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SPACES OF SET-VALUED FUNCTIONS

BY

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ABSTRACT. If X and Y are topological spaces, the set of all continuous functions from X into CY, the space of nonempty, compact subsets of Y with the finite topology, contains a copy (with singleton sets substituted for points) of Y^X, the continuous point-valued functions from X into Y. It is shown that Y^X is homeomorphic to this copy contained in $(CY)^X$ (where all function spaces are assumed to have the compact-open topology) and that, if X or Y is T_2, $(CY)^X$ is homoemorphic to a subspace of $(CY)^{CX}$. Further, if Y is T_2, then these images of Y^X and $(CY)^X$ are closed in $(CY)^X$ and $(CY)^{CX}$ respectively.

Finally, it is shown that, under certain conditions, some elements of X^Y may be considered as elements of $(CY)^X$ and that the induced 1-1 function between the subspaces is open.

If X and Y are topological spaces then Y^X denotes the space of all continuous functions from X into Y. We shall assume throughout that all function spaces have the compact-open topology, which, we recall, is that topology having as a subbasis $\{(A, W) \mid A \subset X$ is compact, $W \subset Y$ is open$\}$ where $(A, W) \equiv \{f \in Y^X \mid f(a) \in W$ for each $a \in A\}$. CY will designate the space of nonempty, compact subsets of Y and will be assumed to have the finite topology, which is obtained by taking as a subbasis $\{\bar{t}_U \mid U$ is an open subset of $Y\} \cup \{\underline{t}_U \mid U$ is an open subset of $Y\}$, where $\bar{t}_U \equiv \{A \in CY \mid A \subset U\}$ and $\underline{t}_U \equiv \{A \in CY \mid A \cap U \neq \emptyset\}$.[2] A basis for CY is formed by the collection of all subsets of the form $\bar{t}_U \cap (\bigcap_{i=1}^{n} \underline{t}_{V(i)})$ where U and each of $V(1), \cdots, V(n)$ are open in Y.

$(CY)^X$ contains a copy (with singleton sets substituted for points) of Y^X and we first show that Y^X is homeomorphic to this subspace of $(CY)^X$ and that, if X or Y is T_2, $(CY)^X$ is homeomorphic to a subspace of $(CY)^{CX}$. Further, if Y is T_2, then these images of Y^X and $(CY)^X$ are closed in $(CY)^X$ and $(CY)^{CX}$ respectively.

Theorem 1. *Y^X is homeomorphic to a subspace of $(CY)^X$ and, if Y is T_2, this subspace of $(CY)^X$ is closed.*

Proof. Define $\phi: Y^X \to (CY)^X$ by $\phi(f)(x) = \{f(x)\}$. Since $\{\{y\} \mid y \in Y\}$ is a

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[2] Note that we are not assuming elements of CY to be necessarily closed, as is the case with the space of compact subsets of [5].

homeomorphic copy of Y contained in CY, the continuity of f certainly implies that of $\phi(f)$. This shows ϕ to be well defined and that ϕ is 1-1 is clear. Now if (A, W) is a subbasic open set in Y^X then $\phi((A, W)) = (A, \bar{t}_W) \cap \phi(Y^X)$ which, since ϕ is 1-1, shows ϕ to be open. Likewise, if $(A, \mathfrak{U})$ is a subbasic open set in $(CY)^X$ containing $\phi(f)$, then, since A is compact and $\{\bar{t}_U | U \subset Y \text{ is open}\}$ is a basis for the subspace topology of $\{\{y\} | y \in Y\}$ in CY, there exist finitely many open subsets, $U(1), \cdots, U(n)$, of Y such that $\{\phi(f)(a) | a \in A\} \subset \bigcup_{i=1}^n \bar{t}_{U(i)}$ and $\bigcup_{i=1}^n \bar{t}_{U(i)} \subset \mathfrak{U}$. Thus $f \in (A, \bigcup_{i=1}^n U(i))$ and $\phi(A, \bigcup_{i=1}^n U(i)) \subset (A, \mathfrak{U})$ which implies the continuity of ϕ.

