TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 169, July 1972

TRANSVERSE CELLULAR MAPPINGS OF POLYHEDRA

BY

ETHAN AKIN

ABSTRACT. We generalize Marshall Cohen's notion of transverse cellular map to the polyhedral category. They are described by the following:

Proposition. *Let $f\colon K \to L$ be a proper simplicial map of locally finite simplicial complexes. The following are equivalent*:

(1) *The dual cells of the map are all cones.*

(2) *The dual cells of the map are homogeneously collapsible in K.*

(3) *The inclusion of L into the mapping cylinder of f is collared.*

(4) *The mapping cylinder triad (C_f, K, L) is homeomorphic to the product triad $(K \times I; K \times 1, K \times 0)$* rel $K = K \times 1$.

Condition (2) is slightly weaker than f^{-1}(point) is homogeneously collapsible in K. Condition (4) when stated more precisely implies f is homotopic to a homeomorphism. Furthermore, the homeomorphism so defined is unique up to concordance.

The two major applications are first, to develop the proper theory of "attaching one polyhedron to another by a map of a subpolyhedron of the former into the latter". Second, we classify when two maps from X to Y have homeomorphic mapping cylinder triads. This property turns out to be equivalent to the equivalence relation generated by the relation $f \sim g$, where $f, g\colon X \to Y$ means $f = gr$ for $r\colon X \to X$ some transverse cellular map.

Marshall Cohen has developed (see [C_1]) a theory of transverse cellular mappings defined on manifolds. They satisfy a slightly weaker condition than collapsibility of point-inverses. They are close to homeomorphisms in that they share with homeomorphisms the property that their mapping cylinder is a product. Their interest is that they are precisely all the maps which satisfy this property.

In this paper, we generalize the notion of transverse cellularity to proper maps of locally compact polyhedra.

Proposition. *Let $f\colon K \to L$ be a proper, simplicial map of locally finite simplicial complexes. Then the following are equivalent*:

(1) *For every $A \in L$, the dual cell of A with respect to f, $D(A; f)$ (defined to be $f^{-1}D(A; L)$) is homeomorphic to the cone on $\dot{D}(A; f)$* rel $\dot{D}(A; f)$ *(where $\dot{D}(A; f) \equiv f^{-1}\dot{D}(A; L)$).*

Received by the editors June 11, 1970 and, in revised form, May 4, 1971.

AMS 1969 subject classifications. Primary 5701, 5705.

Key words and phrases. Collapsing, transverse cellularity, cone, cone complex, cell-like, mapping cylinder, thickening.

(2) *For every* $A \in L$, $D(A; f) \searrow$ *point homogeneously in* K (*for homogeneous collapsing see* [A]).

(3) *The inclusion* $L \subset C_f$ ($C_f \equiv$ *the simplicial mapping cylinder of* f) *is locally collared.*

(4) $(C_f; K, L)$ *is p. l. homeomorphic to* $(K \times I; K \times 1, K \times 0)$ rel $K = K \times 1$.

A map satisfying these is called transverse cellular.

Note that the third and fourth conditions are independent of the triangulation of f because the mapping cylinder homeomorphism type is a p. l. invariant of the map. So it makes sense to speak of a p. l. map being transverse cellular. Also note that (4) implies $|K|$ is p. l. homeomorphic to $|L|$.

We use transverse cellularity to give the p. l. definition of attaching one polyhedron to another. Let X_0 be a closed subpolyhedron of a locally compact polyhedron X and let $f\colon X_0 \to Y_0$ be a proper p. l. map to another locally compact polyhedron. We define a regular extension of f to be a proper p. l. map $F\colon X \to Y$ and a closed p. l. embedding $i\colon Y_0 \leftarrow Y$ such that

(i) $F^{-1}(i(Y_0)) = X_0$, and

(ii) $F \mid X_0 = i \circ f$, and

(iii) $F\colon X - X_0 \to Y - Y_0$ is transverse cellular.

We show that such always exist and prove the required uniqueness theorem:

Proposition. *Let*

$$X \xrightarrow{F_1} Y_1 \xleftarrow{i_1} Y_0 \quad \text{and} \quad X \xrightarrow{F_2} Y_2 \xleftarrow{i_2} Y_0$$

be regular extensions of $f\colon X_0 \to Y_0$ *where* $X_0 \subset X$.

There exists a regular extension $X \xrightarrow{F} Y \xleftarrow{i} Y_0$ *of* f *and transverse cellular maps* $G_\alpha\colon Y \to Y_\alpha$, $\alpha = 1, 2$, *with* $G_\alpha^{-1}(i_\alpha(Y_0)) = i(Y_0)$ *and making the following diagram commute.*

$$\begin{tikzcd}
& X \arrow[dl, "F_1"'] \arrow[d, "F"] \arrow[dr, "F_2"] & \\
Y_1 & Y \arrow[l, "G_1"'] \arrow[r, "G_2"] & Y_2 \\
& Y_0 \arrow[ul, "i_1"] \arrow[u, "i"] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

In particular, Y_1 *and* Y_2 *are p. l. homeomorphic.*

The other major application is to a classification of mapping cylinders. Define a relation $\approx_e$ between p. l. maps. If $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to$

Y_1 then $f_0 \approx_e f_1$ if $Y_0 = Y_1$ and there exists a transverse cellular map r: $X_0 \to X_1$ satisfying $f_0 = f_1 \circ r$. $\approx_e$ generates an equivalence relation which we will call $\approx$.

Proposition. *Let f_0: $X_0 \to Y$ and f_1: $X_1 \to Y$ be proper p. l. maps. Then the following are equivalent*:

(1) $f_0 \approx f_1$.

(2) *There exists a p. l. homeomorphism h:* $(C_{f_0}, X_0) \cong (C_{f_1}, X_1)$ rel Y.

Oddly enough, if one varies the range by a transverse cellular map, too, then one gets a diametrically opposed result.

Define $f_0 \leftrightarrow_e f_1$ for f_α: $X_\alpha \to Y_\alpha$, $\alpha = 0, 1$, to mean there exist transverse cellular maps d: $X_0 \to X_1$ and r: $Y_0 \to Y_1$ with $f_1 \circ d = r \circ f_0$. Call the generated equivalence relation $\leftrightarrow$.

Proposition. $f_0 \leftrightarrow f_1$ *if and only if there exist homeomorphisms d: $X_0 \to X_1$ and r: $Y_0 \to Y_1$ such that f_1 is homotopic to $r \circ f_0 \circ d^{-1}$.*

I would like at this point to express my thanks to the referee, perhaps on the readers' behalf as well as my own. His detailed comments—running in length to half that of the manuscript—resulted in a complete recasting of the paper and the correction of a serious error.

I. **Preliminaries.** We will be looking at the piecewise linear topology of locally compact polyhedra. Unless otherwise stated all polyhedra are locally compact, subpolyhedra are closed and maps are p. l. and proper (inverses of compact sets are compact). The corresponding simplicial category is that of locally finite simplicial complexes and proper simplicial maps. For general background as well as relations between infinite complexes and locally compact polyhedra in particular, see Hudson's book [H]. However, we will also draw heavily on the results in and the notation of Cohen's paper on regular neighborhoods [C_2] and my thesis [A]. In particular, we will use the notation for intrinsic dimension developed in the latter. We define $d(x; X)$ to be the intrinsic dimension of x in X; $I^i(X)$, the intrinsic i-skeleton, is $\{x \in X: d(x; X) \leq i\}$; and in a complex K a simplex A is said to be a "nice face" of B if A is a face of B and $d(x; X)$ is constant as x varies over interior $A \cup$ interior B.

We christen by the name "the Alexander trick" the many results which come from the fact that for compact polyhedra X and Y the product of the cones: $(cX) \times (cY)$ *is homeomorphic* to the cone $c'[(cX \times Y) \cup (X \times cY)]$ rel $(cX \times Y) \cup (X \times cY)$ and the fact that $(cX \times Y) \cup (X \times cY)$ is homeomorphic to the join, $X * Y$. In particular, for $Y = \{0\}$, cY is an interval I and so $(cX) \times I \cong c'((cX \times 0) \cup (X \times I))$ rel $(cX \times 0) \cup (X \times I)$. Also, $(cX) \times I \cong c'((cX \times \dot{I}) \cup (X \times I))$, using $Y = \{0, 1\}$.

While on the subject of cones, we need to introduce the convention of the empty polyhedron 0, satisfying cone 0 = cone point.

In developing a category approach to mapping cylinders, we need the right category.

We introduce the category of "simplicial maps with chosen deriveds".

Objects. Simplicial maps $f\colon K \to L$ together with a choice of derived subdivisions on K and L, such that $f\colon \eta K \to \eta L$ is also simplicial. (When ηK is a derived subdivision of K and $A \in K$, we will use ηA for the point-choice in A of the subdivision, e.g. the "barycenter".)

Morphisms. A morphism $G\colon f_0 \to f_1$ where $f_\alpha\colon K_\alpha \to L_\alpha$, $\alpha = 0, 1$, is a pair of simplicial maps (written $G = (G_d, G_r)$) $G_d\colon K_0 \to K_1$ and $G_r\colon L_0 \to L_1$ also simplicial with respect to the chosen deriveds and such that $f_1 G_d = G_r f_0$.

Examples. If K is a complex, K_0 a subcomplex and ηK is a derived subdivision of K then the identity and inclusion maps $\mathrm{id}_K\colon K \to K$ and inc: $K_0 \to K$ with the derived η chosen on domain and range are objects of this category. Furthermore, if $f\colon K \to L$ is an object of the category with deriveds ηK and ηL, then there are natural morphisms relating the identity maps and f:

$$d_f = (\mathrm{id}_K, f)\colon \mathrm{id}_K \to f, \qquad r_f = (f, \mathrm{id}_L)\colon f \to \mathrm{id}_L.$$

There are also obvious functors from this category to the simplicial category: "Domain" and "Range" and on the derived level "η Domain" and "η Range".

II. Cone complexes.

Definition. A cone complex $\mathfrak{S}$ on a locally compact polyhedron X is a locally finite covering of X by compact polyhedra (called the cells of $\mathfrak{S}$) with a boundary map ∂ defined on $\mathfrak{S}$, satisfying

(a) For each $\sigma \in \mathfrak{S}$, $\partial\sigma \subset \sigma$ and is a union of (necessarily finitely many) cells of $\mathfrak{S}$.

(b) For σ, τ distinct elements of $\mathfrak{S}$, $\mathring{\sigma} \cap \mathring{\tau} = \emptyset$ where $\mathring{\sigma}$ is defined to be $\sigma - \partial\sigma$.

(c) For each $\sigma \in \mathfrak{S}$, there exists a homeomorphism $\sigma \cong$ cone $\partial\sigma$ (rel $\partial\sigma$).

(d) If $\sigma \in \mathfrak{S}$ and dimension $\sigma = 0$, then and only then $\partial\sigma = 0$ (the empty polyhedron).

If c is weakened to c'. $\partial\sigma$ is collared in σ, then we call $\mathfrak{S}$ a general complex on X.

If c is strengthened to c''. Each $(\sigma, \partial\sigma)$ pair is a ball and boundary-sphere pair, then (following custom) we call $\mathfrak{S}$ a cell complex on X.

Auxiliary definitions. (1) The incidence relation $\sigma < \tau$ means $\sigma \subset \partial\tau$.

(2) If A is a subset of X, then "σ meets A" means $\mathring{\sigma} \cap A \neq \emptyset$. Note that

from (a) and (b), if σ meets τ then $\sigma = \tau$ or $\sigma < \tau$. (Note "σ meets τ" is not symmetric.)

(3) By (a) and (b) for each $x \in X$ there is a unique $\sigma \in \mathfrak{S}$ such that σ meets x, i.e. $x \in \mathring{\sigma}$. This σ is called the carrier of x.

(4) If dimension $\sigma = 0$, then we call σ a vertex. Note that for a cone complex $\partial\sigma = 0$ implies σ is a point by (c) and our convention about the empty polyhedron.

Definition. Let $\mathfrak{S}$ be a cone complex on X. A choice of homeomorphism f_σ: $\sigma \cong$ cone $\partial\sigma$ rel $\partial\sigma$ is called a structuring of σ. Such a choice for each σ in $\mathfrak{S}$ is called a structuring of $\mathfrak{S}$. $\mathfrak{S}$ together with a structuring is called a structured cone complex.

Example. If K is a locally finite simplicial complex, then the dual cells of the complex form a cone complex on $|K|$ with a natural structuring given by the fact that $D(A; K) = \eta(A) * \dot{D}(A; K)$.

Remarks. For a cone complex there exist many different structurings and, in particular, at least one. However, by the Alexander trick any two structurings are isotopic, i.e. if f_σ, f'_σ: $\sigma \cong$ cone $\partial\sigma$ rel $\partial\sigma$ then f_σ is isotopic to f'_σ. In fact, by the next remark, the isotopy of $f'^{-1}_\sigma f_\sigma$ to the identity on σ rel $\partial\sigma$ extends to an ambient isotopy of X.

For a general complex, if f: $\sigma \overset{\cong}{\to} \sigma$ rel $\partial\sigma$ and is isotopic to id_σ rel $\partial\sigma$, then the isotopy and, a fortiori f, extends to X. Just extend the isotopy up the cells by induction on dimension, noting that an isotopy of $\mathrm{id}_{\partial\tau}$ extends to an isotopy of id_τ by the fact that $\partial\tau \subset \tau$ is collared. Thus, if $x, y \in \mathring{\sigma}$ are joinable by a path x_t in $\mathring{\sigma}$ with $d(x_t; \sigma)$ constant, then by [A, Definition II. 7] $d(x_t; X)$ is constant.

Definition. If $\mathfrak{S}$ and $\mathfrak{D}$ are cone complexes on X and Y respectively, a map γ: $\mathfrak{S} \to \mathfrak{D}$ is called a cone map if it preserves incidence, i.e. $\sigma < \tau$ implies $\gamma(\sigma) < \gamma(\tau)$, and if γ (vertex) is a vertex.

With this definition, cone-complexes and structured cone complexes become categories. We can similarly define the category of general decompositions and general decomposition maps.

Proposition 1. *There is a covariant realization functor from structured cone complexes to the p. l. category defined as follows: Let* $(\mathfrak{S}^i, \{f^i_\sigma\})$ *be structured cone complexes on* X^i *for* $i = 0, 1$ *and* γ: $\mathfrak{S}^0 \to \mathfrak{S}^1$ *be a cone map.*

Then define $|(\mathfrak{S}^i, \{f^i_\sigma\})| = X^i$ *and* $|\gamma|$ *to satisfy*

$$|\gamma| \mid \sigma = (f^1_{\gamma(\sigma)})^{-1} \circ (\text{cone } |\gamma| \mid \partial\sigma) \circ (f^0_\sigma).$$

Hence, if γ: $\mathfrak{S}^0 \to \mathfrak{S}^1$ *is a cone map, then by picking structurings we can find a map* f: $X^0 \to X^1$ *carried by* γ*, i.e.* $f(\mathring{\sigma}) \subset [\gamma(\sigma)]^\circ$.

Proof. Let $|\gamma| = \gamma$ on vertices and proceed by induction up the cells of $\mathfrak{S}$.

Remarks. (1) If $\gamma\colon \mathfrak{S} \to \mathfrak{D}$ and we pick different structurings $\{f_\sigma\}$, $\{f'_\sigma\}$ on $\mathfrak{S}$ and $\{g_\tau\}$, $\{g'_\tau\}$ on $\mathfrak{D}$ then realizing the commutative diagram

$$\begin{tikzcd}
(\mathfrak{S}, \{f_\sigma\}) \arrow[r, "\gamma"] \arrow[d, "\mathrm{id}"'] & (\mathfrak{D}, \{g_\tau\}) \arrow[d, "\mathrm{id}"] \\
(\mathfrak{S}, \{f'_\sigma\}) \arrow[r, "\gamma"] & (\mathfrak{D}, \{g'_\tau\})
\end{tikzcd}$$

shows that the two realizations of γ differ by homeomorphisms, which are ambiently isotopic to the identities with isotopies carried by $\mathfrak{S}$ and $\mathfrak{D}$.

(2) If $g\colon X \to Y$ is a homeomorphism carried by an isomorphism $\gamma\colon \mathfrak{S} \to \mathfrak{D}$, and $\{g_\tau\}$ is a structuring on $\mathfrak{D}$ then $f_\sigma = (\text{cone } g|\partial\sigma)^{-1} \circ g_{\gamma(\sigma)} \circ g$ is a structuring on $\mathfrak{S}$, with $|\gamma| = g$.

