TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 169, July 1972

ON THE FINITELY GENERATED SUBGROUPS OF AN AMALGAMATED PRODUCT OF TWO GROUPS

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In Memoriam Hanna Neumann

ABSTRACT. Sufficient conditions are found for the free product G of two groups A and B with an amalgamated subgroup U to have the properties (1) that the intersection of each pair of finitely generated subgroups of G is again finitely generated, and (2) that every finitely generated subgroup containing a nontrivial subnormal subgroup of G has finite index in G. The known results that Fuchsian groups and free products (under the obvious conditions on the factors) have properties (1) and (2) follow as instances of the main result.

1. **Introduction.** This paper extends the class of groups G for which the following two properties are known to hold:

(1) every pair of finitely generated subgroups of G intersect in a finitely generated subgroup (briefly, G has the finitely generated intersection property);

(2) no finitely generated subgroup of G of infinite index contains a nontrivial subnormal subgroup of G.

Our main result (Theorem 2.3) gives sufficient conditions for the generalized free product $(A*B;\ U)$ of groups A and B amalgamating the subgroup U to have properties (1) and (2). That some restriction is necessary is shown by the group $\langle x, y \mid x^2 = y^3 \rangle$, which has neither property (Karrass and Solitar [8], Moldavanskiĭ [10]). The proof of Theorem 2.3 utilises the algebraic methods developed by Karrass and Solitar [8]. The following result is a particular case of that theorem. We call a subgroup U of a group A *isolated* if, whenever $a^n \in U$ for any nonzero integer n and any $a \in A$, then $a \in U$.

1.1. **Theorem.** *If $G = (A*B;\ U)$ where A is free and U is infinite cyclic and isolated in A, then*

(i) *if B has the finitely generated intersection property, so does G;*

(ii) *if $U \neq A$, any finitely generated subgroup of G containing a nontrivial subnormal subgroup of G has finite index.*

Received by the editors July 15, 1971.

AMS 1970 subject classifications. Primary 20E30; Secondary 20E05, 20E40.

Key words and phrases. Amalgamated product, generalized free product, malnormal subgroup, subnormal subgroup, finitely generated intersection property, Kuroš rewriting process, Schreier system, compatible regular extended Schreier system, double ended coset.

This implies in particular the result of Greenberg [3] that the Fuchsian groups (i.e. the discrete subgroups of $LF(2, R)$, the group of all 2×2 matrices over the reals with determinant $+1$) have properties (1) and (2). For, as Karrass and Solitar have pointed out, it is not difficult to show that a finite extension G of a group with property (1) again has property (1), and that certainly if the nontrivial subnormal subgroups of G are infinite, the same is true as regards property (2). Now the finitely generated infinite Fuchsian groups are either generalized free products of the sort described in Theorem 1.1 or finite extensions of such generalized free products, and their nontrivial subnormal subgroups are infinite: the first claim was originally proved by R. Fricke and F. Klein by geometrical methods, and recently by Hoare, Karrass and Solitar [5] using methods of combinatorial group theory; the second statement has recently been proved also by Hoare, Karrass and Solitar.(1)

Our Theorem 2.3 is however of wider applicability. Thus it contains as a special case the result that a free product has property (1) if the factors do (Baumslag [1]), and has property (2) if there are at least two nontrivial factors (Karrass and Solitar [7]).

Property (1) was first considered by Howson [6], who established it for free groups. Property (2) originated with Schreier's well-known result that a finitely generated normal subgroup of a free group necessarily has finite index.

I am indebted to A. Karrass and D. Solitar for introducing me to the problem and for many helpful comments.

This research was partially supported by a grant from the National Research Council of Canada.

2. **Statement of results.** For concise statements of the results a few preliminaries are necessary.

For the usual definition, and existence and uniqueness, of a generalized free product see e.g. [9]. Alternatively one may define it by the following well-known characterizing property (which will be all that is used directly in the sequel). Recall first that a *left transversal* for a subgroup H of a group G is a complete set of representatives of left cosets gH $(g \in G)$.