Suppose that Y is T_2 and let g be an element of $(CY)^X - \phi(Y^X)$. Then there is a point x of X and distinct points y_1 and y_2 of Y such that $\{y_1, y_2\} \subset g(x)$. Choose disjoint open subsets $U(1)$ and $U(2)$ of Y such that $y_1 \in U(1)$ and $y_2 \in U(2)$ and then $g \in (\{x\}, \underline{t}_{U(1)} \cap \underline{t}_{U(2)})$. Since $(\{x\}, \underline{t}_{U(1)} \cap \underline{t}_{U(2)}) \cap \phi(Y^X) = \emptyset$, $\phi(Y^X)$ is closed in $(CY)^X$.

Lemma 1. *If either of X or Y is T_2 and if $\mathfrak{B}$ is a subbasis for Y, then $\{(A, B) | A$ is a compact subset of X and B is an element of $\mathfrak{B}\}$ is a subbasis for the compact-open topology on Y^X.*

Lemma 2. *CY is T_2 if and only if Y is T_2 and CY is regular (T_1 and T_3) if and only if Y is regular.*

For Lemma 1 see [2] and [3], and for Lemma 2 see [5]. (It is not necessary for compact subsets of Y to be closed in order to prove either part of Lemma 2.)

Corollary. *$(CY)^X$ is T_2 if and only if Y is T_2 and $(CY)^X$ is regular if and only if Y is regular.*

Proof. This follows from Lemma 2 and the fact that $(CY)^X$ is T_2 if and only if CY is T_2 and $(CY)^X$ is regular if and only if CY is regular.

Theorem 2. *If X or Y is T_2, then $\mathfrak{B}[(CY)^X] \equiv \{(A, \bar{t}_U) | A \subset X$ is compact and $U \subset Y$ is open$\} \cup \{(A, \underline{t}_U) | A \subset X$ is compact, $U \subset Y$ is open$\}$ is a subbasis for $(CY)^X$.*(3)

Theorem 2 is an immediate consequence of Lemmas 1 and 2.

The following example demonstrates that if neither X nor Y is T_2 then $\mathfrak{B}[(CY)^X]$ need not be a subbasis for $(CY)^X$.

Example 1. Let R and S be two disjoint, countably infinite, dense subsets of $(0,1)$.

We define a topology $\mathfrak{T}_{[0,1]}$ on $[0,1]$ such that $\mathfrak{T}_{[0,1]} | (0,1)$ is the usual

(3) For the case that Y is T_2, Theorem 2 is shown in [2].

topology and such that $\{1\} \subset U$ ($\{0\} \subset U$), where U is an element of $\mathcal{T}_{[0,1]}$, if and only if U is open in the usual topology for $[0,1]$ and $R \subset U$ ($S \subset U$). With this topology $[0,1]$ is a compact, T_0 space. Let $(X, \mathcal{T}_X)$ be $[0,1] - (R \cup S)$ with the subspace topology inherited from $([0,1], \mathcal{T}_{[0,1]})$. $(X, \mathcal{T}_X)$ is a compact, T_1 space.

Let $Y' = ([2, 3] \cup \{4\} \cup \{5\})$ and define a topology $\mathcal{T}_{Y'}$ on Y' such that $\mathcal{T}_{Y'}|[2, 3)$ is the usual topology and such that $\{3\}$, $\{4\}$, or $\{5\}$ is a subset of U, where U is an element of $\mathcal{T}_{Y'}$, if and only if U is open in the usual topology for $([2, 3] \cup \{4\} \cup \{5\})$ and $\{2\} \subset U$. Let $(Y, \mathcal{T}_Y)$ be $Y' - \{2\}$ with the subspace topology inherited from $(Y', \mathcal{T}_{Y'})$. We note that $(Y', \mathcal{T}_{Y'})$ is a compact, T_0 space and that $(Y, \mathcal{T}_Y)$ is a compact, T_1 space. Further $(2, 3]$, $\{4\} \cup (2, 3]$, and $\{5\} \cup (2, 3]$ are compact subsets of $(Y, \mathcal{T}_Y)$.

Let $f: X \to CY$ be defined by $f(q) = (2, 3]$ if and only if $q \in ((0, 1) \cap X)$, $f(0) = (2, 3] \cup \{4\}$, $f(1) = (2, 3] \cup \{5\}$. It is easily shown that f is continuous and therefore $f \in (CY)^X$.