(3) If $g\colon X \to Y$ is a map carried by γ then g is homotopic to $|\gamma|$ with the homotopy carried by γ.

Subcomplexes.

Definition. Let $\mathfrak{S}$ be a complex of X and X_0 a subpolyhedron of X. "$\mathfrak{S}$ restricts to X_0", or "$\mathfrak{S}$ induces a complex on X_0" if for each $\sigma \in \mathfrak{S}$ such that σ meets X_0, $\sigma \cap X_0 \cong \text{cone } (\partial\sigma \cap X_0)$ rel $\partial\sigma \cap X_0$. Then we define $\mathfrak{S}|X_0 = \{\sigma \cap X_0 \colon \sigma \in \mathfrak{S} \text{ meeting } X_0\}$ with $\partial(\sigma \cap X_0) = (\partial\sigma) \cap X_0$.

(*Convention.* If $\partial\sigma \cap X_0$ is empty, we say $\partial\sigma \cap X_0 = 0$ so $\sigma \cap X_0$ is a point of $\mathring{\sigma}$, a vertex of $\mathfrak{S}|X_0$.)

If $\mathfrak{S}$ restricts to X_0 and for each σ meeting X_0, $(\sigma, \sigma \cap X_0) \cong \text{cone } (\partial\sigma, \partial\sigma \cap X_0)$ rel $\partial\sigma$ then we say $\mathfrak{S}$ induces a cone complex on the pair (X, X_0).

Example. If σ meets X_0 implies $\sigma \subset X_0$, i.e. X_0 is a union of elements of $\mathfrak{S}$ then $\mathfrak{S}$ induces a complex on (X, X_0).

The definition of "$\mathfrak{S}$ induces a complex on (X, X_0)" is precisely what is needed to obtain a structuring on the pair: That is, a structuring of $\mathfrak{S}$ such that $f_\sigma(\sigma \cap X_0) = \text{cone } (\partial\sigma \cap X_0)$ when σ meets X_0. The analogue of Proposition 1 is true.

Similarly, $\mathfrak{S}$ induces a complex on $(X; \{X_\alpha\})$ $\{X_\alpha\}$ a family of subpolyhedra if for $\sigma \in \mathfrak{S}$, there exists $f_\sigma\colon (\sigma, \{\sigma \cap X_{\alpha'}\}) = \text{cone } (\partial\sigma, \{\partial\sigma \cap X_{\alpha'}\})$ rel $\partial\sigma$ where $\{X_{\alpha'}\}$ is the subclass consisting of those X_α's which σ meets. Again the analogue of Proposition 1 holds.

If $\mathfrak{S}$ induces a complex on X_0, then whether $\mathfrak{S}$ induces a complex on the pair is a collection of weak unknotting problems for cones: For each σ meeting X_0, is the pair of cones $(\sigma, \sigma \cap X_0)$ a cone pair? For example, if $X_0 \cap \sigma$ is a point p then the question is whether p can be a cone point for σ (which amounts to: does it have a minimal intrinsic dimension in $\overset{\circ}{\sigma}$?). This weak unknotting question is examined in [A].

Extending structurings. In applications the following problem arises: Let $\mathfrak{S}$ be a complex on X inducing a complex $\mathfrak{S}_0$ on X_0 and assume that we are given a structuring on $\mathfrak{S}_0$, when can we find a structuring on $\mathfrak{S}$ which restricts to the given one on $\mathfrak{S}_0$?

Clearly, it is necessary that $\mathfrak{S}$ induce a complex on the pair (X, X_0) but in fact more is required. This question is equivalent to a collection of unknotting problems for embedding of cones.

Let $f\colon cX \to cY$ be an embedding with $f^{-1}(Y) = X$. Lickorish [L] has taught us what is means for f to be unknotted: "f unknots" means there exists $h\colon cY \cong cY$ rel Y such that $h \circ f = c(f|X)$. If $X \subset Y$ and $f|X$ is the inclusion, then f is unknotted iff there exists $g\colon cY \cong cY$ rel Y extending f. For h unknots f iff h^{-1} extends f in this case. Thus we have

Proposition 2. *Let $\mathfrak{S}$ be a cone complex on X, restricting to X_0 and let $\{f_{\sigma'}\colon \sigma' \in \mathfrak{S}|X_0\}$ be a structuring on $\mathfrak{S}|X_0$. Then the following are equivalent*:

(1) *The given structuring of $\mathfrak{S}|X_0$ extends to one of $\mathfrak{S}$, i.e. for all $\sigma \in \mathfrak{S}$ meeting X_0, $f_{(\sigma\cap X_0)}\colon \sigma \cap X_0 \overset{\cong}{\rightarrow}$* cone $(\partial\sigma \cap X_0)$ rel $\partial\sigma \cap X_0$ *extends to a g_σ: $\sigma \overset{\cong}{\rightarrow}$* cone $\partial\sigma$ rel $\partial\sigma$.

(2) *There exists $\{h_\sigma\colon \sigma \in \mathfrak{S}\}$ a structuring of $\mathfrak{S}$, with $h_\sigma \circ (f_{\sigma\cap X_0})^{-1}$:* cone $(\partial\sigma \cap X_0) \to$ cone $\partial\sigma$ *is unknotted for all σ meeting X_0.*

(3) *For every structuring $\{h_\sigma\colon \sigma \in \mathfrak{S}\}$ of $\mathfrak{S}$, $h_\sigma \circ (f_{\sigma\cap X_0})^{-1}$ is unknotted for all σ meeting X_0.*

Remark. In applying this result we will use the criteria for unknotting developed in [A] in terms of homogeneous collapsing of sets and conewise homogeneity of the map.

III. **Mapping cylinders.** We now review the theory of mapping cylinders due to Marshall Cohen ([C_1], [C_3]). It is perhaps worth the bother of introducing category jargon in order to note which constructions are canonical. One of the themes of this section is that many p. l. constructions are done by choosing a triangulation and then using a functorial simplicial construction. The choice of triangulation destroys the naturality of the construction. This is usually recovered, but only to a limited degree, by a corresponding uniqueness theorem which usually states that the construction is independent—in some sense—of the choice of triangulation. Thus, for example, "stellar neighborhood of K_0 in K" (where K_0 is a subcomplex of K) is a functor on the category of pairs of simplicial complexes. The corresponding p. l. notion of a regular neighborhood of a subpolyhedron is obtained by choosing a particular triangulation of the polyhedral pair (X, X_0); namely, a first derived of one where X_0 is triangulated as a full subcomplex of X (see [C_2], the truly enlightened approach to regular neighborhoods). The uniqueness theorem for regular neighborhoods says that the result is, up to ambient isotopy, independent of the choice of triangulation. However, naturality with respect to mappings is lost in the transition to the p. l. category and to get it in any particular case, we choose our triangulations more carefully (i.e. to make the map in question simplicial) and use the naturality on the simplicial level. With these remarks in mind we turn to the mapping cylinder.

If $f\colon K \to L$ is a proper simplicial map of locally finite complexes and ηK, ηL are derived subdivisions with respect to which f is still simplicial, then we define the mapping cylinder of f, C_f, to be the subcomplex of $L * \eta K$:

$$C_f = \{A\eta B_0 \cdots \eta B_n\colon B_0 < B_1 < \cdots < B_n \in K \text{ and } fB_0 \geq A \in L\} \cup L.$$

Subdividing $L * \eta K$ to $\eta L * \eta K$ induces a subdivision ηC_f on the mapping cylinder:

$$\eta C_f = \{\eta A_0 \cdots \eta A_k \eta B_0 \cdots \eta B_n\colon$$
$$A_0 < \cdots < A_k \in L,\ A_k \leq fB_0,\ B_0 < \cdots < B_n \in K\} \cup \eta L.$$

If K and L are infinite then the joins are not locally finite. However, the mapping cylinders are because f is proper. For let $L = \bigcup_{i=1}^\infty L_i$ with L_i finite complexes and let $K_i = f^{-1}L_i$ (also finite because f is proper). Then $K = \bigcup_{i=1}^\infty K_i$, $C_f = \bigcup_{i=1}^\infty C_f|K_i$ and $\eta C_f = \bigcup_{i=1}^\infty \eta C_f|K_i$. If each term of the L-sequence is contained in the interior of the next then the K, C_f and ηC_f sequences also satisfy this. But obtaining such a sequence for C_f and ηC_f implies that they are locally finite.

The mapping cylinder C_f and the subdivided mapping cylinder ηC_f define functors from the category of simplicial maps with chosen deriveds to the usual simplicial category. The functor applied to the morphism $G\colon f_0 \to f_1$ written $C(G)\colon C_{f_0} \to C_{f_1}$ or $\eta C(G)\colon \eta C_{f_0} \to \eta C_{f_1}$ is just the restriction of $G_r * \eta G_d$ or $\eta G_r * \eta G_d$ to the subcomplex of the join. Note that if $A \leq f_0 B$ then $G_r A \leq G_r f_0 B = f_1 G_d B$ so $C(G)$ actually does map C_{f_0} to C_{f_1}. Note that $C(G)$ and $\eta C(G)$ are proper maps, by a variation of the above union argument. Also note that $\eta C(G)$ is just the map $C(G)$ on the subdivided mapping cylinders, more precisely, they induce the same map of underlying polyhedra.

Recall the functors "Domain", "Range", "η Domain" and "η Range" from simplicial maps to the simplicial category. There are a host of natural transformations relating the mapping cylinder to their functors.

The retraction of the mapping cylinder onto its base $P_f\colon \eta C_f \to \eta L$ defined by

$$P_f(\eta A_0 \cdots \eta A_k \eta B_0 \cdots \eta B_n) = \eta A_0 \cdots \eta A_k \eta f B_0 \cdots \eta f B_n$$

is a natural transformation $P_f\colon \eta C_f \to \eta\,\mathrm{Range}\, f$. There are also obvious inclusion transformations:

$$\eta\,\mathrm{Domain}\, f \to C_f,\ \eta\mathrm{Domain}\, f \to \eta C_f,\ \mathrm{Range}\, f \to C_f \text{ and } \eta\ \mathrm{Range}\, f \to \eta C_f.$$

Furthermore, $\eta\,\mathrm{Range}\, f \to \eta C_f \xrightarrow{P_f} \eta\,\mathrm{Range}\, f$ is the identity and $\eta\,\mathrm{Domain}\, f \to \eta C_f \xrightarrow{P_f} \eta\,\mathrm{Range}\, f$ is just f.

Finally, we recall that for two ordered simplicial complexes there is an associated triangulation of the product. In the case of $I = [0, 1]$ with $0 < 1$ and ηK with the incidence ordering a general simplex of $\eta K \times I$ is of the form:

$$(\eta A_0, 0) \ldots (\eta A_k, 0)(\eta A_{k+1}, 1) \ldots (\eta A_n, 1) \text{ with } A_0 < \cdots < A_k \leq A_{k+1} < \cdots < A_n \in K.$$

Thus, there is a natural isomorphism: $j_K\colon \eta C_{\mathrm{id}_K} \cong \eta K \times I$ on the full subcategory determined by the identity maps. Note that the projection P_{id_K} is associated under this isomorphism with the projection onto the first coordinates $\pi_K\colon \eta K \times I \to \eta K$, and we get an important commutative diagram:

$$(*)\qquad \begin{tikzcd}
\eta K \times I \arrow[r, phantom, "\overset{j_K^{-1}}{\cong}"] \arrow[dr, "\pi_K"'] & \eta C_{\mathrm{id}_K} \arrow[r, "\eta C(d_f)"] \arrow[d, "P_{\mathrm{id}_K}"] & \eta C_f \arrow[r, "\eta C(r_f)"] \arrow[dr, "P_f"] & \eta C_{\mathrm{id}_L} \arrow[r, phantom, "\overset{j_L}{\cong}"] \arrow[d, "P_{\mathrm{id}_L}"'] & \eta L \times I \arrow[dl, "\pi_L"] \\
 & \eta K \arrow[rr, "f"] & & \eta L &
\end{tikzcd}$$

where the map across the top is $f \times I$.

It is important to know that the mapping cylinder cannot be defined functorially for the p. l. category. This is because the obvious candidate for the mapping cylinder for $f\colon X \to Y$, i.e. $\bigcup\{[f(x), x] \subset Y * X\colon x \in X\} \cup Y$ is not, in general, a subpolyhedron of $Y * X$, e.g. if f is id_I. However, following the program which introduced the section we can, with Cohen, speak of a mapping cylinder of a p. l. map, i.e. choose a triangulation of f and use the simplicial mapping cylinder. Cohen's proof of the associated uniqueness theorem is an elegant application of the cell complex method and transverse cellularity.

Proposition 1. *Let $f\colon K \to L$ be a simplicial map and $f^*\colon K^* \to L^*$ a subdivision of f. Choose deriveds to obtain the mapping cylinders C_f and C_{f^*}.*

There exists a p. l. homeomorphism $H\colon C_f \to C_{f^*}$ rel $K \cup L$ *satisfying $H(C_f|K_0) = C_{f^*}|K_0^*$ for all subcomplexes K_0 of K (where K_0^* is the subdivision of K_0 induced by K^*).*

Proof. (See [C_1, Proposition 9.51].)

The relation between mapping cylinders and regular neighborhoods is rather close.

If V is a regular neighborhood of L in C_f, where $f\colon K \to L$, then $(V, \dot V)$

$\cong (C_f, K)$ rel L ([C_1, 9.3] and uniqueness of regular neighborhoods). For the relation the other way see [C_1, 9.7] or Proposition VI. 4.

Given a complex L and a choice of derived ηL, we can cut L up into a cone complex using the dual cells of L. Cohen defines an associated structure on K and C_f, where $f\colon K \to L$ is a simplicial map with chosen deriveds. For $A \in L$:

$$D(A; f) = f^{-1}D(A; L) = \{\eta B_0 \cdots \eta B_n\colon A \le fB_0 \text{ and } B_0 < \cdots < B_n \in K\},$$

$$\dot{D}(A; f) = f^{-1}\dot{D}(A; L) = \{\eta B_0 \cdots \eta B_n\colon A < fB_0 \text{ and } B_0 < \cdots < B_n \in K\},$$

$$Q(A; f) = P_f^{-1}D(A; L) = \{\eta A_0 \cdots \eta A_k\colon \eta B_0 \cdots \eta B_k \in \eta C_f\colon A \le A_0\},$$

$$\dot{Q}(A; f) = P_f^{-1}\dot{D}(A; L) = \{\eta A_0 \cdots \eta A_k\colon \eta B_0 \cdots \eta B_k \in \eta C_f\colon A < A_0\}.$$

From the definition we obtain several important facts.

(D1) $D(A; f) = Lk(A; C_f) \cap \eta K$.

(D2) Let $\overline{A}$ be a top dimensional face of A. Then

$$(D, \dot{D})(A; f) = (N, \dot{N})(f^{-1}\eta A; \dot{D}(\overline{A}; f)).$$

In particular, if A is a vertex of L, this is

$$(D, \dot{D})(A; f) = (N, \dot{N})(f^{-1}A; \eta K).$$

(D3) $Q(A; f) = \eta A * (\dot{Q}(A; f) \cup D(A; f))$ and there is a simplicial isomorphism rel $D(A; f)$ of $\dot{Q}(A; f) \cup D(A; f)$ with the subdivision of $Lk(A; C_f)$ induced by ηC_f. It is defined by the identity on $D(A; f)$ and the well-known isomorphism between $\dot{D}(A; L)$ and $\eta Lk(A; L)$.

(D4) The decomposition is natural in the sense that $(*)$ induces the following commutative diagram for any $A \in L$.

$$\begin{tikzcd}
{[(D, \dot{D})(A; f)] \times I} \arrow[r, "\eta C(d_f) j_k^{-1}"] \arrow[d, "\pi_K"'] & (Q, \dot{Q})(A; f) \arrow[r, "j_L \eta C(r_f)"] \arrow[d, "P_f"] & {[(D, \dot{D})(A; L)] \times I} \arrow[dl, "\pi_L"] \\
(D, \dot{D})(A; f) \arrow[r, "f"] & (D, \dot{D})(A; L) &
\end{tikzcd}$$

One of the crucial facts about this decomposition is that the pieces are nested regular neighborhoods [C_1, Proposition 5.6] and [C_3, Lemma 1.2].