2.1. **Lemma** (cf. [9, **Theorem** 4.4]). *A group G is a free product of two subgroups A and B amalgamating a subgroup U contained in $A \cap B$ if and only if, for each pair T_A, T_B of left transversals for U in A, B respectively, every element $g \in G$ can be uniquely expressed in the form*

$$g = t_1 \cdots t_n u, \tag{3}$$

where $n \geq 0$, $u \in U$, $t_i \in (T_A \cup T_B)\backslash U$ $(i = 1, \cdots, n)$ and t_i, t_{i+1} do not both

(1) *On subgroups of infinite index in Fuchsian groups*, Math. Z. (to appear).

belong to T_A *nor to* T_B $(i = 1, \cdots, n-1)$.

We write $G = (A*B;\ U)$. We shall refer to the right-hand side of (3) as the *canonical form* for g (relative to the pair T_A, T_B). The element g will be said to be of *length* n, to *begin* with t_1 and to *end* with t_n. The t_i will be called *syllables* (A- or B-syllables according as they belong to T_A or T_B), and u will be termed the *U-syllable* of g. Finally, the elements $t_1 \cdots t_i$ $(i = 1, \cdots, n)$ will be called *initial segments* of g.

We shall also need the concept of malnormality. B. Baumslag [2] defined a subgroup U of a group G to be *malnormal* in G if $g^{-1}Ug \cap U$ is trivial for all $g \in G\setminus U$. A nontrivial example relevant to our purposes is that of an isolated cyclic subgroup of a free group (see § 6).

The following lemma furnishes an equivalent condition more convenient for us.

2.2. **Lemma (Solitar).** *A subgroup* U *is malnormal in* G *if and only if there exists a left transversal* T *for* U *in* G, *containing the identity* e, *such that for all* $t \in T\setminus\{e\}$ *and all* $u \in U$, $ut \in T\setminus\{e\}$; *i.e. such that*

$$U(T\setminus\{e\}) = T\setminus\{e\}. \tag{4}$$

Proof. Suppose U is malnormal in G and let T_1 be a complete set of representatives of the double cosets UgU $(g \in G)$, including e. Then $T = U(T_1\setminus\{e\}) \cup \{e\}$ is a left transversal for U in A; for, if $t_1, t_2 \in T_1\setminus\{e\}$ and $u_1, u_2 \in U$ are such that $(u_1t_1)^{-1}u_2t_2 \in U$, then $t_1^{-1}u_1^{-1}u_2t_2 \in U$, whence $t_1 = t_2$, and thence $u_1 = u_2$ by the malnormality of U. Thus T satisfies (4). The converse is equally straightforward and we omit the proof.

In addition to malnormality the following condition will be imposed. We shall say that a subgroup U of G is *finitely involved* in a subset $S \subseteq G$ with respect to a left transversal T for U in G, if there is a finite subset $V \subseteq U$ such that $S \subseteq TV(S^{-1}S \cap U)$. We shall then say that U is *finitely involved throughout* G if there exists a left transversal T for U in G such that, for every finitely generated subgroup $H \leq G$ and every $g \in G$, U is finitely involved in Hg, i.e. there is a finite subset V of U, depending on Hg, such that

$$Hg \subseteq TV(g^{-1}Hg \cap U). \tag{5}$$

Clearly a finite subgroup of a group G is finitely involved throughout G. However, for us the infinite case is more interesting: we shall prove in § 6 that any isolated infinite cycle in a free group F is finitely involved throughout F. In fact by Theorem 6.1 such an infinite cycle satisfies the more stringent requirements of our main result, which we can now formulate.

2.3. **Theorem.** *Let* $G = (A*B;\ U)$ *where* U *is malnormal in* A *and finitely involved throughout* A, *and has the additional property that there is a left transversal*

T for U in A satisfying simultaneously conditions (4) *and* (5). *The following conclusions are valid*:

(i) *if every subgroup of U (including U itself) is finitely generated, and A, B have the finitely generated intersection property, then so does G*;

(ii) *if $U \neq A$, B and H is a finitely generated subgroup of G containing a subgroup N, $N \nsubseteq U$, N subnormal in G, then H has finite index.*

Theorem 1.1 follows at once from this theorem and Theorem 6.1.