Now $(2, 3]$, $(2, 3] \cup \{4\}$, and $(2, 3] \cup \{5\}$ are all open in $(Y, \mathcal{T}_Y)$. Thus $[(\bar{t}_{(2,3] \cup \{4\}} \cap \underline{t}_{(2,3]}) \cup (\bar{t}_{(2,3] \cup \{5\}} \cap \underline{t}_{(2,3]})] \equiv Z$ is an open subset of CX. Further, it is clear that $f \in (X, Z)$, open in $(CY)^X$, since X is compact and $f(q) \in Z$, for each $q \in X$.

To show $\mathcal{B}[(CY)^X]$ is not a subbasis for $(CY)^X$ it will be sufficient to show that every finite intersection of elements of $\mathcal{B}[(CY)^X]$ containing f contains an element of $(CY)^X$ not contained in (X, Z). Further, we should first note that $g \in (CY)^X$ is contained in (X, Z) only if $g(q)$ is a subset of $(2, 3] \cup \{4\}$ or $g(q)$ is a subset of $(2, 3] \cup \{5\}$, for each $q \in X$.

Let $B = (\bigcap_{i=1}^n (A_i, \bar{t}_{W(i)})) \cap (\bigcap_{j=1}^m (D_j, \underline{t}_{V(j)}))$ be a finite intersection of elements of $\mathcal{B}[(CY)^X]$ containing f. Without loss of generality, we may assume no A_i contains both 0 and 1 else $(A_i, \bar{t}_{W(i)})$ would equal $(CY)^X$.

Assuming no A_i contains both 0 and 1, we first claim $X \not\subset \bigcup_{i=1}^n A_i$. Suppose $X \subset \bigcup_{i=1}^n A_i$. Then let K = the union of all A_i such that $1 \notin A_i$, $1 \leq i \leq n$, and let L = the union of all A_i such that $0 \notin A_i$, $1 \leq i \leq n$. Neither K nor L is empty, $0 \in K$, $1 \in L$, $K \cup L = X$, and both K and L are compact. Let q be an element of R and let $U_1, \cdots, U_n, \cdots$ be a nested sequence of open subsets of $([0, 1], \mathcal{T}_{[0, 1]})$ such that $\bigcap_{r=1}^\infty U_r = q$. For the sequence $\{U_r \cap X\}_{r=1}^\infty$ of open subsets of $(X, \mathcal{T}_X)$ it either is, or is not, the case that there exists an n such that, for each $m > n$, $U_m \cap X \subset L$. If there exists such an n, then an open cover of L having no finite subcover can be constructed and, if there does not exist such an n one can do the same for K. Thus the assumption that $X \subset \bigcup_{i=1}^n A_i$ leads to a contradiction, and, therefore, there exists a point $\tilde{x}$ of X such that $\tilde{x} \notin \bigcup_{i=1}^n A_i$.

Define the function $g: X \to CY$ by $g(\tilde{x}) = Y$ and $g(q) = f(q)$ if $q \neq \tilde{x}$. Since X is T_1 and f is continuous, it follows that g is continuous. Hence $g \in (CY)^X$.

From the definition of g, $g \in \bigcap_{i=1}^n (A_i, \bar{t}_{W(i)})$, for $g|A_i = f|A_i$, $1 \leq i \leq n$. For each $q \in X$, $g(q) \cap (2, 3] \neq \emptyset$, and, since any open subset of Y intersects $(2, 3]$, it follows that $g \in \bigcap_{j=1}^m (D_j, \underline{t}_{V(j)})$, and therefore $g \in B$.

Since $g(\tilde{x})$ is an element of neither $\bar{t}_{(2,3]\cup\{4\}}$ nor $\bar{t}_{(2,3]\cup\{5\}}$, g is evidently not an element of (X, Z). Thus $\mathcal{B}[(CY)^X]$ is not a subbasis for the compact-open topology on $(CY)^X$.

Evidently Y being T_1 does not imply that CY is T_1 for, in the above example, the element $(2, 3]$ of CY is not closed.(4)

To prove the following lemmas, see Theorem 2.5.2 and Corollary 9.6 of [5] respectively. (That compact sets be closed is not necessary in either.)