Proposition 2. *Let $A \in L$ and $\overline{A}$ a top-dimensional face of A. Then $D(A; f)$ is a regular neighborhood of $f^{-1}\eta A$ in $\dot{D}(\overline{A}; f)$ with boundary $\dot{D}(A; f)$.*

So in particular, if $|K|$ is a p. l. manifold so are all the $D(A; f)$'s and $\dot{D}(A; f)$'s.

Remark. The canonical nature of the proof makes the relative theorem an easy variation. Thus, if K_0 is a subcomplex and $L_0 = f(K_0)$ then for $A \in K_0$ the pair $(D(A; f), D(A; f|K_0))$ is a regular neighborhood of $f^{-1}\eta A$ in $(\dot{D}(\bar{A}; f), \dot{D}(A; f|K_0))$.

From this result we obtain several facts about this decomposition.

(D5) The dual cells $\{D(A; f): A \in L\} \cup \{Q(A; f): A \in L\}$ give a general decomposition of $\mathfrak{S}_f(\overline{(C_f - L)}, K)$ (note $\overline{(C_f - L)} = C_f$ iff f is onto and, in fact, $\overline{(C_f - L)} = C_{f'} \subset C_f$ where $f'\colon K \to f(K)$ is the same as f). Recall that a general decomposition satisfies the axioms of a cone-complex decomposition except that $\sigma \cong$ cone $\partial\sigma$ is weakened to $\partial\sigma \subset \sigma$ is collared. The boundary is defined by $\partial D(A; f) = \dot{D}(A; f)$ and $\partial Q(A; f) = \dot{Q}(A; f) \cup D(A; f)$.

(D6) Note that the intrinsic dimension in $\dot{D}(A; f)$ and in $|K|$ are essentially the same. More precisely, if $\bar{A}$ is a top dimensional face of A, then

$$d(x; \dot{D}(A; f)) = d(x; \dot{D}(\bar{A}; f)) - 1, \quad x \in \dot{D}(A; f),$$

$$d(x; D(A; f)) = d(x; \dot{D}(\bar{A}; f)), \qquad x \in D(A; f) - \dot{D}(A; f)$$

(because the boundary of a regular neighborhood is collared) and this implies by induction on dim A,

$$d(x; \dot{D}(A; f)) = d(x; K) - (\dim A + 1).$$

(D7) We occasionally need the fact that the dual cells of the complex K refine the dual cells of the map f:

$$D(A; f) = \bigcup \{D(B; K): fB = A\}.$$

IV. **Transverse cellular mappings.** We now develop in detail the theory of transverse cellular mappings. This is the natural meeting point for the last two sections as this is precisely the case where the dual cell decomposition is a cone complex on the domain.

Definition. An onto simplicial map $f\colon K \to L$ is called transverse cellular if for each $A \in L$, $D(A; f) \cong$ cone $\dot{D}(A; f)$ rel $\dot{D}(A; f)$.

Proposition 1. *Let $f\colon K \to L$ be an onto simplicial map with deriveds chosen so that $f\colon \eta K \to \eta L$ is simplicial. If f is transverse cellular, then there is a p. l. homeomorphism*

$$c\colon (C_f; K, L) \cong (L \times I; L \times 1, L \times 0) \quad \text{rel } L = L \times 0$$

carried by the isomorphism of cone complexes:

$$\gamma_f(Q(A;\, f)) = D(A;\, L) \times I \cong (\eta A,\, 0) * [\dot{D}(A;\, L) \times I \cup D(A;\, L) \times 1],$$
$$\gamma_f(D(A;\, f)) = D(A;\, L) \times 1.$$

Since $j_L \circ C(r_f)$ *is also carried by this complex map, it is homotopic* rel L *to the homeomorphism through maps carried by* γ_f.

In particular, f *is homotopic to a homeomorphism through maps carried by the restriction of* γ_f *taking dual cells of* f *to those of* L.

Proof. Note that if we use the natural structuring on $Q(A;\, f) = \eta A * \partial Q(A;\, f)$, then $\mathfrak{S}_f$ is a cone complex on the pair $(C_f,\, L)$ and $\mathfrak{S}_f|L$ is just the dual cells of L with the natural structuring. So the homeomorphism $|\gamma_f|$ will be the identity on L.

Relating cone complexes $\{D(A;\, f) \times I\}$ on $K \times I$ to $\mathfrak{S}_f$ gives a similar theorem changing $C(d_f) \circ j_K^{-1}$ to a homeomorphism of $(K \times I;\, K \times 1,\, K \times 0)$ to $(C_f;\, K,\, L)$ rel $K \times 1 = K$.

This generalizes in a straightforward way to pairs. Let $f\colon (K,\, K_0) \to (L,\, L_0)$ be an onto simplicial map with $L_0 = f(K_0)$ ($K_0 = f^{-1}L_0$ is *not* necessary). Then $\mathfrak{S}_f|C_{f|K_0} = \mathfrak{S}_{f|K_0}$, interpreted as an equality of general complexes. Because $(Q(A;\, f),\, Q(A;\, f|K_0))$ is always a cone pair by (D3) of §III, the only cone problems involve the dual cells.

Assume $f\colon K \to L$ is transverse cellular, and so $\mathfrak{S}_f$ is a cone complex. Then $f\colon K_0 \to L_0$ is transverse cellular iff $\mathfrak{S}_f$ induces a cone complex on C_{f_0}. So we define "$f\colon (K,\, K_0) \to (L,\, L_0)$ is transverse cellular" to mean $\mathfrak{S}_f$ induces a cone complex on the pair $(C_f,\, C_{f|K_0})$.

If $f\colon (K,\, K_0) \to (L,\, L_0)$ is a transverse cellular, then by using a structuring on the complex pair, we can obtain a homeomorphism $c\colon C_f \to L \times I$ as in Proposition 1 satisfy in addition: $c(C_{f|K_0}) = L_0 \times I$. In particular, f can be homotoped to a homeomorphism as a map of pairs.

Thus going from transverse cellularity of $f\colon K \to L$ and $f\colon K_0 \to L_0$ to transverse cellularity of $f\colon (K,\, K_0) \to (L,\, L_0)$ is equivalent to the usual collection of weak unknotting problems of whether the pair of cones $(D(A;\, f),\, D(A;\, f|K_0))$ is a cone pair. We shall later see that if it happens that $K_0 = f^{-1}(L_0)$ then $f\colon (K,\, K_0) \to (L,\, L_0)$ is transverse cellular if f is transverse cellular on each term of the pair.

Proposition 1 generalizes for families of subcomplexes. Transverse cellularity is defined by the condition that $\mathfrak{S}_f$ induces a cone complex on the family of mapping cylinders.

There is an important sharpening of the relative case:

Proposition 2. *Let* $f\colon (K,\, K_0) \to (L,\, L_0)$ *be a transverse cellular map with* $f|K_0$ *an isomorphism, then there is a homeomorphism*

$$c\colon (C_f;\ K,\ L) \cong (L \times I;\ L \times 1,\ L \times 0)) \text{ rel } L = L \times 0$$

carried by γ_f *and equal to* $j_{L_0} \circ C(r_f|K_0)$ *on* $C_f|K_0$. *The homotopy of* c *with* $j_L \circ C(r_f)$ *can be taken* rel $C_f|K_0 \cup L$.

Proof. In this case if $A \in L_0$ and $B = (f|K_0)^{-1}A$ then $D(A; f) \cap K_0 = D(A; f|K_0) = \eta B * \dot{D}(A; f|K_0)$. Since $D(A; f|K_0)$ is a subcomplex of $D(A; f)$, the inclusion map $\eta B * \dot{D}(A; f|K_0) \to^{\text{inc}} D(A; f)$ is conewise homogeneous (each open cone-line lies entirely in one simplex). Hence if q: $(D(A; f), D(A; f|K_0)) \cong \text{cone }(\dot{D}(\bar{A}; f), \dot{D}(A; f|K_0))$ rel $\dot{D}(A; f)$ (recall $f\colon (K, K_0) \to (L, L_0)$ is transverse cellular), $q \circ \text{inc}$ unknots [A, Corollary IV. 8] and hence we can find a structuring on $D(A; f)$ which extends the natural structuring on $D(A; f|K_0)$. Using this structuring we sharpen the relative case of Proposition 1, because the homeomorphism we obtain of $C_f|K_0 \cong L_0 \times I$ by using the natural structuring is $j_{L_0} \circ C(r_f|K_0)$.

Criteria for transverse cellularity. The definition of transverse cellularity as it stands appears rather ad hoc. Its usefulness depends on relating the definition to regular neighborhood theory to give a more directly verifiable condition than the definition.

Proposition 3. *Let* $f\colon K \to L$ *be an onto simplicial map. Then the following are equivalent*:

(1) f *is transverse cellular.*

(2) *For all* $A \in L$, $D(A; f)$ *is a regular neighborhood of one of its points in* $\dot{D}(\bar{A}; f)$ *where* $\bar{A}$ *is a top dimensional face of* A.

(3) *For all* $A \in L$, $D(A; f) \searrow$ *point homogeneously in* K.

Let $f\colon (K, K_0) \to (L, L_0)$ *be an onto simplicial map of pairs such that* $f\colon K \to L$ *and* $f\colon K_0 \to L_0$ *are transverse cellular. Then the following are equivalent*:

(1) $f(K, K_0) \to (L, L_0)$ *is transverse cellular.*

(2) *For all* $A \in L_0$, $(D(A; f), D(A; f|K_0))$ *is a regular neighborhood of a point of* $D(A; f|K_0)$ *in* $(\dot{D}(\bar{A}; f), \dot{D}(\bar{A}; f|K_0))$.

(3) *For all* $A \in L_0$, $D(A; f) \searrow D(A; f|K_0)$ *homogeneously in* K.

Proof. Recall that by Proposition III. 2, $D(A; f)$ is a regular neighborhood of $f^{-1}\eta A$ in $\dot{D}(\bar{A}; f)$ with boundary $\dot{D}(A; f)$. Then the following are equivalent:

(1) $D(A; f) \cong \text{cone } \dot{D}(A; f)$ rel $\dot{D}(A; f)$.

(2) $D(A; f)$ is a regular neighborhood of a point in $\dot{D}(\bar{A}; f)$.

(3) $D(A; f) \searrow$ point homogeneously in $\dot{D}(\bar{A}; f)$.

(3) $\Rightarrow$ (2) by [A, III. 11d]. (2) $\Rightarrow$ (1) by uniqueness of regular neighborhoods. (1) $\Rightarrow$ (3) by collapsing down by the cone isomorphism. Homogeneity of the collapse in $\dot{D}(\bar{A}; f)$ follows because $\dot{D}(A; f)$ is bicollared.

Applying (D6) of §III, we can replace in (3), "homogeneously in $\dot{D}(\bar{A}; f)$" by "homogeneously in K".

In the case of pairs, (2) $\Rightarrow$ (1) is again uniqueness of regular neighborhoods. (1) $\Rightarrow$ (3) is collapsing a cone to a subcone, again using (D6) of §III to relate intrinsic dimension conditions. To obtain (3) $\Rightarrow$ (2), we first recall the remark that follows Proposition III.2, that the pair $D(A; (f, f|K_0))$ is a regular neighborhood pair of $f^{-1}\eta A$ in $\dot{D}(\bar{A}; (f, f|K_0))$. Then by [A, III. 11d], $D(A; f)$ is a regular neighborhood of $D(A; f|K_0)$ mod $\dot{D}(A; f|K_0)$ in $\dot{D}(\bar{A}; f)$, and since $f|K_0$ is transverse cellular, $D(A; f|K_0)$ is a regular neighborhood of a point in $\dot{D}(\bar{A}; f|K_0)$. (2) follows from [C_2, 7.9 b].

Corollary 4. *Let $f: K \to L$ be an onto simplicial map and for each $x \in L$, $f^{-1}(x) \searrow$ point homogeneously in K, then f is transverse cellular.*

Let $f: (K, K_0) \to (L, L_0)$ be an onto simplicial map with $f|K$ and $f|K_0$ transverse cellular and let $f^{-1}(x) \searrow (f|K_0)^{-1}(x)$ homogeneously in K. Then f is transverse cellular as a map of pairs.

Proof. $D(A; f)$ is a regular neighborhood of a point by [A, III.6]. For pairs, recall the remark following Proposition III.2 and apply, analogous to the absolute case, [A, strengthened version of Theorem III.6 described on pp. 439–440].

Two results from this corollary are

Corollary 5. *Let $f: (K, K_0) \to (L, L_0)$ be a simplicial map with $f: K \to L$ and $f: K_0 \to L_0$ transverse cellular and assume $K_0 = f^{-1}L_0$. Then $f: (K, K_0) \to (L, L_0)$ is transverse cellular.*

Corollary 6. *Let $f: (B, S) \to (X, Y)$ be an onto p. l. map with $f^{-1}Y = S$ and such that B is a p. l. ball with boundary S and such that for $x \in X$, $f^{-1}(x)$ is collapsible. Then any triangulation of f is transverse cellular and hence X is a p. l. ball homeomorphic to B with boundary Y.*

Because of Corollary 4, we ask the following: $X_0 \searrow$ point homogeneously in X, and $p \in X_0$; then does $X_0 \searrow p$ homogeneously in X?

Lemma 7. *Let $X_0 \subset X$ and assume X_0 collapses to a point homogeneously in X. Then for $p \in X_0$, $X_0 \searrow_h p$ in X iff $d(p; X) = \min \{d(x; X): x \in X_0\}$.*

Proof. Define $m(X_0; X) = \min \{d(x; X): x \in X_0\}$. Then if $X_0 \searrow_h X_1$ in X then $m(X_0; X) = m(X_1; X)$, for it is easily seen to be true for elementary geometrical homogeneous collapses. Hence, if $X_0 \searrow_h p$ in X, then $m(X_0; X) = m(p; X) = d(p; X)$.

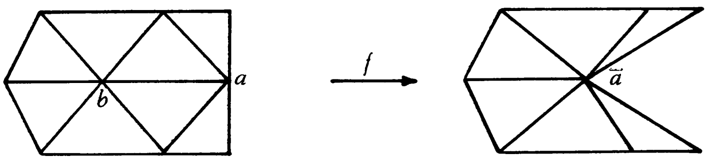
Conversely, if $m = m(X_0; X)$ then since $X_0 \searrow$ pt homogeneously and since [A, III. 10] this can be done in order of decreasing intrinsic dimensions, $X_0 \searrow_h X_0 \cap I^m(X) = X_0 \cap (I^m(X) - I^{m-1}(X)) \searrow$ pt in X. Now if $d(p; X) =$

$m(X_0; X)$ then $p \in X_0 \cap I^m(X)$ which is collapsible and hence $X_0 \cap I^m(X) \searrow p$. Since $X_0 \cap I^m(X) = X_0 \cap (I^m(X) - I^{m-1}(X))$ (m is minimal), this collapse is homogeneous in X.

Corollary 8. *Let $f\colon K \to L$ be an onto simplicial map with $f^{-1}(x) \searrow$ point homogeneously in K for each $x \in L$ and let $K_0 \subset K$ such that $f|K_0$ is a homeomorphism onto $f(K_0) = L_0$. Then if for each $y \in K_0$, $d(y; K) = \min\{d(z; K)\colon z \in f^{-1}f(y)\}$, then $f\colon (K, K_0) \to (L, L_0)$ is transverse cellular.*

Proof. Lemma 7 and Corollary 4.

The necessity for homogeneous collapsing is illustrated by the map from the disc shown below so that $f^{-1}(\bar{a}) = [a, b]$, so $f|b$ is an isomorphism,



but clearly no homeomorphism of balls will take the interior point b to the boundary point $\bar{a}$. Note that the collapse $[a, b] \searrow b$ is not homogeneous.

We now show that despite its simplicial definition, transverse cellularity is a p. l. phenomenon. To do this, we use the fact that the simplicial mapping cylinder is a p. l. invariant and then follow Cohen by proving a strong converse to Proposition 1.