3. **Lemmas.** The following lemma is crucial to the proof of Theorem 2.3. Broadly speaking it shows that the restriction given by the hypothesis of Theorem 2.3 on the way U is embedded as a subgroup of a single factor of $G = (A*B; U)$ implies a property similar to finite involvement of U in certain subsets of the whole group G.

3.1. **Lemma.** *Suppose that $G = (A * B; U)$ where U and A satisfy the hypothesis of Theorem* 2.3. *Let T_A be a left transversal for U in A satisfying* (4) *and* (5), *and let T_B be any left transversal for U in B, containing e. Let H be any finitely generated subgroup of G and let g be any element of G. Then if D denotes the set of all elements of Hg ending in an element of $T_A \setminus \{e\}$ and D_1 denotes the set obtained from D by deleting the U-syllables from the ends of the elements of D, there exists a finite subset $V \subseteq U$ such that*

$$D \subseteq D_1 V (g^{-1}Hg \cap U). \tag{6}$$

For the proof we need some of the technique developed by Karrass and Solitar [8]. In order to elicit the structure of the subgroups of $(A*B; U)$ they introduced the concept of a compatible regular extended Schreier system (or cress) for a subgroup. This has evolved from the idea of a Schreier transversal for a subgroup of a free group. In the case that U is trivial the cress becomes the regular extended Schreier system employed in e.g. [9] to prove the Kuroš subgroup theorem for free products. We give a definition in terms of the canonical forms of elements. This gives the definition greater concreteness at considerable cost in generality: however it suffices for our purposes.

3.2. **Definition** (cf. **Karrass and Solitar** [8]). A *compatible regular extended Schreier system* (cress) for a subgroup H of $G = (A*B; U)$ relative to left transversals T_A, T_B for U in A, B respectively, both containing e, is a pair $\{C_A, C_B\}$ of right transversals for H in G, with the following properties:

(i) for all $g \in C_A \cup C_B$, where $g = t_1 \cdots t_n u$ in canonical form,

(a) if $g \in C_A$, then $gu^{-1} \in C_A$ (and similarly for C_B);

(b) if $u = e$ and $t_n \in T_A$, then $g, gt_n^{-1} \in C_A$ (and similarly, if $t_n \in T_B$, then

g, $gt_n^{-1} \in C_B$);

(c) if $gu^{-1} \in C_A \cap C_B$, then $g \in C_A \cap C_B$;

(ii) if S_A is the set consisting of e and all $e \neq g \in C_A$ which have $u = e$ and $t_n \in T_B$, then S_A is a complete double coset representative system for G modulo (H, A) (and similarly for S_B);

(iii) if R_A is the set of all $g \in C_A$ which have $u = e$, then R_A is a complete double coset representative system for G modulo (H, U) (and similarly for R_B).

We remark that this fulfills the conditions defining a cress in [8, §5]. To see this let all nontrivial elements of U be, in the terminology of [8], generating symbols of U, and let the nontrivial elements of $T_A \cup U$, $T_B \cup U$ be the generating symbols of A, B respectively. Conditions (1) to (4) of [8, §5] are then readily verified as being equivalent to the above. The proof of the existence of a cress as defined above is the same as that of [8, Lemma 6].

We now choose a particular generating set for H in terms of a given cress $\{C_A, C_B\}$ for H in G. For each $g \in G$ let $g\phi_A$ denote the representative in C_A of Hg, and define the coset representative function $\phi_B : G \to C_B$, analogously. For each $k \in C_A$, $x \in A$, write

$$s_A(k, x) = kx((kx)\phi_A)^{-1},$$

and define $s_B(k, x)$ for $k \in C_B$, $x \in B$, analogously. For each $d \in C_B$ define

$$t(d) = d(d\phi_A)^{-1}.$$

The *Kuroš rewriting process* for H in terms of these s- and t-symbols is then defined as follows. Suppose $h \in H$ has canonical form $t_1 \cdots t_n u$, and write

$$d_i^A = (t_1 \cdots t_i)\phi_A, \quad d_i^B = (t_1 \cdots t_i)\phi_B \qquad (i = 1, \cdots, n).$$