Lemma 3. *If X is any topological space and $\mathfrak{A}$ is a compact subset of CX, then $\bigcup_{A \in \mathfrak{A}} A$, is a compact subset of X.*

Lemma 4. *If $f \in (CY)^X$ and $A \subset X$ is compact, then $\bigcup_{x \in A} f(x)$ is a compact subset of Y.*

Theorem 3. *$(CY)^X$ is homeomorphic to a subspace of $(CY)^{CX}$, with the compact-open topology, provided that either of X or Y is T_2.*

Proof. Define a function $\Sigma: (CY)^X \to (CY)^{CX}$ by $\Sigma(f)(A) = D$ where $A \in CX$, and $D \in CY$ such that $D = \bigcup_{x \in A} f(x)$. D is a compact subset of Y by Lemma 4 and thus, if $f \in (CY)^X$, then $\Sigma(f)$ is a function from CX to CY, and we first show that $\Sigma(f)$ is actually an element of $(CY)^{CX}$.(5)

It suffices to prove that the pre-images of subbasic open sets are open. Let $\bar{t}_U$ be a subbasic open set of CY containing $\Sigma(f)(A)$ where $A \in CX$. Since f is continuous, $\{x | f(x) \in \bar{t}_U\} \equiv Z$ is open in X. Hence $\bar{t}_Z$ is open in CX, and, since $f(x) \in \bar{t}_U$, for each $x \in A$, $A \in \bar{t}_Z$. Clearly, if $B \in \bar{t}_Z$ then $f(b) \in \bar{t}_U$ for each $b \in B$ and hence $\Sigma(f)(B) \in \bar{t}_U$. Similarly, if $\Sigma(f)(A) \in \underline{t}_V$, where $V \subset Y$ is open, then there is an element a of A such that $f(a) \in \underline{t}_V$. Since $\{x | f(x) \in \underline{t}_V\} \equiv W$ is open in X, $\underline{t}_W$ is an open subset of CX containing A and $\Sigma(f)(\underline{t}_W) \subset \underline{t}_V$. Thus $\Sigma(f) \in (CY)^{CX}$.

Clearly Σ is 1-1.

Now we shall show that Σ is continuous. Since X or Y is T_2, we know that CX or CY is T_2. Thus by Lemma 1, $\{(\mathfrak{A}, \bar{t}_U) | \mathfrak{A} \subset CX$ is compact and $U \subset Y$ is open$\} \cup \{(\mathfrak{A}, \underline{t}_U) | \mathfrak{A} \subset CX$ is compact and $U \subset Y$ open$\}$ forms a subbasis for $(CY)^{CX}$.

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(4) Compare with Theorem 4.9.2 of [5].

(5) It is shown in Lemma A of [6] that if f is an element of Y^X then $\Sigma \circ \phi(f)$, where ϕ is as in Theorem 1, is an element of $(CY)^{CX}$.

Let $(\mathfrak{A}, \bar{t}_U)$ and $(\mathfrak{D}, \underline{t}_V)$ be subbasis elements of $(CY)^{CX}$. If $f \in (CY)^X$ such that $\Sigma(f) \in (\mathfrak{A}, \bar{t}_U)$ then, by Lemma 3, $\bigcup_{A\in\mathfrak{A}} A \equiv \mathfrak{A}^*$ is a compact subset of X. Further, if $g \in (CY)^X$, $\Sigma(g) \in (\mathfrak{A}, \bar{t}_U)$ if and only if $g(x) \in \bar{t}_U$ for each $x \in \mathfrak{A}^*$. But this is true if and only if $g \in (\mathfrak{A}^*, \bar{t}_U)$. Hence $f \in (\mathfrak{A}^*, \bar{t}_U)$, open in $(CY)^X$, and $\Sigma(\mathfrak{A}^*, \bar{t}_U) \subset (\mathfrak{A}, \bar{t}_U)$.