Proposition 9 (closed). *Let $f\colon K \to L$ be a simplicial mapping and Y a subpolyhedron of L, then $P_f^{-1}(Y)$ is p. l. homeomorphic to a mapping cylinder for $f|f^{-1}(Y)$ rel $f^{-1}(Y) \cup Y$.*

Proof. Let $P_f^{-1}(Y) \cap K = f^{-1}(Y) = X$. Triangulate $f|X$ to obtain $\hat{f}\colon K_0 \to L_0$ simplicial with K_0 (resp. L_0) a refinement of K (resp. L). Note that we do not assume that K_0 is obtained from a subdivision of K.

We define isomorphic cell complexes on $P_f^{-1}(L_0)$ and $C_{\hat{f}|K_0}$ whose realization will give the desired homeomorphism.

Let $\sigma \in L_0$, with $A \in L$ the carrier of σ and let $B \in K$ be such that $fB = A$. That the complexes exist follows from the sublemma:

Sublemma. (1) $C_{\hat{f}|f^{-1}(\sigma)\cap B}$ *is a* $\dim(f^{-1}(\sigma) \cap B) + 1$ *p. l. ball with boundary* $= C_{\hat{f}|f^{-1}(\dot{\sigma})\cap B} \cup C_{\hat{f}|f^{-1}(\sigma)\cap \dot{B}} \cup (f^{-1}(\sigma) \cap B) \cup \sigma$.

(2) $P_f^{-1}(\sigma) \cap C_{f|B}$ *is a* $\dim(f^{-1}(\sigma) \cap B) + 1$ *p. l. ball with boundary* $= (P_f^{-1}(\dot{\sigma}) \cap C_{f|B}) \cup (P_f^{-1}(\sigma) \cap C_{f|\dot{B}}) \cup (f^{-1}(\sigma) \cap B) \cup \sigma$.

Assuming the sublemma we compare the following complexes by the obvious isomorphism and obtain the homeomorphism:

$\{P_f^{-1}(\sigma) \cap C_{f|B}, f^{-1}(\sigma) \cap B, \sigma\colon\ \sigma \in L_0$ and $fB = \mathrm{Car}\,(\sigma)$ in $L\}$

(decomposing $(P_f^{-1}(L_0); K_0, L_0)$).

$\{C_{\hat{f}|f^{-1}(\sigma)\cap B}, f^{-1}(\sigma) \cap B, \sigma\colon \sigma \in L_0$ and $fB = \mathrm{Car}\,(\sigma)$ in $L\}$

(decomposing $(C\hat{f}|K_0; K_0, L_0)$).

The key to proving the sublemma was given in the proof of $[C_1, 9.5]$. In fact, since $f\colon f^{-1}(\sigma) \cap B \to \sigma$ is an onto linear map of a convex cell, for the proof of part 1, we need only note that $\partial(f^{-1}(\sigma) \cap B) = (f^{-1}(\dot{\sigma}) \cap B) \cup (f^{-1}(\sigma) \cap \dot{B})$, and apply Corollary 6 to $(f^{-1}(\sigma) \cap B) \times I \to C_{\hat{f}|f^{-1}(\sigma)\cap B}$, obtained from $(*)$ of §III.

To prove the second part, we again use the commutative diagram $(*)$ which restricts to the following:

$$\begin{array}{ccc} B \times I & \xrightarrow{C(d_f)\circ j_B^{-1}} & C_{f|B} \\ \big\downarrow{\scriptstyle \pi_B} & & \big\downarrow{\scriptstyle P_f} \\ B & \xrightarrow{\ \ f\ \ } & A \end{array}$$

Hence, $C(d_f) \circ j_B^{-1}$ restricts to a map:

$$((f^{-1}(\sigma) \cap B) \times I;\ (f^{-1}(\dot{\sigma}) \cap B) \times I,\ (f^{-1}(\sigma) \cap \dot{B}) \times I,$$
$$(f^{-1}(\sigma) \cap B) \times 1,\ (f^{-1}(\sigma) \cap B) \times 0)$$
$$\to (P_f^{-1}(\sigma) \cap C_{f|B};\ P_f^{-1}(\dot{\sigma}) \cap C_{f|B},\ P_f^{-1}(\sigma) \cap C_{f|\dot{B}},\ f^{-1}(\sigma) \cap B,\ \sigma).$$

Furthermore, the inverse of each point is collapsible by $[C_1, 3.3]$ or see §VI. The result follows from Corollary 6.

Proposition 9 **(open).** *Let $f\colon K \to L$ be a simplicial mapping and U an open subset of L, then $P_f^{-1}(U)$ is p. l. homeomorphic to a mapping cylinder for $f|f^{-1}(U)$* rel *$U \cup f^{-1}(U)$.*

Proof. Triangulate with refinements of K and L to obtain $\hat{f}\colon f^{-1}(U) \to U$ and a mapping cylinder $C_{\hat{f}}$. We want to obtain a homeomorphism of $P_f^{-1}(U)$ with $C_{\hat{f}}$. To do this we use the Morton Brown infinite union swindle and repeatedly apply Propositon 9 (closed).

Let $U = \bigcup_{i=1}^{\infty} V_i$, $\{V_i\}$ an increasing sequence of finite subcomplexes of U. Since each V_i is compact, Proposition 9 (closed) applies and we have, for each i, $k_i\colon P_f^{-1}(V_i) \cong P_{\hat{f}}^{-1}(V_i) = C_{\hat{f}|f^{-1}(V_i)}$ rel $V_i \cup f^{-1}(V_i)$. We refer to the proof of Proposition 9 (closed) to show that we can assume $k_{i+1}|P_f^{-1}(V_i) = k_i$. Then $\bigcup_{i=1}^{\infty} k_i\colon P_f^{-1}(U) \cong C_{\hat{f}}$. To refine the proof of Proposition 9 (closed) note that the cell-complexes constructed for $P_f^{-1}(V_{i+1})$ and $C_{\hat{f}|f^{-1}(V_{i+1})}$ have as subcomplexes respectively the cell complexes for $P_f^{-1}(V_i)$ and $C_{\hat{f}|f^{-1}(V_i)}$. So if k_i: $P_f^{-1}(V_i) \to C_{\hat{f}|f^{-1}(V_i)}$ has been defined preserving this complex isomorphism, k_{i+1} can be realized to extend k_i.

Now we obtain a p. l. invariant criterion for transverse cellularity.

Proposition 10. *If f: $K \to L$ is a simplicial map, with L_0 a subcomplex of L and $K_0 = f^{-1}L_0$, then the following are equivalent.*

(1) $D(A; f)$ is a cone on $\dot{D}(A; f)$ for all $A \in L - L_0$.

(2) C_f is locally collared on $L - L_0$.

(3) For any choice of triangulations, f: $|K - K_0| \to |L - L_0|$ is transverse cellular.

(4) There exists a triangulation of f: $|K - K_0| \to |L - L_0|$ which is transverse cellular.

Proof. (2) $\Rightarrow$ (1) by induction on dim L.

By local collaring, we have for $A \in L - L_0$ the isomorph of its link—recall property (D3) of §III—is a cone, i.e. $\dot{Q}(A; f) \cup D(A; f) \cong$ cone $\dot{D}(A; L)$ rel $\dot{D}(A; L)$, implying in particular $D(A; f) \neq \emptyset$ so f is onto $L - L_0$.

Since $\dot{Q}(A; f) = P_f^{-1}\dot{D}(A; L)$ it follows from Proposition 9 (closed) that $(\dot{Q}(A; f); \dot{D}(A; f), \dot{D}(A; L))$ is homeomorphic to a mapping cylinder of $f|\dot{D}(A; f)$. Hence, by inductive hypothesis $f|\dot{D}(A; f)$ is transverse cellular (since $\dot{D}(A; L) \subset \dot{Q}(A; f)$ is collared) and so by Proposition 1, $(\dot{Q}(A; f); \dot{D}(A; f), \dot{D}(A; L))$ is homeomorphic to $(\dot{D}(A; f) \times I; \dot{D}(A; f) \times 0, \dot{D}(A; f) \times 1)$. Hence the cone $\dot{Q}(A; f) \cup D(A; f)$ is just $D(A; f)$ with an exterior collar. So $D(A; f) \cong \dot{Q}(A; f) \cup D(A; f)$ and is hence a cone.

(1) $\Rightarrow$ (2). If $D(A; f)$ is a cone for all $A \in L - L_0$ then γ_f gives a cone complex isomorphism between $\dot{Q}(A; f) \cup D(A; f)$ and $(\dot{D}(A; L) \times I) \cup (D(A; L) \times 1)$ indexed by $\{B: A < B\}$. So $Lk(A; C_f)$ is a cone by §III (D3). Essentially, we just reverse the above argument.

To relate these to (3) and (4), note that by Proposition 9 (open) $P_f^{-1}(L - L_0)$ is homeomorphic rel $|K - K_0| \cup |L - L_0|$ to any mapping cylinder for f: $|K - K_0| \to |L - L_0|$. Hence, (4) $\Rightarrow$ (2) a fortiori by Proposition 1.

Finally, given (2), let $\hat{f}$ be any triangulation of $f||K - K_0|$. By Proposition 9 (open) again, $P_f^{-1}(L - L_0)$ is homeomorphic to $C_{\hat{f}}$ and hence $|L - L_0| \subset C_{\hat{f}}$ is collared. Now (2) $\Rightarrow$ (1) applied to the absolute case of $\hat{f}$ gives $\hat{f}$ transverse cellular. Thus (2) $\Rightarrow$ (3).

Clearly, (3) $\Rightarrow$ (4).

Remark. From the proof of (2) $\Rightarrow$ (1) we obtain a useful fact about transverse cellularity on the dual cells. Let f: $K \to L$ be simplicial and, $A \in L$ and $Lk(A; C_f) \cong$ cone $Lk(A; L)$ rel $Lk(A; L)$; then

(1) $f|\dot{D}(A; f)$ is transverse cellular.

(2) $D(A; f)$ is a cone on $\dot{D}(A; f)$.

(3) $f|D(A; f)$ is transverse cellular.

(1) and (2) are directly contained in the above proof. From it we obtain an isomorphism

$$(\dot{Q}(A; f) \cup D(A; f); D(A; f), \dot{D}(A; L))$$

$$\cong ((\dot{D}(A; f) \times I) \cup (D(A; f) \times 1); D(A; f) \times 1, \dot{D}(A; f) \times 0);$$

coning this and using the Alexander trick, we get a homeomorphism

$$(Q(A; f); D(A; f), D(A; L)) \cong (D(A; f) \times I; D(A; f) \times 1, D(A; f) \times 0).$$

Since $Q(A; f)$ is by Proposition 9 a mapping cylinder for $f|D(A; f)$, Proposition 10 implies that $f|D(A; f)$ is transverse cellular.

We have used the invariance of the mapping cylinder and of the property of local collaring to obtain invariance of transverse cellularity. Note that the logic of this argument depended on the obvious p. l. invariance of the property $f^{-1}(x) \searrow$ point homogeneously, which is stronger than transverse cellularity. The dependence comes from repeated applications of Corollary 6, in Proposition 9 for example.

From Corollary 10, we can define a p. l. map $f\colon X \to Y$ to be transverse cellular if it is so for any triangulation.

Quasi-concordance and quasi-isotopy. We also obtain a method for making constructions of transverse cellular mappings. Recall that a general decomposition of a space is a collection of pieces that fit together like a cone complex but $\partial\sigma$ is only collared in σ instead of $\sigma \cong$ cone $\partial\sigma$. An isomorphism of general decompositions is a bijection preserving incidence in each direction.

Proposition 11. *Let $f\colon X \to Y$ be an onto p. l. map carried by an isomorphism $\gamma\colon \mathfrak{S} \to \mathfrak{D}$ with $\mathfrak{S}$, $\mathfrak{D}$ general complexes on X and Y respectively and $f^{-1}(\partial\gamma\sigma) = \partial\sigma$. Assume that $f\colon \sigma \to \gamma(\sigma)$ is transverse cellular for each $\sigma \in \mathfrak{S}$. Then f is transverse cellular.*

Proof. Proceed by induction on the dimension of cells in $\mathfrak{S}$. Assume $\dim X = n$ and that there is only one n-dimensional cell. Let σ be the top dimensional cell of $\mathfrak{S}$ and $X_\sigma = \mathrm{Cl}(X - \sigma) = \bigcup(\mathfrak{S} - \sigma)$, and $Y_\sigma = \mathrm{Cl}(Y - \gamma\sigma) = \bigcup(\mathfrak{D} - \gamma\sigma)$. By inductive hypothesis $f\colon X_\sigma \to Y_\sigma$ and $f\colon \partial\sigma \to \gamma\partial\sigma = \partial\gamma\sigma$ are transverse cellular. Since $f^{-1}\gamma\partial\sigma = \partial\sigma$, $f\colon (\sigma, \partial\sigma) \to (\gamma\sigma, \partial\gamma\sigma)$ and $f\colon (X_\sigma, \partial\sigma) \to (X_{\gamma\sigma}, \partial\gamma\sigma)$ are transverse cellular by Corollary 5. Triangulating so that everything is subcomplex, we obtain isomorphisms

$$j_1\colon (C_{f|X_\sigma}, C_{f|\partial\sigma}) \cong (Y_\sigma, \gamma\partial\sigma) \times I \quad \text{rel } Y_\sigma = Y_\sigma \times 0,$$

$$j_2\colon (C_{f|\sigma}, C_{f|\partial\sigma}) \cong (\gamma\sigma, \gamma\partial\sigma) \times I \quad \text{rel } \gamma\sigma = \gamma\sigma \times 0.$$

Since $\partial\sigma \subset \sigma$ is collared, we can change j_2 to agree with j_1 on $C_{f|\partial\sigma}$. (Extend the concordance of $\mathrm{id}_{\gamma\partial\sigma}$ given by $j_1 j_2^{-1}$ to a concordance of $\mathrm{id}_{\gamma\sigma}$ and compose with j_2. Note that if A is compact and $B \subset A$ is collared then any concordance of id_B extends to one of id_A because $(A \times I;\ A \times 0 \cup B \times I,\ A \times 1) \cong (A \times I;\ A \times 0,\ A \times 1 \cup B \times I)$.) Glue them together to obtain $j\colon\ C_f \cong Y \times I$ rel $Y = Y \times 0$ and hence f is transverse cellular by Proposition 10.

In general, just deal with all of the n-dimensional cells at the same time as above.

Remarks. Alternative to the assumption $f^{-1}(\partial\gamma\sigma) = \partial\sigma$, we can just assume $f\colon\ (\sigma, \partial\sigma) \longrightarrow (\gamma\sigma, \partial\gamma\sigma)$ is transverse cellular.

For a converse to this theorem, see Proposition V.9.

Corollary 12. *Let $f\colon\ X \longrightarrow Y$, $g\colon\ Y \longrightarrow Z$ be transverse cellular p. l. maps. Then gf is transverse cellular.*

Proof. Triangulate to obtain $K \xrightarrow{f} L \xrightarrow{g} M$ simplicial. Since g is transverse cellular, we have $D(A;\ g) = \operatorname{cone} \dot{D}(A;\ g)$ for $A \in M$. By Remark 3 after Proposition 10, $f|D(B;\ f)$ for each B in L is transverse cellular. $D(A;\ g)$ and $\dot{D}(A;\ g)$ for each A in M is a union of dual cells of L by (D7) of §III and hence $D(A;\ gf) = f^{-1}D(A;\ g)$ is a union of dual cells of f. So $f\colon\ (D(A;\ gf),\ \dot{D}(A; gf)) \longrightarrow (D(A;\ g),\ \dot{D}(A;\ g))$ is transverse cellular by the above proposition and Corollary 5. Since $(D(A;\ g),\ \dot{D}(A;\ g))$ is a cone pair, $(D(A;\ gf),\ \dot{D}(A;\ gf))$ is by Proposition 1.

In applying Proposition 11 to cone complexes to make inductive construction, we will need the following:

Lemma 13. *Let $f\colon\ X \longrightarrow Y$ be a transverse cellular map. Then $cf\colon\ (cX,\ X) \longrightarrow (cY,\ Y)$ (the cone of f) is transverse cellular.*

Proof. By Corollary 5, we need only show $cf\colon\ cX \longrightarrow cY$ is transverse cellular. Triangulate so that $f\colon\ K \longrightarrow L$ is simplicial. Choose deriveds so that $f\colon\ \eta cK \longrightarrow \eta cL$ is simplicial. $f^{-1}(L) = K$ so for $A \in L$, $D(A;\ cf) \searrow D(A;\ f) \searrow$ point homogeneously. $(cf)^{-1}(c) = c$ so $D(c;\ cf) \searrow (cf)^{-1}c = c$ homogeneously.