The rewritten expression for h (cf. [9, p. 230]) is obtained by replacing t_i $(i = 1, \cdots, n-1)$ by

$$t(d_{i-1}^B) \cdot s_A(d_{i-1}^A, t_i) \cdot t(d_i^B)^{-1} \quad \text{if } t_i \in T_A; \tag{7}$$

and by

$$s_B(d_{i-1}^B, t_i) \quad \text{if } t_i \in T_B; \tag{8}$$

and replacing $t_n u$ by

$$t(d_{n-1}^B) \cdot s(d_n^A, t_n u) \quad \text{if } t_n \in T_A;$$

and by

$$s(d_n^B, t_n u) \quad \text{if } t_n \in T_B.$$

It is not difficult to verify that these replacements leave h unchanged. Notice also that if $d \in C_B$ has the form ru_1 where $r \in R_B$, $u_1 \in U$, then (cf. [8, p. 243])

$$t(d) = ru_1((ru_1)\phi_A)^{-1} = r(r\phi_A)^{-1} \cdot (r\phi_A u_1((ru_1)\phi_A)^{-1});$$

that is

$$t(d) = t(r) \cdot s_A(r\phi_A, u_1). \tag{9}$$

It follows that H is generated by the $t(r)$ $(r \in R_B)$ and the s-symbols. Consider the element $s_A(k, x)$, where $x \in A$, and $k \in C_A$ has the form dqu, where $d \in S_A$, $q \in T_A$ (and $dq \in R_A$) and $u \in U$. It follows from (ii) of Definition 3.2 that $(dqux)\phi_A = dq_1u_1$ where $q_1 \in T_A$, $u_1 \in U$. Thus

$$s_A(k, x) = d(quxu_1^{-1}q_1^{-1})\, d^{-1} \in dAd^{-1} \cap H.$$

Similarly, when $k \in C_B$ and $x \in B$, we have $s_B(k, x) \in d_1Bd_1^{-1} \cap H$ where $d_1 \in S_B$ is the representative of the double coset HkB. Write

$$R = \{t(r) \mid r \in R_B\},$$
$$Q_A = \{d \mid d \in S_A,\ dAd^{-1} \cap H \nleq dUd^{-1}\},$$
$$Q_B = \{d \mid d \in S_B,\ dBd^{-1} \cap H \nleq dUd^{-1}\}.$$

The above facts are summarised and supplemented in the following lemma.

3.3. **Lemma.** *Let $H \leq G = (A*B;\ U)$ and let $\{C_A, C_B\}$ be a cress for H relative to T_A, T_B. Then H is generated by the set R together with $H \cap U$ and all subgroups $dAd^{-1} \cap H$ $(d \in Q_A)$ and $dBd^{-1} \cap H$ $(d \in Q_B)$. Furthermore, if every subgroup of U (including U itself) is finitely generated, then H is finitely generated if and only if R, Q_A and Q_B are finite, and for all $g \in G$, $gAg^{-1} \cap H$ is finitely generated.*

The proof of the first part of this lemma is sketched above (see also [8, Lemma 7]). The second part is immediate from Lemma 3 and Theorems 4, 5 of [8].

Write $R_1 = \{r, r\phi_A \mid r \in R_B,\ t(r) \neq e\}$, and write P for the set of all A-syllables of elements of the set $R_1 \cup Q_A \cup Q_B$.