Now $\Sigma(f) \in (\mathfrak{D}, \underline{t}_V)$ implies that if $D \in \mathfrak{D}$ then there exists some $x \in \mathfrak{D}$ such that $f(x) \in \underline{t}_V$. By Lemma 3, $\bigcup_{D\in\mathfrak{D}} D \equiv \mathfrak{D}^*$ is a compact subset of X and $\{x \mid f(x) \in \underline{t}_V\} \equiv W$ is an open subset of X. Further, $\mathfrak{D}^* \cap W \neq \emptyset$ since $D \in \mathfrak{D}$ implies that $\Sigma(f)(D) \in \underline{t}_V$ which implies that there exists some $x \in D$ such that $f(x) \in \underline{t}_V$. But this says that $x \in W$, and, since $x \in \mathfrak{D}^*$, we have $x \in (\mathfrak{D}^* \cap W)$. Hence for each $D \in \mathfrak{D}$, there exists some $x \in D$ such that $x \in (\mathfrak{D}^* \cap W)$. Since $\mathfrak{D}^*$ is compact, it follows that, if X is T_2, $\mathfrak{D}^*$, as a subspace of X, is regular. Therefore, since $\mathfrak{D}^* \cap W$ is open in the subspace topology of $\mathfrak{D}^*$, there exists for each $x \in (\mathfrak{D}^* \cap W)$, a subset $O(x)$ of X, open in the subspace topology of $\mathfrak{D}^*$, such that $x \in O(x) \subset \overline{O(x)} \subset (\mathfrak{D}^* \cap W)$. Note that the closure of $O(x)$, in X, is contained in $\mathfrak{D}^*$ since $\mathfrak{D}^*$ being compact and X being T_2 implies that $\mathfrak{D}^*$ is closed.

If Y is T_2 then CY is T_2. Further, for $x \in (\mathfrak{D}^* \cap W)$, if q is an element of $[\overline{(\mathfrak{D}^* \cap W)} - (\mathfrak{D}^* \cap W)] \cap \mathfrak{D}^*$ then $f(x) \neq f(q)$, since $f(x) \in \underline{t}_V$ and $f(q) \notin \underline{t}_V$. Thus, since CY is T_2 and f is continuous, there are disjoint sets, $M(x,q)$ and $N(q)$, open in the subspace topology for $\mathfrak{D}^*$ and containing x and q respectively, such that $M(x, q) \subset (\mathfrak{D}^* \cap W)$. Since $\mathfrak{D}^*$ is compact and $[\overline{(\mathfrak{D}^* \cap W)} - (\mathfrak{D}^* \cap W)] \cap \mathfrak{D}^*$ is closed in $\mathfrak{D}^*$, it follows that $[\overline{(\mathfrak{D}^* \cap W)} - (\mathfrak{D}^* \cap W)] \cap \mathfrak{D}^*$ is compact, which implies that there exist $N(q_1), \cdots, N(q_k)$ such that $\bigcup_{i=1}^k N(q_i) \supset [\overline{(\mathfrak{D}^* \cap W)} - (\mathfrak{D}^* \cap W)] \cap \mathfrak{D}^*$. Hence $x \in \bigcap_{i=1}^k M(x, q_i) \equiv O(x) \subset (\overline{O(x)} \cap \mathfrak{D}^*) \subset (\mathfrak{D}^* \cap W)$.

Thus, whether X or Y is T_2 we have, for each $x \in (\mathfrak{D}^* \cap W)$, a subset $O(x)$, open in the subspace topology for $\mathfrak{D}^*$ such that $O(x) \subset (\overline{O(x)} \cap \mathfrak{D}^*) \subset (\mathfrak{D}^* \cap W)$. Each $O(x)$ is open in the subspace topology for $\mathfrak{D}^*$ and hence, for each $x \in (\mathfrak{D}^* \cap W)$, there exists a set $O'(x)$, open in X, such that $O(x) = O'(x) \cap \mathfrak{D}^*$.

Now if $D \in \mathfrak{D}$, $f(D) \in \underline{t}_V$ which implies that $D \cap (\mathfrak{D}^* \cap W) \neq \emptyset$ and thus $D \cap O'(x) \neq \emptyset$ for some $x \in (\mathfrak{D}^* \cap W)$, since each such x in $\mathfrak{D}^* \cap W$ is covered by some $O(x)$. $\{\underline{t}_{O'(x)} \mid x \in (\mathfrak{D}^* \cap W)\}$ is an open cover of $\mathfrak{D}$ and because $\mathfrak{D}$ is compact there exist $x(1), \cdots, x(r)$ elements of $\mathfrak{D}^* \cap W$, such that $\mathfrak{D} \subset \bigcup_{l=1}^r \underline{t}_{O'(x(l))}$. Thus, if $D \in \mathfrak{D}$, $D \cap O'(x(l)) \neq \emptyset$ for some l such that $1 \leq l \leq r$. Since $D \subset \mathfrak{D}^*$, this implies $D \cap [(O'(x(l))) \cap \mathfrak{D}^*] \neq \emptyset$.