Consider the simplicial maps $q\colon\ \eta K \longrightarrow \eta cK$ and $q\colon\ \eta L \longrightarrow \eta cL$ where $q(\eta A) = \eta cA$ for $A \in K$ or L. Note that $q \circ f = cf \circ q$, and $q(D(A;\ L)) = D(cA;\ cL)$. Hence, $q\colon\ D(A;\ f) \cong D(cA;\ cf)$. Hence $D(cA;\ cf) \searrow$ point. Since the collapse of $D(A;\ f)$ is homogeneous in K and since $d(q(x);\ cK) = \dim(x;\ K) + 1$ for all x, the corresponding collapse of $D(cA;\ cf)$ is homogeneous in cK.

Transverse cellularity follows from Proposition 3.

An application is the following uniqueness theory for the homeomorphism associated to a transverse cellular map.

We say that f_0, f_1: $X \to Y$ are quasi-concordant if there exists F: $(X \times I;\ X \times 0,\ X \times 1) \to (Y \times I;\ Y \times 0,\ Y \times 1)$ transverse cellular with $F^{-1}(X \times i) = X \times i$ and $F_i = f_i$, $i = 0, 1$. If F is also level-preserving, we will call F a quasi-isotopy. The importance of this equivalence relation is the following:

Lemma 14. *Let* f: $K \to L$ *be a transverse cellular simplicial map and let* $\hat{f}$: $K \to L$ *be a p.l. homeomorphism carried by* γ_f. *Then* f *and* $\hat{f}$ *are quasi-isotopic.*

Proof. Consider the cone complexes $\{D(A; f) \times i,\ D(A; f) \times I\colon\ A \in L,\ i = 0, 1\}$ and $\{D(A; L) \times i,\ D(A; L) \times I\colon\ A \in L,\ i = 0, 1\}$ on $K \times I$ and $L \times I$ respectively and let γ be the obvious isomorphism of cone complexes. Extend $f \times 0$ and and $\hat{f} \times 1$ to a map of $K \times I \to L \times I$, carried by γ, by proceeding up the cells coning inductively with cone points at the ½ level using the Alexander trick. F is transverse cellular on $D(A; f) \times 1$'s and is so on the $D(A; f) \times 0$'s by the remarks following Proposition 10. On the remaining cells, the $D(A; f) \times I$'s, F is transverse cellular by induction, i.e. F is transverse cellular on $\dot{D}(A; f) \times I \cup D(A; f) \times \dot{I}$ by induction and Proposition 11, so it is on $D(A; f) \times I$ by Lemma 13. Another application of Proposition 11 proves that F is transverse cellular on $X \times I$.

Note that F as constructed is level-preserving, i.e. F is a quasi-isotopy.

Proposition 15. (1) *Two homeomorphisms are quasi-concordant iff they are concordant.*

(2) *Two homeomorphisms associated to a transverse cellular map are concordant.*

(3) *A homeomorphism is quasi-concordant to a transverse cellular map iff it is concordant to an associated homeomorphism.*

(4) *Transverse cellular maps are quasi-concordant iff associated homeomorphisms are concordant.*

Proof. Proposition 2.

For a strengthening of these results in the compact case to their analogues for quasi-isotopies see the last section. Also we will show that our definition of quasi-isotopy is equivalent to the more natural definition: a level-preserving map F with F_t transverse cellular for all t (Corollary V. 10).

V. **Regular extensions.** We construct for the p. l. category the proper way of attaching one polyhedron by a map of a subpolyhedron to another one. Topologically if f: $X_0 \to Y$ with $X_0 \subset X$ then we can think of $X \cup_f Y$ as $X/f \cup Y$ where X/f is obtained by "crushing" X_0 by the equivalence relation $x \equiv x'$ means $f(x) = f(x')$, and where the union, $X/f \cup Y$ has intersection $X_0/f = f(X_0)$. We generalize this to the simplicial category following Cohen.

We call a subcomplex K_0 of K well situated if $A \in K$ with $A \cap K_0 = \emptyset$ implies $Lk(A; K) \cap K_0$ is empty or a simplex. For example, if K_0 is a full subcomplex of K, then K_0 is well situated in $\eta_{K_0}K$ where η_{K_0} is a derived modulo K_0.

Suppose that K_0 is well situated in $K = N(K_0; K) \cup P$, $(P \cap N(K_0; K) = \dot{N}(K_0; K))$ and that f: $K_0 \to L_0$ is a simplicial mapping, then the stellar extension F: $K \to L$ is defined to be the unique, simplicial map of $K \to (L_0 * \dot{N}(K_0; K)) \cup P$ which is the identity on P and f on K_0, where L is defined to be the image of this map $\bigcup L_0$. Note that L is locally finite and F is proper.

Example. $C(d_f)$: $C_{\mathrm{id}_K} \to C_f$ is a stellar extension of f: $K \to L$ (K regarded as range of id_K).

The key property of stellar extensions, which relates them to transverse cellular maps is given by the following results of Cohen which calculates the preimages of points under a stellar extension [C_1, 3.3].

Proposition 1. *Let K_0 be a well-situated subcomplex of K, f: $K_0 \to L_0$ a simplicial map and F: $K \to L$ the stellar extension of f. Then*:

(1) *for $y \in L_0$, $F^{-1}(y) = f^{-1}(y)$; $y \in P$, $F^{-1}(y) = y$; $y \in N(L_0; L) - (L_0 \cup P)$, $F^{-1}(y)$ is a convex cell.*

(2) *If $K = \eta_{K_0}\overline{K}$ then for $y \in N(L_0; L) - (L_0 \cup P)$, $F^{-1}(y)$ is contained in the interior of a simplex of $\overline{K}$ and so F: $|K - K_0| \to |L - L_0|$ is transverse cellular.*

Proof. (1) is [C_1, 3.3] and the first part of (2) is an easy extension of that proof. The transverse cellularity follows from Corollary IV.4.

Just as in the case of regular neighborhood theory where we must sharpen the notion of stellar neighborhood to obtain p. l. invariance so also in the case of stellar extension of a map.

If K_0 is a full subcomplex of K, f: $K_0 \to L_0$ an onto simplicial map and $\eta_{K_0}K$ a derived mod K_0, then the stellar extension, F: $\eta_{K_0}K \to L$, associated to f, is called a derived stellar extension, or a derived extension of f. Using derived extensions we will be able to develop the corresponding p. l. theory.

Note that if $K = \Delta^2$, $K_0 = \Delta^1$ and $L_0 = \Delta^0$ then the image of K under the stellar extension of f: $K_0 \to L_0$ is a one simplex but the image after going to the derived is a two ball.

The theorem motivates our p. l. definition.

Definition. Let X_0 be a subpolyhedron of X and f: $X_0 \to Y_0$ a p. l. map. A regular extension of f is a map F: $X \to Y$ and an embedding i: $Y_0 \to Y$ satisfying

(1) $F|X_0 = i \circ f$ and $F^{-1}(Y_0) = X_0$.

(2) $F\colon X - X_0 \to Y - Y_0$ is transverse cellular.

We have seen that any derived stellar extension of a triangulation of f gives a regular extension. This is "essentially" all there are.

Let $F_1\colon K \to L_1$ be a simplicial map with a full subcomplex L_0 of L_1 and $K_0 = F_1^{-1}(L_0)$ (necessarily a full subcomplex of K). Choose deriveds mod K_0 and L_0 such that $F_1\colon \tau_{K_0}K \to \tau_{L_0}L_1$ is simplicial, then we can factor F_1, simplically, through the derived stellar extension $F\colon \tau_{K_0}K \to L$ of $F_1|K_0\colon K_0 \to L_0$. That is, there is a unique G, simplicial and making the following diagram commute:

$$\begin{tikzcd} \tau_{K_0}K \arrow[rr, "F_1"] \arrow[dr, "F"'] & & \tau_{L_0}L_1 \\ & L \arrow[ur, "G"'] & \end{tikzcd}$$

We use a special case of the following situation:

Lemma 2. *Let $X \to^{F_1} Y_1 \leftarrow^{i_1} Y_0$ and $X \to^{F} Y \leftarrow^{i} Y_0$ are regular extensions of $f\colon X_0 \to Y_0$ and $G\colon Y \to Y_1$ such that $G^{-1}i_1Y_0 = iY_0$ and the following commutes:*

$$\begin{tikzcd} X \arrow[r, "F_1"] \arrow[dr, "F"'] & Y_1 & Y_0 \arrow[l, "i_1"'] \arrow[dl, "i"] \\ & Y \arrow[u, "G"] & \end{tikzcd}$$

Then G is transverse cellular.

Proof. Identify Y_0 with its images under i, i_1 and triangulate to obtain the simplicial diagram

$$\begin{tikzcd} (K, K_0) \arrow[rr, "F_1"] \arrow[dr, "F"'] & & (L_1, L_0) \\ & (L, L_0) \arrow[ur, "G"'] & \end{tikzcd}$$

Let $A \in L_1$, then since $F_1 = GF$, the definition of dual cells gives the commutative diagram:

$$\begin{tikzcd} (D, \dot{D})(A; F_1) \arrow[rr, "F_1"] \arrow[dr, "F"'] & & (D, \dot{D})(A; L_1) \\ & (D, \dot{D})(A; G) \arrow[ur, "G"'] & \end{tikzcd}$$

(1) $A \in L_0$, then since $G|L_0 = \mathrm{id}_{L_0}$ and $G^{-1}(L_0) = L_0$, $(D, \dot{D})(A; G) = (D, \dot{D})(A; L)$ and is consequently a cone pair. For later reference note that in

this case it then follows that $(D, \dot{D})\,(A;\, F_1) = (D, \dot{D})\,(A;\, F)$.

(2) $A \in L_1 - L_0$, then by Proposition IV.10 $(D, \dot{D})\,(A;\, F_1)$ is a cone pair. But $D(A;\, G)$ decomposes to the cone-complex $\{D(B;\, L)\colon\ A \leq GB\}$ and by applying F^{-1}, we decompose $D(A;\, F_1)$ by $\{D(B;\, F)\colon\ A \leq GB\}$, which is also a cone complex by Proposition IV.10 applied to F. (Compare this with the proof of Proposition IV.12.) Realizing the obvious isomorphism gives a homeomorphism of pairs $(D, \dot{D})\,(A;\, F_1) \cong (D, \dot{D})\,(A;\, G)$. Since the former is a cone pair, the latter is.

Corollary 3. *Let $f\colon\ X \to Y$ and $g\colon\ Y \to Z$ with f and gf transverse cellular, then g is transverse cellular.*

Proof. f and gf are regular extensions of the "empty map" and Proposition 2 applies with $Y_0 = 0$.

Combining Corollary 3 and Corollary IV.12 we see that if f and g are transverse cellular then gf is and if f and gf are transverse cellular then g is. However, g and gf may be transverse cellular and f not so. For let Y be a manifold (with $\partial Y = 0$) and B be a codimension 0 ball in Y. If g is a regular extension of $B \to$ point then g has collapsible point-inverses and is thus transverse cellular. Now if $f\colon\ Y \to Y$ with $f^{-1}(B) = B$ and $f|Y - B =$ identity then $gf = g$ and is thus transverse cellular. But since f can do anything inside the ball it clearly need not be transverse cellular itself.

We now apply Lemma 2 to obtain the "uniqueness theorem" for regular extensions.

Proposition 4. *Let X_0 be a subpolyhedron of X and $f\colon\ X_0 \to Y_0$ be a p. l. map.*

(1) A p. l. map $F_1\colon\ X \to Y_1$ and an embedding $i\colon\ Y_0 \to Y_1$ satisfying $F_1|X_0 = i \circ f$ and $F^{-1}(Y_0) = X_0$ is a regular extension of f iff there exists a derived stellar extension of a triangulation of f, $F\colon\ X \to Y$, and a transverse cellular map $G\colon\ Y \to Y_1$ with $G^{-1}(i(Y_0)) = Y_0$ and $GF = F_1$.

(2) Let $F_1\colon\ X \to Y_1$, $i_1\colon\ Y_0 \to Y_1$ and $F_2\colon\ X \to Y_1$, $i_2\colon\ Y_0 \to Y_2$ be regular extensions of f. Then there exists a regular extension $F\colon\ X \to Y$, $i\colon\ Y_0 \to Y$ and transverse cellular maps $G_\alpha\colon\ Y \to Y_\alpha$ with $G_\alpha^{-1}(i_\alpha(Y_0)) = i(Y_0)$ for $\alpha = 1, 2$ and making the following diagram commute:

$$\begin{tikzcd}
& X \arrow[dl, "F_1"'] \arrow[d, "F"] \arrow[dr, "F_2"] & \\
Y_1 & Y \arrow[l, "G_1"'] \arrow[r, "G_2"] & Y_2 \\
& Y_0 \arrow[ul, "i_1"] \arrow[u, "i"'] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

Proof. (1) follows from Lemma 2 one way and Corollary IV.12, the other.

(2) follows directly from Lemma 2, just triangulate so that F_1 and F_2 are simultaneously simplicial.

Remark. For a rather different "uniqueness theorem" for regular extensions see Proposition VI.9.

One might suspect that we have allowed too much with transverse cellular maps and that if we restricted our definition to triangulations of derived stellar extensions we might be able to get a uniqueness theorem relating two extensions by conjugation with a homeomorphism. This is not usually possible as the following example shows.

Let $(K, K_0) = (\Delta^2, \Delta^1)$, $L_0 = \Delta^0$ and K^* equals stellar subdivision of K at Δ^1. Then if $F\colon \eta_{\Delta^1} K \to L$ and $F^*\colon \eta_{\Delta^1} K^* \to L^*$ are the derived stellar extensions then the singularity sets are not homeomorphic. $S(F)$ is a 2-simplex and $S(F^*)$ is two 2-simplices joined at a vertex.

Of occasional usefulness is the following feeble analogue of Cohen's powerful result for regular neighborhoods [C_2, 6.1].

Lemma 5 (Stellar Extension Lemma). *Let K_0 be well situated in K and $f\colon K_0 \to L_0$ a simplicial map with $F\colon K \to L$ the stellar extension of f. Any simplex $A \in N(K_0; K) - (K_0 \cup \dot{N})$ can be written $A = BC$ where $B = A \cap K_0$. If for any such A all faces $\tilde{B}C$, where $\tilde{B} < B$ and $f(\tilde{B}) = f(B)$, are nice faces (see* [A, II 12]), *then $F\colon |K| \to |L|$ is a regular extension of f.*

Proof. Referring to [C_1, 3.3] as in the proof of part (2) of Proposition 1, we see that this is precisely the condition needed to insure constant intrinsic dimension of all convex cell $F^{-1}(y)$ for $y \in N(L_0; L) - (L_0 \cup \dot{N})$.

Regular extensions provide a tool for constructing transverse cellular maps.

Proposition 6. *Let X_0 be a subpolyhedron of X and $f\colon X_0 \to Y_0$ a p. l. map. Let $K \xrightarrow{F} L \xleftarrow{i} L_0$ be the triangulation of a regular extension of f, i.e. $|K| = X$, $|L_0| = Y$ and $|K_0| = X_0$. Then the following are equivalent.*

(1) *For all $A \in i(L_0)$, $D(A; F) \searrow$ point homogeneously in X.*

(2) *$F\colon K \to L$ is transverse cellular.*

(3) *For every regular extension of f, $X \xrightarrow{F_1} Y_1 \xleftarrow{i_1} Y_0$, F_1 is transverse cellular.*

Proof. By Proposition IV. 10 (1) is equivalent to (2). Clearly, (3) implies (2).

(2) $\Rightarrow$ (3). By Proposition 4.(2) we can obtain a regular extension $X \xrightarrow{F_2} Y_2 \xleftarrow{i_2} Y_0$ "between" F and F_1. We will show that F_2 is transverse cellular, whence F_1 will be by Corollary IV.12.

Triangulate so that F_2, F and the connecting transverse cellular map $G\colon Y_2 \to |L|$ are simplicial. For σ a simplex of $|L_0|$, we know that $D(i\sigma; F)$ is a cone on its boundary and we must show this for $D(i_2\sigma;\ F_2)$. But just as in the proof of Lemma 2, part (1) $(D,\dot{D})\ (i_2\sigma,\ F_2) = (D,\dot{D})\ (i\sigma,\ F)$. So F_2 satisfies condition (1) and as (1) $\Rightarrow$ (2), F_2 is transverse cellular.