3.4. **Corollary.** *The subgroup $H \leq (A*B;\ U)$ is finitely generated only if P is finite.*

3.5. **Lemma.** *Let $H \leq G = (A*B;\ U)$ and let $\{C_A, C_B\}$ be a cress for H relative to T_A, T_B. Suppose that $g \in R_A$ and $h \in H$ are such that, in canonical form, $hg = t_1 \cdots t_n u \notin U$, where $e \neq t_n \in T_A$. Then $t_n u = u_1 q_1^{-1} a q_2$, where $u_1 \in U$, $q_1 \in P \cup \{e\}$, either $q_2 = e$ or q_2 is the last syllable of g, and $a \in A \cap d^{-1}Hd$ where $dq_2 = g$.*

Proof. Firstly we apply to $hg = t_1 \cdots t_n u$ a rewriting process similar to the Kuroš rewriting process described above. To be specific, we replace t_i $(i = 1, \cdots, n-1)$ by the appropriate one of the products (7), (8), but $t_n u$ is replaced by $t(d^B_{n-1}) \cdot s_A(d^A_{n-1}, t_n u) \cdot g$. This process does not alter hg: for example

$$t(d^B_{n-1}) \cdot s_A(d^A_{n-1}, t_n u) \cdot g = d^B_{n-1}(d^A_{n-1})^{-1} d^A_{n-1} t_n u((d^A_{n-1} t_n u)\phi_A)^{-1} \cdot g = d^B_{n-1} \cdot t_n u, \tag{10}$$

since $(d^A_{n-1} t_n u)\phi_A = (hg)\phi_A = g$; and then d^B_{n-1} cancels with part of the element (8) replacing t_{n-1}, and so on.

Write $d^A_{n-1} = dq_1 u_1$ where $d \in S_A$ is the representative of the double coset $Hd^A_{n-1}A$, $q_1 \in T_A$ and $u_1 \in U$. Then by Definition 3.2(ii), $g = dq_2$ where $q_2 \in T_A$, since $HgA = Hd^A_{n-1}A$. From (10) we have

$$t_n u = (d^A_{n-1})^{-1} \cdot d^A_{n-1} t_n u g^{-1} \cdot g = u_1^{-1} q_1^{-1} d^{-1} \cdot d(q_1 u_1 t_n u q_2^{-1}) d^{-1} \cdot dq_2.$$

Write $a = q_1 u_1 t_n u q_2^{-1}$. Then $a \in A \cap d^{-1}Hd$ and $t_n u = u_1^{-1} q_1^{-1} a q_2$. It remains to show only that if $q_1 \neq e$ then $q_1 \in P$.

Thus suppose $q_1 \neq e$. If $d^B_{n-1} = d_1 p_1 v_1 \neq d^A_{n-1} = dq_1 u_1$, where $d_1 \in S_B$, $p_1 \in T_B$, $v_1 \in U$, then by Definition 3.2(iii) $(d_1 p_1)\phi_A = dq_1 w_1$ where $w_1 \in U$ and $d_1 p_1 \neq dq_1 w_1$. Then we should have $dq_1 w_1 \in R_1$, whence $q_1 \in P$. Thus we may assume that $d^B_{n-1} = d^A_{n-1}$. We next prove that $n > 2$. If $n = 1$ then $d^A_{n-1} = e$, which is impossible since d^A_{n-1} has q_1 as its last syllable. If $n = 2$ then $t_1 \in T_B$ and $d^B_{n-1} = dq_1 u_1^{-1} = d^B_1 = t_1 \phi_B$. Thus $Hdq_1 B = Ht_1 B = HB$, whence, by 3.2, $dq_1 = e$, which is impossible since $q_1 \neq e$.

For all odd j such that $1 \leq j \leq n-1$ write $d^A_{n-j} = d_j q_j u_j$ where $d_j \in S_A$, $q_j \in T_A$ and $u_j \in U$; for j even $(1 \leq j \leq n-1)$ write $d^B_{n-j} = d_j q_j u_j$ where $d_j \in S_B$, $q_j \in T_B$ and $u_j \in U$. Note that $t_{n-j} \in T_B$ or T_A according as j is odd or even. We shall prove that if j is odd $(1 \leq j \leq n-2)$ and $d^B_{n-j} = d^A_{n-j} = d_j q_j u_j$ where $q_j \neq e$, then either $d_j q_j$ is an initial segment of an element of $R_1 \cup Q_A \cup Q_B$, or $d^A_{n-j-1} = d^B_{n-j-1} = d_j q_j q_{j+1} u_{j+1}$ where $e \neq q_{j+1} \in T_B$; and that if j is even $(1 \leq j \leq n-2)$ the same statement is true except that $e \neq q_{j+1} \in T_A$.