Thus each D contains a point of $O(x(l))$ for some l such that $1 \leq l \leq r$ where $D \in \mathfrak{D}$. Now $E \equiv \bigcup_{l=1}^r (\overline{O(x(l))} \cap \mathfrak{D}^*)$ is closed in the subspace $\mathfrak{D}^*$, and hence compact, and is also contained in W. Thus, for each $x \in E$, $f(x) \in \underline{t}_V$, and further, if $D \in \mathfrak{D}$, there exists some $x \in D$ such that $x \in E$. Thus $f \in (E, \underline{t}_V)$ and, if $g \in (E, \underline{t}_V)$, then for each $D \in \mathfrak{D}$, there exists an element $x \in D$ such that $x \in E$ and hence $g(x) \in \underline{t}_V$. This implies that $\Sigma(g) \in (\mathfrak{D}, \underline{t}_V)$. Hence $f \in (E, \underline{t}_V)$ and $\Sigma((E, \underline{t}_V)) \subset (\mathfrak{D}, \underline{t}_V)$. This shows Σ to be continuous.

Let (A, W) be a subbasic open set of $(CY)^X$, and then $\mathfrak{A} \equiv \{\{x\} \in CX \mid x$ is a point of $A\}$ is a compact subset of CX. If $f \in (A, W)$, then from the definition of Σ, $\Sigma(f) \in (\mathfrak{A}, W)$. Further, it is clear that if $\Sigma(f) \in (\mathfrak{A}, W)$ then $f(x) \in W$, for each $\{x\} \in \mathfrak{A}$, which implies that $f \in (A, W)$. We have shown $\Sigma((A, W)) = (\mathfrak{A}, W) \cap \Sigma((CY)^X)$ which is open in $(CY)^{CX} \cap \Sigma((CY)^X)$, and, since Σ is 1-1, this shows Σ to be open.

Let $\mathcal{C}$ denote the image of Σ. An element of $(CY)^{CX}$ will be called *consistent* if and only if it is an element of $\mathcal{C}$. An element of $(CY)^{CX} - \mathcal{C}$ will be called *inconsistent.*

Corollary. *An element f of $(CY)^{CX}$ is consistent if and only if, for each $A \in CX$, $f(A) = D$, where $D = \bigcup_{x \in A} f(\{x\})$.*

Easy examples, where X and Y are finite discrete spaces, show that $(CY)^{CX} - \mathcal{C}$ may be nonempty. We may also observe that $\mathcal{C}$ is not necessarily closed by letting X be a two point discrete space and Y the Sierpinski space.(6) However, if Y is T_2 we have the following theorem.

Theorem 4. *If Y is T_2 then $\mathcal{C}$ is a closed subset of $(CY)^{CX}$.*

Proof. Suppose f is an inconsistent element of $(CY)^{CX}$. Then there is an element A of CX such that $f(A) \neq \bigcup_{x \in A} f(\{x\})$.

Assume there is a point p of Y such that $p \in f(A)$ but $p \notin \bigcup_{x \in A} f(\{x\})$. Now $\{\{x\} \mid x \in A\} \equiv \mathfrak{A}$ is a compact subset of CX and this implies that $f(\mathfrak{A}) = \{f(\{x\}) \mid x \in A\}$ is a compact subset of CY. Hence, by Lemma 3, $\bigcup_{x \in A} f(\{x\})$ is a compact subset of Y. Therefore, since Y is T_2, there exist disjoint open sets, U and V, of Y such that $p \in U$ and $\bigcup_{x \in A} f(\{x\}) \subset V$. Then f is an element of $(\{A\}, \underline{t}_U) \cap (\mathfrak{A}, \overline{t}_V)$, which is an open subset of $(CY)^{CX}$ and clearly contains no element of $\mathcal{C}$.

Now assume there is a point p of Y such that $p \in \bigcup_{x \in A} f(\{x\})$ but $p \notin f(A)$. Then there is an element x of X such that $x \in A$ and $p \in f(\{x\})$. Since $f(A)$ is compact and Y is T_2, there exist disjoint open sets, U and V, of Y such that $f(A) \subset U$ and $p \in V$. Therefore, f is an element of $(\{A\}, \overline{t}_U) \cap (\{\{x\}\}, \underline{t}_V)$, an open subset of $(CY)^{CX}$ which contains no element of $\mathcal{C}$. Thus $(CY)^{CX} - \mathcal{C}$ is open which completes the proof.