Proposition 7. *Let K_0 be a full subcomplex of K, and $f\colon\ K_0 \to L_0$ a simplicial map. If for all $A \in L_0$, $D(A;\ f) \searrow$ point homogeneously in K, then any regular extension of f is transverse cellular.*

Proof. By Proposition 6 we need only show the derived stellar extension F: $r_{K_0}K \to L$ satisfies $D(A;\ F) \searrow$ point homogeneously in K for $A \in L_0$. But $D(A;\ F) \searrow D(A;\ f)$ homogeneously in K because $F^{-1}\eta A = f^{-1}\eta A$ and the relative version of Proposition III.2, mentioned in the remark following it, imply that $D(A;\ (F, f))$ is a regular neighborhood pair of $f^{-1}\eta A$ in $\dot{D}(\overline{A};\ (F,\ f))$. Collapsing follows [$C_2$, 7.10 a] and [A, III. 11].

Beware: Homogeneity of a collapse in K and in K_0 are independent conditions. In particular, $f\colon\ K_0 \to L_0$ transverse cellular need not imply F transverse cellular.

We conclude this section with the converse of IV.12 promised earlier.

Proposition 8. *Let $f\colon\ X \to Y$ be a transverse cellular map with $U \subset Y$ subpolyhedron, $V = f^{-1}(U) \subset X$ with* $\mathrm{Frontier}_Y\ U \equiv \dot{U}$ *and* $\mathrm{Frontier}_X\ V \equiv \dot{V}$. *Assume the following conditions hold.*

(a) $\dot{V} = f^{-1}(\dot{U})$.

(b) *Let* $C(U;\ Y) = Y - \mathrm{Int}\ U = (Y - U) \cup \dot{U}$ *similarly* $C(V;\ X)$, *then*

$$U \supset \dot{U} \subset C(U;\ Y), \qquad V \supset \dot{V} \subset C(V;\ X)$$

are assumed to be locally collared inclusions.

Then, $f\colon\ (X;\ V,\ C(V;\ X),\ \dot{V}) \to (Y;\ U,\ C(U;\ Y),\ \dot{U})$ *is transverse cellular.*

Proof. Triangulate to obtain $f\colon\ K \to L$ simplicial with $L_0 = U$ a subcomplex and hence $\dot{L}_0 = \dot{U}$ a complex. Let $K_0 = V = f^{-1}(L_0)$ and $\dot{K}_0 = \dot{V} = f^{-1}(\dot{L}_0)$.

If $A \in L - \dot{L}_0$ then

$$D(A;\ f) = \begin{cases} D(A;\ f|K_0), & A \in L_0, \\ D(A;\ f|C(K_0;\ K)), & A \notin L_0. \end{cases}$$

Hence, since f is transverse cellular each of these two collapses as required. We need only worry about simplices of $\dot{L}_0$. In fact, because $f^{-1}(\dot{L}_0) = \dot{K}_0$ we know $D(A;\ f) \searrow D(A;\ f|\dot{K}_0)$ homogeneously in K and hence $D(A;\ f|K_0) \searrow D(A;\ f|\dot{K}_0)$ and $D(A;\ f|C(K_0;\ K)) \searrow D(A;\ f|\dot{K}_0)$ (by local collaring $D(A;\ f) - D(A;\ f|\dot{K}_0)$ has two components with these closures). So it suffices to know $f|\dot{K}_0\colon\ \dot{K}_0 \to \dot{L}_0$ is transverse cellular.

Now we use the natural retractions: T_1: $N(\dot{K}_0; \eta K_0) \to \eta \dot{K}_0$ and T_2: $N(\dot{L}_0; \eta L_0) \to \eta \dot{L}_0$. We have the commutative diagram:

$$\begin{array}{ccc} \dot{N}(\dot{K}_0; \eta K_0) & \xrightarrow{f|\dot{N}} & \dot{N}(\dot{L}_0; \eta L_0) \\ {\scriptstyle T_1}\downarrow & & \downarrow{\scriptstyle T_2} \\ \dot{K}_0 & \xrightarrow{f|\dot{K}_0} & \dot{L}_0 \end{array}$$

Now $f|\dot{N}$ is the union of maps $f|D(A; f)$ for $A \in N(\dot{L}_0; L_0) - (\dot{N} \cup \dot{L}_0)$. Since f is transverse cellular on $(D(A; f), \dot{D}(A; f))$ it is transverse cellular on $\dot{N}(\dot{K}_0; \eta K_0)$ by the remarks to Proposition IV.11.

On the other hand the maps T_1 and T_2 are transverse cellular, by Corollary VI.5 of the next section, and so, by Corollary 3 $f|\dot{K}_0$ is also transverse cellular.

Remarks. (1) Note that in this proof if we only assumed $\dot{U} \subset U$ and $\dot{V} \subset V$ collared then we would obtain f: $(V, \dot{V}) \to (U, \dot{U})$ transverse cellular. We only use local collaring on the outside to insure that the homogeneous (in K_0) collapse of $D(A; f|\dot{K}_0) \searrow$ point is also homogeneous in $(C(K_0; K), \dot{K}_0)$.

(2) If f: $K \to L$ is a transverse cellular simplicial map with chosen derived and L_0 is a full subcomplex of L, $f^{-1}L_0 = K_0$, then the theorem applies with $U = N(L_0; \eta L)$ and $V = N(K_0; \eta K)$.

(3) Iterating the result gives a theorem in the following situation. Let f: $X \to Y$ be a transverse cellular map and define sequences: $Y = \dot{Y}_{-1} \supset Y_0 \supset \dot{Y}_0 \supset Y_1 \supset \dot{Y}_1 \supset \cdots \supset Y_n \supset \dot{Y}_n$ and $X_i = f^{-1}(Y_i)$, $\dot{X}_i = f^{-1}(\dot{Y}_i)$ where Y_i is a regular neighborhood with boundary $\dot{Y}_i$ in $\dot{Y}_{i-1}$ for $i = 0, \cdots, n$ and similarly for X_i (respectively, $Y_i - \dot{Y}_i$ is open in $\dot{Y}_{i-1}$ with $\dot{Y}_i \subset Y_i$ collared for $i = 0, \cdots, n$ and similarly for X_i). By Proposition 8,

$$f\colon (\dot{X}_{i-1}; X_i, \overline{\dot{X}_{i-1} - X_i}, \dot{X}_i) \to (\dot{Y}_{i-1}; Y_i, \overline{\dot{Y}_{i-1} - Y_i}, \dot{Y}_i)$$

is transverse cellular for $i = 0, \cdots, n$ (respectively, by Remark (1) f: $(X_i, \dot{X}_i) \to (Y_i, \dot{Y}_i)$ is transverse cellular for $i = 0, \cdots, n$). From this follows the promised converse:

Proposition 9. *Let f: $X \to Y$ be a transverse cellular map and assume γ: $\mathfrak{S} \to \mathfrak{D}$ is an isomorphism of general decompositions carrying f, i.e. $f(\sigma) = \gamma(\sigma)$, and $f^{-1}(\partial\gamma\sigma) = \partial\sigma$. Then $f|\sigma$: $(\sigma, \partial\sigma) \to (\gamma\sigma, \partial\gamma\sigma)$ is transverse cellular.*

This allows us also to verify the satisfying definition of quasi-isotopy mentioned after Proposition IV.15.

Corollary 10. *Let F: $X \times I \to Y \times I$ be a map with $F^{-1}(Y \times i) = X \times i$ for $i = 0, 1$. Then the following are equivalent:*

(1) *F is transverse cellular.*

(2) *F: $(X \times I; X \times 0, X \times 1) \to (Y \times I; Y \times 0, Y \times 1)$ is transverse cellular.*

If F is level-preserving and X and Y are compact these are equivalent to:

(3) *F_t: $X \to Y$ is transverse cellular for all $t \in I$.*

Proof. (1) $\Rightarrow$ (2) by Remark (1) after Proposition 8 letting $(U, \dot{U}) = (Y \times I, Y \times \dot{I})$. Now assume F level-preserving.

(1) $\Rightarrow$ (3). Let J be any subinterval of I, then F: $(X \times J, X \times \dot{J}) \to (Y \times J, Y \times \dot{J})$ is transverse cellular by Remark (1) after Proposition 8. A fortiori, F_t: $X \to Y$ is transverse cellular for each t in I.

(3) $\Rightarrow$ (1). Triangulate so that $X \times I \xrightarrow{F} Y \times I \xrightarrow{\pi} I$ is simplicial (this needs compactness, else π is not proper). Choose deriveds preserving the diagram. If $A \in Y \times I$ with $\pi A = t$ a zero simplex, then $D(A; f) \searrow D(A; F|X \times t)$ homogeneously in $X \times I$, as $F^{-1} Y \times t = X \times t$. Since F_t is transverse cellular, $D(A; F|X \times t) \searrow$ point homogeneously in $X \times t$ and hence in $X \times I$. If $\pi A = J$ a one simplex, then since $F_{\eta J}$ is transverse cellular and $D(A; F) \subset X \times \eta J$, F: $D(A; F) \to D(A; L \times I)$ is transverse cellular by Proposition 9, using $\{D(B; F): \pi B = J\}$ and $\{D(B; L \times I): \pi B = J\}$ as general complexes on $X \times \eta J$ and $Y \times \eta J$ respectively. Since $D(A; L \times I)$ is a cone on $\dot{D}$, $D(A; F)$ is.

VI. **Classification of mapping cylinders.** We begin by applying the result on regular extensions to mapping cylinders. First, the relation between regular extensions and mapping cylinders.

Proposition 1. *Let f: $K \to L$ and g: $L \to M$ be simplicial maps with chosen deriveds, with g onto. Call $\bar{g}$: $f \to g \circ f$ the morphism defined as the pair (id_K, g). The map $C(\bar{g})$: $C_f \to C_{g \circ f}$ is a regular extension of g: $L \to M$.*

Proof. We apply the criterion of Lemma V.5 and we must show for $A\eta C_0 \cdots \eta C_k \in C_f$ and $\tilde{A} < A$ such that $g(A) = g(\tilde{A})$, that $\tilde{A}\eta C_0 \cdots \eta C_k$ is a nice face of $A\eta C_0 \cdots \eta C_k$ in C_f (since this is for any g, the condition $g(\tilde{A}) = g(A)$ gives no information, and we do not use it). But

$$Lk(A\eta C_0 \cdots \eta C_k; C_f) = Lk(A\eta C_0; C_{f|C_0}) * Lk(\eta C_0 \cdots \eta C_k; D(C_0; K)),$$

$$Lk(\tilde{A}\eta C_0 \cdots \eta C_k; C_f) = Lk(\tilde{A}\eta C_0; C_{f|C_0}) * Lk(\eta C_0 \cdots \eta C_k; D(C_0; K)).$$

The latter join factor is held in common. So to prove the result, we need only show that $\tilde{A}\eta C_0$ is a nice face of $A\eta C_0$ in $C_{f|C_0}$. But $C_{f|C_0}$ is a $\dim C_0 + 1$ ball with boundary $C_0 \cup fC_0 \cup C_{f|\partial C_0}$ so both $\tilde{A}\eta C_0$ and $A\eta C_0$ are interior simplices.

Note here—as I failed to in my original manuscript—that g can be transverse cellular without $C(\bar{g})$ being transverse cellular. I was not wary of the misuse of

Proposition V.7, mentioned after its proof. However, a weaker result is true and useful.

Corollary 2. *Let g: $K \to L$ be a transverse cellular simplicial map. Then $\eta C(d_g) \circ j_K^{-1}$: $\eta K \times I \to \eta C_g$ is transverse cellular.*

Proof. j_K is an isomorphism and $C(d_g)$ is transverse cellular, by the above Propositions V.6 and V.7. Note that for $A \in L$, $D(A; g)$ collapses homogeneously in $K \times 1$ by transverse cellularity and hence in $C(\mathrm{id}_K)$ because $K \times 1$ is collared in $C(\mathrm{id}_K)$.

There is a corresponding morphism $\bar{f}$: $g \circ f \to g$ given by the pair of maps (f, id_M) but the corresponding map $C(\bar{f})$: $C_{g \circ f} \to C_g$ is in general not a regular extension of $\bar{f}$ if f is not transverse cellular. For in particular if $C(\bar{f})$ were a regular extension then it would be transverse cellular on a complement of a regular neighborhood of ηK to the complement of a regular neighborhood of ηL. By restricting to the boundaries we would obtain a transverse cellular map of K onto L and hence that K is homeomorphic to L. Oddly, the analogue of Corollary 2 can be strengthened.

Proposition 3. *Let f: $K \to L$ and g: $L \to M$ be simplicial maps with chosen derived, with f transverse cellular. Then $C(\bar{f})$: $C_{g \circ f} \to C_g$ is transverse cellular.*

Proof. $C(\bar{f})$ preserves the decomposition maps:

$$D(A; g \circ f) \to D(A; g), \qquad Q(A; g \circ f) \to Q(A; g), \quad \text{for } A \in M$$

and $C(\bar{f})$ on $Q(A; g \circ f)$ is the cone on $C(\bar{f})|\dot{Q}(A; g \circ f) \cup D(A; g \circ f)$.

Since $D(A; g)$ is a union of $D(B; L)$'s, $D(A; g \circ f)$ is a union of $D(B; f)$'s. Now f is transverse cellular mapping $D(B; f) \to D(B; L)$ for $B \in L$ by Proposition IV.10 and remarks thereafter. Hence, by Proposition IV.11, f: $D(A; g \circ f) \to D(A; g)$ is transverse cellular.

Hence, $C(\bar{f})$ is transverse cellular on each $D(A; g \circ f)$ and hence by Proposition IV.11 and Lemma IV.13 applied to an induction up the dimension of the $Q(A; g \circ f)$'s, $C(\bar{f})$ is transverse cellular on each of these pieces. The map is then transverse cellular on the whole thing by yet another application of Proposition IV.11.

We now prove the mapping cylinder regular neighborhood relation that we mentioned earlier. It is essentially [C_1, 9.7], but his retraction is a little different and for later work we must have the result for the natural retraction of a simplicial neighborhood.

Proposition 4. *Let K_0 be a full subcomplex of K and let T: $N(K_0; \eta K) \to \eta K_0$ be the simplicial retraction defined by $T(\eta A) = \eta(A \cap K_0)$ for $A \in N(K_0; K)$.*

Then $C_{T|\dot{N}(K_0;\eta K)} \cong N(K_0;\, \eta K)$ rel $K_0 \cup \dot{N}$.

Proof. Define A^η, for $A \in K$, to be the subcomplex of ηK obtained by subdividing the simplex A. Let $U = N(K_0;\, K) - (K_0 \cup \dot{N}(K_0;\, K))$.

By [C_2, 2.16] the following is a cell-complex on $N(K_0;\, \eta K)$:

$$\{N(K_0 \cap A;\, A^\eta)\colon A \in U\} \cup \{\dot{N}(K_0 \cap A;\, A^\eta)\colon A \in U\} \cup K_0,$$

and we claim that the following is a cell-complex on $C_{T|\dot{N}}$.

$$\{C_{T|\dot{N}(K_0 \cap A;\, A^\eta)}\colon A \in U\} \cup \{\dot{N}(K_0 \cap A;\, A^\eta)\colon A \in U\} \cup K_0.$$

The result will then follow that by realizing the obvious cell-complex isomorphism.

To show that we have a cell-complex on $C_{T|\dot{N}}$ we must show that $C_{T|\dot{N}(K_0 \cap A;\, A^\eta)}$ is a dim A ball with boundary equal to $C_{T|\dot{N}(K_0 \cap A;\, \partial A^\eta)} \cup (K_0 \cap A) \cup \dot{N}(K_0 \cap A;\, A^\eta)$. For this we show that $T\colon \dot{N}(K_0 \cap A;\, A^\eta) \to K_0 \cap A$ has collapsible point-inverses. This and Proposition 1 allow us to apply Corollary IV.6 to the map

$$\dot{N}(K_0 \cap A;\, A^\eta) \times I \xrightarrow{C(d_T)} C_{T|\dot{N}(K_0 \cap A;\, A^\eta)}$$

as in the proof of Proposition IV.9 (closed).

We are reduced to looking at the map $T\colon \dot{N}(K_0 \cap A;\, A^\eta) \to A \cap K_0$.