Thus suppose j is odd $(1 \leq j \leq n-2)$, $e \neq q_j \in T_A$, $d_j q_j$ is not an initial segment of any element of $R_1 \cup Q_A \cup Q_B$, and $d^B_{n-j} = d^A_{n-j}$. By Definition 3.2(ii), since $d^B_{n-j} = d^A_{n-j} = d_j q_j u_j$, we must have $d^B_{n-j-1} = d_j q_j p_{j+1} v_{j+1}$, where $p_{j+1} \in T_B$ and $v_{j+1} \in U$. Then

$$\begin{aligned} s_B(d^B_{n-j-1}, t_{n-j}) &= d_j q_j (p_{j+1} v_{j+1} t_{n-j} u_j^{-1})(d_j q_j)^{-1} \\ &= d_j q_j b (d_j q_j)^{-1}, \quad \text{say.} \end{aligned}$$

Since $t_{n-j} \notin U$ it follows that $b \notin U$ if $p_{j+1} = e$. But then $d_jq_j \in Q_B$. We must therefore have $p_{j+1} \neq e$. Next suppose that $d^A_{n-j-1} \neq d^B_{n-j-1}$. Again by Definition 3.2 (ii) this implies that $d_jq_jp_{j+1} \neq (d_jq_jp_{j+1})\phi_A$ whence d_jq_j is an initial segment of an element of R_1. Therefore $d_jq_jp_{j+1} = d_{j+1}q_{j+1}$ (i.e. $p_{j+1} = q_{j+1}$ and $d_jq_j = d_{j+1}$), and $d^A_{n-j-1} = d_{j+1}q_{j+1}u_{j+1}$, where $e \neq q_{j+1} \in T_B$. A similar argument disposes of the case j even.

We now apply this to complete the proof. We have by assumption that $d^B_{n-1} = d_1q_1u_1$ where $e \neq q_1 \in T_A$. By what we have just proved, if d_1q_1 were not an initial segment of any element of $R_1 \cup Q_A \cup Q_B$, we should have $d^B_{n-2} = d^A_{n-2} = d_1q_1q_2u_2$ where $e \neq q_2 \in T_B$; and then since $d_1q_1q_2$ cannot be such an initial segment, it would follow that $d^B_{n-3} = d^A_{n-3} = d_1q_1q_2q_3u_3$, where $e \neq q_3 \in T_A$, and so on. We should finally arrive at the situation where $d^B_1 = d^A_1 = d_1q_1 \cdots q_{n-1}u_{n-1}$. This is impossible since if $t_1\phi_A = t_1\phi_B$, then by 3.2 (ii), this element can have length at most 1, whereas $d_1q_1 \cdots q_{n-1}u_{n-1}$ has length at least 2. This completes the proof of the lemma.

Proof of Lemma 3.1. Suppose that $g_1 \in Hg\backslash U$ has the canonical form $g_1 = t_1 \cdots t_nu$, where $t_n \in T_A$, and that we have a cress for H in G, relative to T_A, T_B. Let $g_2 \in R_A$ be the representative of the double coset HgU. Then $g = hg_2u_1$ where $h \in H$ and $u_1 \in U$. By Lemma 3.5,

$$t_nu = u_2q_1^{-1}aq_2u_1 = u_2q_1^{-1}aq_1(q_1^{-1}q_2u_1), \tag{11}$$

where $u_2 \in U$, $q_1 \in P \cup \{e\}$, either $q_2 = e$ or q_2 is the final syllable of g_2, and $a \in A \cap d^{-1}Hd$ where $g_2 = dq_2$.