(6) By the "Sierpinski space" we mean a topological space consisting of two points, x_1 and x_2, with the totality of open sets being $\{x_1, x_2\}$, $\{x_1\}$, and $\emptyset$.

If $f \in X^Y$ then f^{-1} is a set-valued function from the image of f to Y. Theorem 5 shows how, under certain conditions, some elements of X^Y may be considered as elements of $(CY)^X$. However, the function between the two subspaces is not necessarily a homeomorphism.

Let $D_X^Y \equiv \{f \in X^Y | f$ is open, closed, and onto$\}$ and let $\mathfrak{D}_Y^X \equiv \{f \in (CY)^X | f$ is disjointly 1-1, $\bigcup_{x \in M} f(x)$ is open (closed) whenever $M \subset X$ is open (closed), and $\bigcup_{x \in X} f(x) = Y\}$. (By "disjointly 1-1" we simply mean that if x_1 and x_2 are distinct elements of X then $f(x_1) \cap f(x_2) = \emptyset$.)

Theorem 5. *If Y is compact, X or Y is T_2, and $D_X^Y \neq \emptyset$, then ψ, defined by $\psi(f)(x) = f^{-1}(x)$, is a* 1-1, *open function from D_X^Y onto $\mathfrak{D}_Y^X$.*

Proof. If f is in D_X^Y, it is easily verified that $\psi(f) \in \mathfrak{D}_Y^X$.(7)

Let $h \in \mathfrak{D}_Y^X$ and define $f: Y \longrightarrow X$ by $f(y) = x$ if and only if $y \in h(x)$. Then $f \in D_X^Y$ and $\psi(f) = h$. Clearly ψ is 1-1.

We have only to show that ψ is open: Let (A, W) be a subbasis element of X^Y and let $f \in (D_X^Y \cap (A, W))$. Now since W is open in X, and $f: Y \longrightarrow X$ is continuous and onto, it follows that X must be compact and that $X - W$ must be compact. Hence $(X - W, \bar{t}_{Y-\bar{A}})$ is open in $(CY)^X$. Assume X is T_2. If $x' \in (X - W)$ and $x \in f(A)$ there exist disjoint open sets $U(x)$ and $V(x, x')$ containing x and x' respectively and, since $\psi(f) = f^{-1}$ is an element of $\mathfrak{D}_Y^X$, it follows that $\bigcup_{p \in U(x)} \psi(f)(p)$ and $\bigcup_{p \in V(x,x')} \psi(f)(p)$ are disjoint open sets, containing $\psi(f)(x)$ and $\psi(f)(x')$, respectively. By repeating this process for all $x \in f(A)$, we observe that $\{\bigcup_{p \in U(x)} \psi(f)(p) | x \in f(A)\}$ is an open cover of A and since A is compact there exist $x_1, \cdots, x_m$ elements of $f(A)$ such that $\bigcup_{j=1}^m (\bigcup_{p \in U(x_j)} \psi(f)(p)) \supset A$. But

$$\bigcap_{j=1}^{m} \left(\bigcup_{p \in V(x_j, x')} \psi(f)(p) \right) \supset \psi(f)(x')$$

and

$$\left[\bigcup_{j=1}^{m} \left(\bigcup_{p \in U(x_j)} \psi(f)(p) \right) \right] \cap \left[\bigcap_{j=1}^{m} \left(\bigcup_{p \in V(x_j, x')} \psi(f)(p) \right) \right] = \emptyset.$$

This implies that no point of $\psi(f)(x')$ is in $\bar{A}$, for each $x' \in (X - W)$. Therefore, $\psi(f) \in (X - W, \bar{t}_{Y-\bar{A}})$ if X is T_2.

Now assume Y is T_2. Then $A = \bar{A}$ since A is compact, and thus $(X - W, \bar{t}_{Y-\bar{A}}) = (X - W, \bar{t}_{Y-A})$. But $x \in (X - W)$ implies that $x \notin f(A)$, which implies

(7) It is essentially shown in [5] that $\psi(f)$ is continuous if and only if f is open and closed (where the notation f^{-1*} is used instead of our $\psi(f)$).

$\psi(f)(x) \cap A = f^{-1}(x) \cap A = \emptyset$. Hence $\psi(f) \in (X - W, \bar{t}_{Y-A})$. Therefore, if either of X or Y is T_2, $\psi(f) \in (X - W, \bar{t}_{Y-\bar{A}})$.

