like to define a map $\partial: \mathbf{W}_*(Z_2) \rightarrow \mathbf{O}_*(Z_2)$, of degree -1 , which associates to the class of (W, t, f) the class of $(N, t|N)$, where N is a submanifold of W invariant under t and dual to $w_1(W)$. This makes sense if, for some point $* \in S^1$, f is equivariantly homotopic to a map $g: W \rightarrow S^1$ which is transverse regular at $*$, for then $N = g^{-1}(*)$ will have the needed properties (see Wall's original construction [8, §1]).

Lemma 4.1. *Let (M, t) be a manifold with involution, X any manifold, and $* \in X$. Any equivariant map $f: (M, t) \rightarrow (X, 1_X)$ can be equivariantly homotoped to a map g which is transverse regular at $*$. Further, if $A \subseteq M$ is an invariant closed set and f is transverse regular at $*$ for points of A , the homotopy can be kept fixed on a neighborhood of A .*

The technique of proof is standard; for a more extensive use see Stong [7, Lemma 4.2]. Briefly, we homotope f to be transverse regular on the fixed set F of t . Since F is an equivariant strong neighborhood deformation retract, we may assume f to be transverse regular on a neighborhood of F . We may extend the result to all of M if we can prove the theorem under the assumption that t is free. In that case, divide out the action of t , obtaining $b: M/t \rightarrow X$. b can be made transverse regular via some $H: M/t \times I \rightarrow X$. If $\pi: M \times I \rightarrow M/t \times I$ is the projection, $H\pi: M \times I \rightarrow X$ is the desired equivariant homotopy. ***

Using this lemma, it is easy to prove that ∂ is well defined. We now prove that there is a partial Rohlin sequence relating $\mathbf{W}_*(Z_2)$ and $\mathbf{O}_*(Z_2)$.

Theorem 4.2. Let $p: \mathbf{O}_*(Z_2) \rightarrow \mathbf{W}_*(Z_2)$ be defined by assigning to the class of (M, t) the class of $(M, t, *: M \rightarrow \text{point} \in S^1)$. There is an exact sequence

$$\mathbf{O}_*(Z_2) \xrightarrow{p} \mathbf{W}_*^I \xrightarrow{\partial} \mathbf{O}_*(Z_2) \xrightarrow{2} \mathbf{O}_*(Z_2),$$

where 2 represents multiplication by 2.

Note. There is clearly an inclusion $\text{Im } 2 \subset \text{Ker } p$, but Conner has shown it to be proper [2, Theorem 5.8].

Proof. If W is orientable, $w_1(W) = 0$ and its dual submanifold is $\emptyset$, so $\partial p = 0$.

If $\partial(w) = 0$, and w is represented by (W, t, f) with f transverse regular at $* \in S^1$, then $N = f^{-1}(*)$ bounds some oriented manifold M with involution m . Now the normal bundle of N in W is trivial by construction, so one can form (S, s) from $(W \times I, t \times 1_I)$ and $(M \times I, m \times 1_I)$ by identifying an equivariant tubular neighborhood D of $N \times 1$ in $W \times 1$ with $(N \times I, t|N \times 1_I) \subset (M \times I, m \times 1_I)$. The result is a cobordism between $(W, t, f) = (W, t, f) \times 0$ and an object of the form $(\text{cl}(W - D) \cup (M \times \partial I), t', f')$. We may assume, from the construction, that $\text{Im } f'$ misses $*$, so this object lies in $\text{Im } p$.

To show that $2\partial = 0$, let $\partial(W, t, f) = (N, t|N)$ as before, and deform f so that, on a tubular neighborhood D of N , f factors through the projection $D \rightarrow N$. If $V \doteq W - \text{Int } D$, then V is orientable, and $\partial(V, t|V) = 2(N, t|N)$.

Finally, suppose (M, t') is an oriented manifold with involution, and $(M, t') \times (S^0, 1) = \partial(U, u)$. Let $* \in S^1$ have a tubular neighborhood D with boundary S^0 . Using a tubular neighborhood of ∂U , we can easily construct a map $f: U \rightarrow S^1 - \text{Int } D \subset S^1$ which is the projection $M \times S^0 \rightarrow S^0$ when restricted to ∂U . Form (W, t) by sewing (U, u) and $(M \times D, t' \times 1_D)$ along the copies of $M \times S^0$, and extend f by the projection $M \times D \rightarrow D$ to define $g: W \rightarrow S^1$. It is clear that $g^*(\iota) = w_1(W)$, g is transverse regular at $*$, $g^{-1}(*) = M$, and g is equivariant, so (M, t') represents a class in $\text{Im } \partial$. ***

In particular, the 2-torsion of $\mathbf{O}_*(Z_2)$ is precisely $\text{Im } \partial$. Since $\mathbf{W}_*^I$ imbeds in $I_*(Z_2)$, which was computed by Conner and Floyd [4], it becomes possible, in principle, to compute the 2-torsion of $\mathbf{O}_*(Z_2)$ from the structure of $I_*(Z_2)$. There are, of course, two highly practical obstacles: the complexity of $I_*(Z_2)$ and the lack of a geometric splitting in the proof of Theorem 3.3.

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