To analyze it, let $A = BD$ with $B = A \cap K_0$. Define a simplicial isomorphism so that the following diagram commutes:

$$\begin{tikzcd} N(B;\, A^\eta) \arrow[rr, "\phi"] \arrow[dr, "T"'] & & B^\eta \times cD^\eta \arrow[dl, "\pi_1"] \\ & B^\eta & \end{tikzcd}$$

Let

$$\phi(\eta(B'D')) = \begin{cases} (\eta(B'),\, \eta(D')), & \text{for } 0 \neq B' \leq B \text{ and } 0 \neq D' \leq D, \\ (\eta(B'),\, c),. & \text{for } 0 = D'. \end{cases}$$

Note that $\phi(\dot{N}(B;\, A^\eta)) = B^\eta \times D^\eta$.

So if $x \in A \cap K_0$, $(T|\dot{N})^{-1}(x)$ is a ball because $(\pi_1 | B^\eta \times D^\eta)^{-1}(x) = x \times D^\eta$ is.

Corollary 5. *Let* (K, K_0) *be a triangulation of* $(X \times J,\, X \times p)$ *for* J *a real interval with endpoint* p. *Assume* K_0 *is full. Then* $T\colon \dot{N}(K_0;\, \eta K) \to \eta K_0$ *is a transverse cellular mapping.*

Proof. By Corollary 4 the mapping cylinder is homeomorphic to the regular neighborhood which is a product. T is transverse cellular by Proposition IV.10.

Remark. Uniqueness of regular neighborhoods gives a homeomorphism h: $X \times J \to X \times J$ rel $X \times p$, with $h(N(K_0; \eta K)) = X \times J'$ ($J' = [p, q]$ a subinterval of J). Then $h|\dot{N}$: $\dot{N}(K_0; \eta K) \to X \times q \to X$ is a homeomorphism and we claim that h is quasi-concordant to $T|\dot{N}$. Consider the composed map:

$$\dot{N}(K_0; \eta K) \times I \xrightarrow{C(d_T)} C_{T|\dot{N}} \cong N(K_0; \eta K) \xrightarrow{h} X \times J' \cong X \times I;$$

where $C(d_T)$ is transverse cellular by Corollary 2 and the last map is induced by an isomorphism of J' with I taking p to 0 and q to 1. Then on 0, this composed map is $T|\dot{N}$ and on 1 it is the above homeomorphism.

We now classify mapping cylinders. Consider p. l. maps from X to Y. Define an equivalence relation $\sim$ generated by $f_0 \sim_e f_1$ meaning there exists a transverse cellular map r: $X \to X$ quasi-concordant to id_X and such that $f_0 = f_1 r$.

Theorem 6. *Let f_0, f_1: $X \to Y$ be p. l. maps. The following are equivalent.*

(1) $f_0 \sim f_1$.

(2) *There exists a homeomorphism h: $C_{f_0} \cong C_{f_1}$* rel $X \cup Y$.

Proof. (1) $\Rightarrow$ (2) We may assume $f_0 \sim_e f_1$ since condition (2) is transitive and symmetric. If $f_0 = f_1 r$, triangulate so that f_1 and r (and hence f_0) are simplicial, by Proposition 3, $C(\bar{r})$: $C_{f_0} \to C_{f_1}$ is transverse cellular. In fact it is a transverse cellular map of $(C_{f_0}; X, Y) \to (C_{f_1}; X, Y)$ which is r on X and id_Y on Y. So by Proposition IV.2, we can change $C(\bar{r})$ rel Y to obtain a homeomorphism $(C_{f_0}, X) \xrightarrow{h_0} (C_{f_1}, X)$. The homeomorphism so obtained on X is quasi-concordant to r and hence to id_X and so is concordant to id_X by Proposition IV.15. Since X is collared in C_{f_0} and C_{f_1}, we can change h_0 by a concordance rel Y to obtain h: $C_{f_0} \cong C_{f_1}$ rel $X \cup Y$.

(2) $\Rightarrow$ (1). Let h: $C_f \cong C_g$ be a homeomorphism rel $X \cup Y$, for f, g: $X \to Y$. For this proof, we let d equal $C(d_f) j_X^{-1}$: $X \times I \to C_f$, and $\bar{d}$ equal $C(d_g) j_X^{-1}$. Subdivide so that the following diagram is simplicial:

$$X \times I \xrightarrow{d} C_f \xrightarrow{h} C_g \xleftarrow{\bar{d}} X \times I$$

and choose deriveds so that it remains so:

$$\eta X \times I \xrightarrow{d} \eta C_f \xrightarrow{h} \bar{\eta} C_g \xleftarrow{\bar{d}} \bar{\eta}(X \times I).$$

Consider the following commutative diagram where the vertical maps are the natural simplicial retractions of Proposition 4 and Corollary 5 (in particular, note that T_2 and T_3 are not P_f and P_g):

$$\begin{tikzcd}
\dot N(X\times 0;\ \eta(X\times I)) \arrow[r, "d"] \arrow[d, "T_1"'] & \dot N(Y;\ \eta C_f) \arrow[r, "h"] \arrow[d, "T_2"'] & \dot N(Y;\ \bar\eta C_g) \arrow[d, "T_3"] & \dot N(X\times 0;\ \bar\eta(X\times I)) \arrow[l, "\bar d"'] \arrow[d, "T_4"] \\
\eta X\times 0 \arrow[r, "f"] & \eta Y \arrow[r, "\mathrm{id}"] & \bar\eta Y & \bar\eta X\times 0 \arrow[l, "g"']
\end{tikzcd}$$

If we had allowed the domain space to vary in defining our equivalence relation we would now be done for then f relates to T_2 relates to T_3 relates to g, as the other maps are transverse cellular. The remainder of the proof is obtaining the sharp form of the result that we stated above.

Essentially, we have cut the diagram of maps at the ϵ-level and looked at the $[0, \epsilon]$ piece.

However, we have now to look at the part "above the cut", the complements of the regular neighborhoods $(C(A;\ \eta B) = \text{closure } (B - N(A;\ \eta B)))$:

$$C(X\times 0;\ \eta(X\times I)) \xrightarrow{d} C(Y;\ \eta C_f) \xrightarrow{h} C(Y;\ \bar\eta C_g) \xleftarrow{\bar d} C(X\times 0;\ \bar\eta(X\times I)).$$

Now by Corollary 2, $d\colon X\times I \to C_f$ is transverse cellular off $X\times 0$ and so Proposition V.8 and Remark (1), thereafter, implies that

$$d\colon (C(X\times 0;\ \eta(X\times I));\ \dot N(X\times 0;\ \eta(X\times I)),\ X\times 1) \to (C(Y;\ \eta C_f);\ \dot N(Y;\ \eta C_f),\ X)$$

is transverse cellular, and similarly for $\bar d$. So we can change d and $\bar d$ rel X to homeomorphisms of triads q and $\bar q$. We will also need homeomorphisms $k, \bar k$: $X\times I \to X\times I$ rel $X\times \dot I$ with $k(X\times[0, \frac12]) \cong N(X\times 0;\ \eta(X\times I))$ and $\bar k(X\times[0, \frac12]) \cong N(X\times 0;\ \bar\eta(X\times I))$.

This enlarges the $[0, \epsilon]$ diagram to the following:

$$\begin{tikzcd}
X\times\frac12 = X \arrow[d, "k"] & & & X = X\times\frac12 \arrow[d, "\bar k"] \\
\dot N(X\times 0;\ \eta(X\times I)) \arrow[r, shift left, "q"] \arrow[r, shift right, "d"'] \arrow[d, "T_1"'] & \dot N(Y;\ \eta C_f) \arrow[r, "h"] \arrow[dr, "T_2"'] & \dot N(Y;\ \bar\eta C_g) \arrow[d, "T_3"] & \dot N(X\times 0;\ \bar\eta(X\times I)) \arrow[l, shift right, "\bar q"'] \arrow[l, shift left, "\bar d"] \arrow[d, "T_4"] \\
X = X\times 0 \arrow[rr, "f"'] & & Y & X\times 0 = X \arrow[l, "g"]
\end{tikzcd}$$

$f \mathrel{{}_e{\sim}} f\circ T_1\circ k = T_2\circ d\circ k = T_3\circ h\circ d\circ k$ where $T_1\circ k$ is quasi-concordant to id_X by the remarks after Corollary 5. $T_3\circ h\circ q\circ k \mathrel{{}_e{\sim}} T_3\circ h\circ q\circ k\circ (k^{-1}q^{-1}dk) = T_3\circ h\circ d\circ k$ for since q is quasi-concordant to d, $k^{-1}q^{-1}dk$ is quasi-concordant to id_X.

Similarly, $g \mathrel{{}_e{\sim}} T_3\circ \bar d\circ \bar k \sim_e T_3\circ\bar q\circ\bar k$.

Finally, $T_3\circ\bar q\circ\bar k \mathrel{{}_e{\sim}} T_3\circ\bar q\circ\bar k\circ(\bar k^{-1}\circ\bar q^{-1}\circ h\circ q\circ k) = T_3\circ h\circ q\circ k$. Now

$\bar{k}^{-1} \circ \bar{q}^{-1} \circ h \circ q \circ k$ is concordant to the identity because it is the restriction to $X \times \frac{1}{2}$ of a homeomorphism $X \times [\frac{1}{2}, 1] \to X \times [\frac{1}{2}, 1]$ which is id_X on 1.

This theorem may be of interest in examining thickenings of polyhedra. A thickening $(m \geq 6)$ of X is a manifold M^m together with an embedding $i\colon X \to \operatorname{Int} M$ which is a simple homotopy equivalence. More generally, an element $\mathfrak{T}^m(X)$ is a regular neighborhood pair (M, X) where M is an m manifold. By Proposition 4, there exists $T\colon M \to X$, a retraction with $M \cong C_{T|\partial M}$ rel $\partial M \cup X$. The above theorem tells us how T can vary, i.e. (M, X) determines T up to equivalence $\sim$. For any real usefulness, though, we would seem to need a theory of when an onto map $T\colon N \to X$, N a manifold $(\partial = 0)$ and X a polyhedron, has C_T a manifold (note that since $N \subset C_T$ is collared so $N \subset \partial C_T$).

Theorem 6 weakens to obtain other equivalence relation results. Let $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$.

(1) Define $f_0 \approx_e f_1$ if $Y_0 = Y_1$ and there exists $r\colon X_0 \to X_1$ transverse cellular with $f_0 = f_1 \circ r$. Call the generated equivalence relation $\approx$.

(2) Define $f_0 \approxeq_e f_1$ if there exists $r\colon X_0 \to X_1$ transverse cellular and $h\colon Y_1 \to Y_0$ a homeomorphism with $f_0 = h \circ f_1 \circ r$. Call the generated equivalence relation $\approxeq$.

Corollary 7. *Let $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$.*

(1) $f_0 \approx f_1$ *iff* $Y_0 = Y_1$ *and there exists a homeomorphism* $h\colon (C_{f_0}, X_0) \to (C_{f_1}, X_1)$ rel $Y_0 = Y_1$.

(2) $f_0 \approxeq f_1$ *iff there exists a homeomorphism* $h\colon (C_{f_0}; X_0, Y_0) \to (C_{f_1}; X_1, Y_1)$.

Proof. That the equivalence relation implies the corresponding mapping cylinder result goes through just as in Theorem 6. The proofs the other way are obtained from Theorem 6. We will do the proof for (1). If h is the homeomorphism of pairs rel Y, then one easily constructs a homeomorphism of C_{f_0} and a mapping cylinder for $f_1 \circ (h|X_0)$ which is rel $X_0 \cup Y_0$. Hence, $f_0 \sim f_1 \circ (h|X_0)$ and $f_1 \;{}_e\!\approx f_1 \circ (h|X_0)$. So $f_0 \approx f_1$.

From Corollary 7 and Proposition IV.10, we obtain a quick proof of an alternate formulation of Corollary V.3:

Corollary 8. *Let $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$ be p. l. maps with $f_0 \approxeq f_1$. If f_0 is transverse cellular then f_1 is.*

We now apply the mapping cylinder results to obtain a different uniqueness theorem for regular extensions. Recall that the uniqueness theorem Proposition V.4 part (2) involves varying the range by a transverse cellular map. Such variation can destroy nice properties of a map in general (see §VII) and it is desirable to have a theorem of the domain variation type.

Proposition 9. *Let* X_0 *be a subpolyhedron of* X *and* $f\colon X_0 \to Y_0$ *be a p. l. map. Let* $X \xrightarrow{F_1} Y_1 \xleftarrow{i_1} Y_0$ *and* $X \xrightarrow{F_2} Y_2 \xleftarrow{i_2} Y_0$ *be regular extensions of* f. *Then* $F_1 \approxeq F_2$.

Proof. By Proposition V.4 part (2), we need only deal with the case where there exists a transverse cellular map $G\colon Y_1 \to Y_2$ with $G^{-1}(i_2(Y_0)) = i_1(Y_0)$, $G \circ i_1 = i_2$ and $F_2 = G \circ F_1$. Triangulate X, Y_1 and Y_2 so that X_0, $i_1(Y_0)$ and $i_2(Y_0)$ are full subcomplexes and F_1 and G are simplicial (and so F_2 is too). Choose deriveds so that everything is simplicial also.

Consider the simplicial map $C(\overline{G})\colon C_{F_1} \to C_{F_2}$.

By Proposition 1, $C(\overline{G})$ is a regular extension. Hence, if A is a simplex of $C_{F_2} - Y_2$, $D(A; C(\overline{G}))$ is a cone on its boundary.

Now if $A \in Y_2 - i_2(Y_0)$, $D(A; C(\overline{G})) \searrow D(A; G)$ homogeneously in C_{F_1} because $C(\overline{G})^{-1}\eta A = G^{-1}\eta A$; so III.2 and [$C_2$, VII.10] apply. Since G is transverse cellular we know that $D(A; G) \searrow$ point homogeneously in Y_1. But $D(A; G) \subset |Y_1 - i_1(Y_0)|$ and the inclusion $Y_1 \subset C_{F_1}$ is locally collared on $|Y_1 - i_1(Y_0)|$ so this collapse is homogeneous in C_{F_1}, too. So $D(A; C(\overline{G}))$ is a cone on its boundary for $A \in Y_2 - i_2(Y_0)$, also.

This leaves the crucial case $A \in i_2(Y_0)$. But since $C(\overline{G})^{-1} i_2(Y_0) = i_1(Y_0)$ and $C(\overline{G}) i_1 | Y_0 = i_2 i_1^{-1}$, we have $(D, \dot{D})(A; C(\overline{G})) = (D, \dot{D})(i_1 i_2^{-1} A; C_{F_1})$, just as in part (1) of the proof of Lemma V.2.

Thus, $C(\overline{G})$ is a transverse cellular map of triads $(C_{F_1}; \eta X, Y_1) \to (C_{F_2}; \eta X, Y_2)$. So we can obtain a homeomorphism

$$(C_{F_1}, Y_1) \underset{r}{\cong} (C_{F_2}, Y_2) \quad \text{rel } X.$$

So by Corollary 7, $F_1 \approxeq F_2$.

VII. **Transverse cellularity and homotopy.** This section has its genesis in the error discovered by the referee.

Let $f_0, f_1\colon X \to Y$. Then we can define a relation $f_0 \leftrightarrow_e f_1$ if there exist transverse cellular maps quasi-concordant to the identity: $d\colon X \to X$ and $r\colon Y \to Y$ with $rf_0 = f_1 d$. Call the generated equivalence relation $\leftrightarrow$.

I originally "proved" Theorem VI.6 using $\leftrightarrow$ instead of $\sim$. However, this is false.

In fact, the following proposition is an almost antithetical result.

Proposition. *Let* $f_0, f_1\colon X \to Y$ *be p. l. maps with* X *and* Y *compact.* $f_0 \leftrightarrow f_1$ *if and only if* f_0 *is homotopic to* f_1.

Proof. $f_0 \leftrightarrow_e f_1$ clearly implies homotopy so $\leftrightarrow$ does.

Conversely, if f_0 is homotopic to f_1 then there exists $f\colon X \times I \to Y \times I$ level-preserving, f_0 on 0 and f_1 on 1. Triangulate so that the diagram $X \times I$

$\xrightarrow{f} Y \times I \xrightarrow{\pi} \tau I$ is simplicial, and assume that the subdivision of I, τI, has vertices $t_0 = 0 < t_1 < \cdots < t_n = 1$. Let $s_i = \frac{1}{2}(t_i + t_{i+1})$, $i = 0, \cdots, n-1$. We claim $f_{s_i} \leftrightarrow_e f_{t_i}$ and $f_{s_i} \leftrightarrow_e f_{t_{i+1}}$. So $f_0 \leftrightarrow f_1$.