Let $g \in ((X - W, \bar{t}_{Y-\bar{A}}) \cap \mathfrak{D}_Y^X)$. We know $\psi^{-1}(g)$ exists and is exactly one element of D_X^Y. Let $y \in A$. Since $g(x) \cap \bar{A} = \emptyset$, for each $x \in (X - W)$, it follows that $\psi^{-1}(g)(y) \in W$. Thus $\psi^{-1}(g) \in (A, W)$ which implies that $g \in \psi((A, W) \cap D_X^Y)$. This shows $((X - W, \bar{t}_{Y-\bar{A}}) \cap \mathfrak{D}_Y^X) \subset \psi((A, W) \cap D_X^Y)$ which, since ψ is 1-1, completes the proof that ψ is an open function.

The following example shows that $\psi: D_X^Y \to \mathfrak{D}_Y^X$ need not be continuous.

Example 2. Let X be the topological space consisting of the set of all real numbers of the form $1/i$, $i = 1, 2, \cdots$, together with 0, with the subspace topology inherited from the reals. Let Y be the topological space consisting of the set of all real numbers of the form $1/i$, $i = 1, 2, \cdots$, or of the form $2 - 1/i$, $i = 1, 2, \cdots$, together with 0 and 2 with the subspace topology inherited from the reals. We note that X and Y are both compact, T_2 spaces.

Define $f: Y \to X$ by $f(y) = y$, $0 \le y \le 1$, and $f(y) = 2 - y$, $1 \le y \le 2$. Observe that f is continuous, open, closed, and onto. Thus $f \in D_X^Y$. Now $\psi(f) \in ((\{0\}, \underline{t}_{(1,2]\cap Y}) \cap \mathfrak{D}_Y^X) \equiv U$, open in $\mathfrak{D}_Y^X$. We will show that given any basic open set, V, of D_X^Y, containing f, there exists an element of $(\mathfrak{D}_Y^X - U)$ contained in $\psi(V)$.

Let $V = (\bigcap_{i=1}^n (A_i, W_i)) \cap D_X^Y$ be a basic open set of D_X^Y containing f. Let $M = \{A_i, 1 \le i \le n \mid 2 \in A_i\}$, $N = \{A_i, 1 \le i \le n \mid 2 \notin A_i\}$.

Let $K = \bigcup_{A_i \in N} A_i$. Assume $K \neq \emptyset$. Then K is a compact subset of $[0, 2) \cap Y$ and hence there is an element k of $(1,2) \cap Y$ such that if $k' \in K$ then $k > k'$. If $K = \emptyset$, choose k to be $1 + \frac{1}{2}$.

Suppose that $M \neq \emptyset$. Then if $A_i \in M$, there is a half-open interval Z_i such that $0 \in Z_i$ and $Z_i \cap X \subset W_i$. Let $Z = (\bigcap_{A_i \in M} Z_i) \cap X$ and choose an element z of Z such that $z \in ((0, 1) \cap X)$. If $M = \emptyset$, choose z to be $\frac{1}{2}$.

Let l be the greater of k and $2 - z$ and define $g: Y \to X$ by $g(y) = f(y)$, $y \le l$, and $g(y) = f(l)$, $l \le y$. Clearly g is continuous, open, closed, and onto, and hence an element of D_X^Y. Also, $g \in (\bigcap_{i=1}^n (A_i, W_i) \cap D_X^Y)$ since $g | \bigcup_{A_i \in N} A_i = f | \bigcup_{A_i \in N} A_i$, $g(y) = f(y)$, for each $y \in ([0, l] \cap Y)$, and $g(y) \in \bigcap_{A_i \in M} W_i$, for each $y \in ([l, 2] \cap Y)$. However, $g^{-1}(0) = \{0\}$ and thus $\psi(g)(0) \notin \underline{t}_{(1,2]\cap Y}$. This implies that $\psi(g) \notin ((\{0\}, \underline{t}_{(1,2]\cap Y}) \cap \mathfrak{D}_Y^X)$. Therefore $\psi(g) \in (\mathfrak{D}_Y^X - U)$, and it follows that $\psi: D_X^Y \to \mathfrak{D}_Y^X$ is not continuous.

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