Derive $X \times I$, $Y \times I$, τI so that the above f and π diagram is still simplicial and $\eta[t_i, t_{i+1}] = s_i$. Then, $N(X \times t_i; \eta(X \times [t_i, t_{i+1}])) = X \times [t_i, s_i]$ with boundary $X \times s_i$. Let T_1: $N(X \times t_i; \eta X \times [t_i, t_{i+1}]) \to X \times t_i$ and T_2: $N(Y \times t_i; \eta(Y \times [t_i, t_{i+1}])) \to Y \times t_i$ be the natural retractions. Then we obtain the commutative diagram:

$$\begin{tikzcd} X \times s_i \arrow[r, "f_{s_i}"] \arrow[d, "T_1"'] & Y \times s_i \arrow[d, "T_2"] \\ X \times t_i \arrow[r, "f_{t_i}"] & Y \times t_i \end{tikzcd}$$

So $f_{s_i} \leftrightarrow_e f_{t_i}$ by Corollary VI.5. Similarly, $f_{s_i} \leftrightarrow_e f_{t_{i+1}}$.

VIII. **Equalizer theorems.** An application of transverse cellularity due in essence to Dancis is to the following conjecture. Given two embeddings g_0 and g_1: $X \to Y$ then a sufficient condition for them to be ambient isotopic should be the existence of a transverse cellular map f: $Y \to Z$ with $fg_0 = fg_1$, i.e. an equalizer, e.g. if g: $X \times I \to M$ is an embedding into a manifold then let f be a regular extension of $\pi_X \circ G^{-1}$: $G(X \times I) \to X$. This we generalize below. As stated this is not quite enough, for if Y is a cone and C is a regular neighborhood of the cone point of Y, and X is a point then if g_0 and g_1 put the point anywhere in the interior of C, then a regular extension to Y of $C \to$ point is an equalizer for g_0 and g_1, but, while isotopic, g_0 and g_1 need not be ambient isotopic.

The proper statement is the following:

Proposition 1. *Let g_0, g_1: $X \to Y$ be embeddings such that there exists f: $Y \to Z$ satisfying*

(1) *$fg_0 = fg_1$ and is an embedding. Let $Z_0 = fg_i(X)$.*

(2) *f: $(Y, g_i(X)) \to (Z, Z_0)$ is transverse cellular for $i = 0, 1$.*

Then g_0 and g_1 are ambient isotopic.

Proof. By Proposition IV.2 there exist $\bar{f}_i$: $Y \to Z$ homeomorphisms associated to f with $\bar{f}_i g_i = fg_i$, $i = 0, 1$. So $\bar{f}_1^{-1}\bar{f}_0$: $Y \to Y$ is a homeomorphism ambient isotopic to the identity taking g_0 to g_1. Both $\bar{f}_i$'s can be assumed carried by the same isomorphism of cone complexes from X to Y, and hence the isotopy result.

A corollary is closer to the original theorem of Dancis.

Corollary 2. *Let g_0, g_1: $X \to Y$ be embeddings such that there exists $Y \supset$*

$Y_0 \supset g_0(X) \cup g_1(X)$ and $f\colon Y_0 \to Z$ satisfying

(1) $fg_0 = fg_1$ and is a homeomorphism of X with Z.

(2) $f^{-1}(x) \searrow f^{-1}(x) \cap g_i(x)$ homogeneously in Y for $x \in Z$ and $i = 0, 1$.

Then g_0 and g_1 are ambient isotopic.

Proof. Any regular extension of f satisfies the conditions of Proposition 1, see Corollary IV.4.

Remark. In applying Corollary 2, we note that Lemma IV.7 implies that $f^{-1}(x) \,{}_h\!\searrow\, f^{-1}(x) \cap g_i(x)$, $i = 0, 1$, iff $f^{-1}(x)$ collapses homogeneously to some point and

$$d(f^{-1}(x) \cap g_i(x);\ Y) = \min\{d(y;\ Y)\colon y \in f^{-1}(x)\}, \qquad i = 0, 1.$$

Corollary 3. *Let $G\colon X \times I \to Y$ be an embedding such that for each $x \in X$, $d(G(x, t);\ Y)$ is constant for t varying in $[0, 1]$, then the embeddings g_0 and g_1 are ambient isotopic.*

Proof. In Corollary 2, let $Y_0 = G(X \times I)$ and $f = \mathrm{proj} \circ G^{-1}\colon\ G(X \times I) \to X$.

IX. Fiberwise transversality. Once upon a time, Dancis asked me if one could prove a level-preserving version of Cohen's theorems on transverse cellularity. It was this question which started me on this work and so it is satisfying to be able to include an affirmative answer.

Lemma 1. *Let X be a compact polyhedron and let $X \xrightarrow{\pi} I$ be a p. l. map such that for each subinterval J of I, the inclusion $\pi^{-1}(\dot{J}) \subset \pi^{-1}(J)$ is locally collared, then there exists a homeomorphism $h\colon\ \pi^{-1}(0) \times I \to X$ rel $\pi^{-1}(0)$ with $\pi h(x, t) = t$, i.e. h preserves levels.*

Proof. A simple variation of the Hudson-Zeeman trick is used to cover isotopies by covering them locally.

First, we define pieces of h, locally. If $J \subset I$ (subinterval) and $a \in \dot{J}$, then by collaring there exists a homeomorphism ($a \in J' \subset J$):

$$k\colon \pi^{-1}(a) \times J' \cong V \ \ \mathrm{rel}\, \pi^{-1}(a)$$

where V is a regular neighborhood of $\pi^{-1}(a)$ in $\pi^{-1}(J)$.

Triangulate the diagram

$$\pi^{-1}(a) \times J' \xrightarrow{k} \pi^{-1}(J') \xrightarrow{\pi} J'$$

and choosing J_0 a small enough subinterval of $J' = [a, b]$, such that if v is a vertex of $\pi^{-1}J_0$ or $k^{-1}\pi^{-1}J_0$ then $\pi v = a$ or $\pi k v = a$. Then we can change k so that it is level-preserving on $\pi^{-1}(a) \times J_0$, following the Hudson-Zeeman technique of [H, 6.7].

Thus, for each t in I, we can find a neighborhood J_t of t in I, and a homeomorphism k_t: $\pi^{-1}(t) \times J_t \longrightarrow \pi^{-1}(J_t)$ level-preserving and rel $\pi^{-1}(t)$. To construct the global homeomorphism we use the compactness of I analogous to the Hudson-Zeeman method for covering isotopies (see [H, 6.12.2]). We can find a sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $t_{i+1} \in J_{t_i}$ for $0 \le i < n$. Inductively we define H_{i+1}: $\pi^{-1}(0) \times [0, t_{i+1}] \longrightarrow \pi^{-1}[0, t_{i+1}]$ to be equal to H_i on $\pi^{-1}(0) \times [0, t_1]$ and by letting $H_{i+1}(x, t)$ equal $k_{t_i}(p_1(k_{t_i}^{-1}H_i(x, t_i)), t)$ for $(x, t) \in \pi^{-1}(0) \times [t_i, t_{i+1}]$. Then $h = H_n$.

Proposition 2. *Let f: $K \times I \longrightarrow L \times I$ be a level-preserving, transverse cellular simplicial map of finite complexes with $L \times I$ triangulated so that the projection π: $L \times I \longrightarrow \sigma I$ is simplicial for some subdivision σI.*

There exists a level-preserving homeomorphism g: $K \times I \longrightarrow L \times I$ carried by γ_f, and if T is a subcomplex of σI such that $f|K \times T$ is a homeomorphism then g can be taken equal to f on $K \times T$.

Proof. By starting at 0 and proceeding up the one-simplices of σI we reduce the situation to a one-simplex $J \in \sigma I$. So we must show that if f: $K \times J \to L \times J \to^{\pi} J$ is simplicial we can find g: $K \times J \overset{\cong}{\to} L \times J$, where if f is a homeomorphism on $K \times a$ or $K \times b$ or both ($[a, b] = J$) then g can be chosen to agree with f there.

For $A \in L \times a$, $\dot{D}(A; L \times J) = \bigcup \{D(B; L \times J)$: $A < B \in L \times a\} \cup C(A; L \times J)$ where (definition) $C(A; L \times J) \equiv \bigcup \{D(B; L \times J)$: $A < B \in L \times J - L \times a\}$. $C(A; L \times J)$ is a cone on $\bigcup\{C(B; L \times J)$: $A < B \in L \times a\}$ because it is the complement of a regular neighborhood of $\dot{D}(A; L \times a)$ in $\dot{D}(A; L \times J)$, see [C_2, 4.2].

f preserves this decomposition, i.e. $\dot{D}(A; f) = \bigcup\{D(B; f)$: $A < B \in L \times a\} \cup C(A; f)$ where $C(A; f) \equiv \bigcup\{D(B; f)$: $A < B \in L \times J - L \times a\}$.

Now $f|C(A; f)$: $C(A; f) \longrightarrow C(A; L \times J)$ is transverse cellular since $f|D(B; f)$ is for each $B \in L \times J$. Since f_a is transverse cellular (Corollary V.10), f: $D(A; f_a) \longrightarrow D(A; L \times a)$ is also. Hence

$$(\dot{D}(A; f); C(A; f), D(A; f_a)) \cong (\dot{D}(A; L \times J); C(A; L \times J), D(A; L \times a))$$

$$\cong ((D(A; L \times a) \times I; D(A; L \times a) \times 1, D(A; L \times a) \times 0))$$

(for the latter homeomorphism see [C_2, 4.2 proof] or [A, 6.2 proof]).

Now πf: $D(A; f) \longrightarrow [a, \eta J]$ and π: $D(A; L) \longrightarrow [a, \eta J]$ are simplicial maps. So $(\pi f)^{-1}[a, t]$ in $D(A; f)$ is a regular neighborhood of $D(A; f_a)$ with boundary $\pi f^{-1}(t)$ and hence $\pi f^{-1}(t)$ is collared on both sides. Similarly, for π: $D(A; L) \longrightarrow [a, \eta J]$.

Thus, both πf: $D(A; f) \longrightarrow [a, \eta J]$ and π: $D(A; L) \longrightarrow [a, \eta J]$ satisfy the hypotheses of Lemma 1. Thus for each $A \in L \times a$, there are level-preserving homeomorphisms:

$$k_A\colon D(A; f_a) \times [a, \eta J] \overset{\cong}{\rightrightarrows} D(A; f) \quad \mathrm{rel}\, D(A; f_a) \times a = D(A; f_a),$$

$$l_A\colon D(A; L \times a) \times [a, \eta J] \overset{\cong}{\rightrightarrows} D(A; L) \quad \mathrm{rel}\, D(A; L \times a) \times a = D(A; L \times a),$$

and similarly for each $A \in L \times b$ we get a homeomorphism involving the interval $[\eta J, b]$.

Now choose any homeomorphism $g\colon K \times \{a, \eta J, b\} \longrightarrow L \times \{a, \eta J, b\}$, carried by

$$D(A; f_a) \longrightarrow D(A; L \times a) \quad \text{for } A \in L \times a,$$

$$D(B; f) \longrightarrow D(B; L \times J) \quad \text{for } B \in L \times J - L \times \dot{J},$$

$$D(A; f_b) \longrightarrow D(A; L \times b) \quad \text{for } A \in L \times b.$$

Choose g equal to f on any piece where f is already a homeomorphism.

Note that $g(C(A; f)) = C(A; L \times J)$ for either $A \in L \times a$ or $A \in L \times b$.

By induction we can extend g in a level-preserving way over $K \times J$. If $g\colon \dot{D}(A; f) \cup D(A; f_a) \overset{\cong}{\rightrightarrows} \dot{D}(A; L \times J) \cup D(A; L \times a)$ is defined either level-preserving for $A \in L \times a$, then

$$l_A^{-1} g k_A\colon (\dot{D}(A; f_a) \times [a, \eta J]) \cup (D(A; f_a) \times \{a, \eta J\}) \\ \longrightarrow (\dot{D}(A; L \times a) \times [a, \eta J]) \cup (D(A; L \times a) \times \{a, \eta J\})$$

is a level-preserving homeomorphism. We can cone this map using the Alexander trick and extend $l_A^{-1} g k_A$ to a level-preserving homeomorphism on $D(A; f_a) \times [a, \eta J]$. Composing with l_A and k_A^{-1} extends g as required.

As a corollary we can sharpen Proposition IV.15 to analogous results concerning quasi-isotopies, in the compact case.

Corollary 3. (1) *Two homeomorphisms are quasi-isotopic iff they are isotopic.*

(2) *Two homeomorphisms associated to a transverse cellular map are isotopic.*

(3) *A homeomorphism is quasi-isotopic to a transverse cellular map iff it is isotopic to an associated homeomorphism.*

(4) *Transverse cellular maps are quasi-isotopic iff associated homeomorphisms are isotopic.*

It seems reasonable that Proposition 2 could be generalized by replacing I by Δ^n, i.e.

Conjecture. Let $K \times \Delta^n \xrightarrow{f} L \times \Delta^n \xrightarrow{\pi} \Delta^n$ be simplicial with f transverse cellular level-preserving. Then there exists a level-preserving homeomorphism $g\colon K \times \Delta^n \longrightarrow L \times \Delta^n$ carried by γ_f and such that if J is a union of faces of Δ^n such that $f|K \times J$ is a homeomorphism then g can be chosen to equal f on $K \times J$.

A consequence of this conjecture, by induction on the simplices of M, would be the same statement with M replacing Δ^n.

Another corollary of this would be the following:

Conjecture. Let p_1: $E_1 \to B$ and p_2: $E_2 \to B$ be locally trivial p. l. maps, and let f: $E_1 \to E_2$ be a fiber-preserving transverse cellular map. Then for any $\epsilon > 0$ there exists a fiber-preserving homeomorphism g: $E_1 \to E_2$ within ϵ of f (for any metric on E_2).

This would follow by triangulating B so that E_1 and E_2 are trivial over each simplex of B, and then subdivided so that the mesh of E_1 and E_2 are small and $E_1 \xrightarrow{f} E_2 \xrightarrow{p_2} B$ is simplicial (hence p_1: $E_1 \to B$ is). Proceed by induction on the simplices of B.

This result seems approachable in another manner. For $A \in B$ let $E_i^A = p_i^{-1}A$. Then f: $(E_1; E_1^A\colon A \in B) \to (E_2; E_2^A\colon A \in B)$ is transverse cellular as a map of families. So by family version of Proposition IV.1, we can find g: $(E_1; E_1^A\colon A \in B) \xrightarrow{\cong} (E_2; E_2^A\colon A \in B)$. It is then a matter of changing g inductively to be level-preserving. Thus we need something like g: $F \times \Delta \to F \times \Delta$ level-preserving on $F \times \dot{\Delta}$ can be changed to a level-preserving homeomorphism.

Also, it should be noted that this family argument applies directly to block-bundles, i.e. a block preserving transverse cellular map can be changed to a block-bundle isomorphism.

BIBLIOGRAPHY

[A] E. Akin, *Manifold phenomena in the theory of polyhedra*, Trans. Amer. Math. Soc. 143 (1969), 413–473. MR 40 #6544.

[C$_1$] M. M. Cohen, *Simplicial structures and transverse cellularity*, Ann. of Math. (2) 85 (1967), 218–245. MR 35 #1037.

[C$_2$] ———, *A general theory of relative regular neighborhoods*, Trans. Amer. Math. Soc. 136 (1969), 189–229. MR 40 #2052.

[C$_3$] ———, *Homeomorphisms between homotopy manifolds and their resolutions*, Invent. Math. 10 (1970), 239–250.

[D] J. Dancis, Unpublished correspondence.

[H] J. F. P. Hudson, *Piecewise linear topology*, Benjamin, New York, 1969. MR 40 #2094.

[L] W. B. R. Lickorish, *The piecewise linear unknotting of cones*, Topology 4 (1965), 67–91. MR 34 #3585.

DEPARTMENT OF MATHEMATICS, THE CITY COLLEGE, NEW YORK, NEW YORK 10031