The element $q_1^{-1}aq_1(q_1^{-1}q_2u_1)$ lies in the coset $(A \cap (dq_1)^{-1}Hdq_1)(q_1^{-1}q_2u_1)$. Since H is finitely generated, by Lemma 3.3 so is $A \cap (dq_1)^{-1}Hdq_1$. Therefore since the transversal T_A satisfies condition (5) of the definition of finite involvement, there exists a finite subset $V \subseteq U$ such that

$$(A\cap(dq_1)^{-1}Hdq_1)(q_1^{-1}q_2u_1) \subseteq T_AV(g^{-1}Hg\cap U). \tag{12}$$

(We have used here that $(q_1^{-1}q_2u_1)^{-1}(A \cap (dq_1)^{-1}Hdq_1)(q_1^{-1}q_2u_1) \cap U = (g_2u_1)^{-1}H(g_2u_1) \cap U = g^{-1}Hg \cap U$.) It is clear from (12) that V depends only on g_2, q_1, q_2 and u_1. However g_2 and u_1 are determined uniquely by g (and if $q_2 \neq e$, then it also is determined by g). It follows from (11) that for each $g_1 = t_1 \cdots t_nu \in Hg\backslash U$, with $t_n \in T_A\backslash\{e\}$, we have

$$t_nu \in u_2T_AV(g^{-1}Hg\cap U),$$

where $V \subseteq U$ is finite and depends only on q_1, q_2 and g. Now P is finite by Corollary 3.4; hence since $q_1 \in P \cup \{e\}$ and since there are only two possibilities for q_2, there are only finitely many pairs q_1, q_2. For each such pair let $V(q_1, q_2) \subseteq U$ be a finite subset satisfying (12). Let V_1 be the union of the $V(q_1, q_2)$, taken over all pairs q_1, q_2. Then for all $g_1 = t_1 \cdots t_n u$ as above, we have

$$t_n u \in UT_A V_1(g^{-1}Hg \cap U).$$

However, since T_A satisfies condition (4), and since $t_n \in T_A \setminus \{e\}$, it follows that $u \in V_1(g^{-1}Hg \cap U)$, from which the desired inclusion (6) follows.

We require one final result. Let $G = (A*B;\ U)$ and let T_A, T_B be left transversals for U in A, B respectively. A subset $X \subseteq G$ is called *double ended* if it contains at least one element with its ending in T_A and at least one element with its ending in T_B.

3.6. Lemma [8, **Lemma 8**]. *Let $H \leq G = (A*B;\ U)$ and suppose Q_A, Q_B and R_1 are defined as above in terms of a cress for H in G. Then $Q_A \cup Q_B \cup R_1$ is finite if and only if the number of double ended (H, U) cosets in G is finite.*

4. The finitely generated intersection property. Theorem 2.3 (i) is an immediate consequence of Lemmas 3.3, 3.6 and the following result.

4.1. Lemma. *Let $G = (A*B;\ U)$ where U and A satisfy the hypothesis of Theorem 2.3, let T_A be a left transversal for U in A satisfying* (4) *and* (5) *and let T_B be any left transversal for U in B, containing e. If H and K are finitely generated subgroups of G, then each intersection of an (H, U) coset with a (K, U) coset contains only finitely many $(H \cap K, U)$ cosets containing elements ending in elements of $T_A \setminus \{e\}$ (and therefore contains only finitely many double ended $(H \cap K, U)$ cosets).*

Proof. Suppose on the contrary that $g_1, g_2 \in G$ are such that $Hg_1U \cap Kg_2U$ contains infinitely many $(H \cap K, U)$ cosets containing elements with endings in T_A. Let $Y = \{y_1, y_2, \cdots\}$ be a set of representatives of a countably infinite set of distinct $(H \cap K, U)$ cosets such that, for all i, y_i ends in an element of $T_A \setminus \{e\}$. We may also assume that, for all i, y_i has U-syllable e since y_i and $y_i u$ represent the same $(H \cap K, U)$ coset for all $u \in U$. Write

$$y_i = h_i g_1 u_i = k_i g_2 v_i \qquad (i = 1, 2, \cdots),$$

where $h_i \in H$, $k_i \in K$, $u_i, v_i \in U$. Let W_H and W_K be left transversals in U for $g_1^{-1}Hg_1 \cap U$ and $g_2^{-1}Hg_2 \cap U$ respectively. Then we may assume that $u_i^{-1} \in W_H$ and $v_i^{-1} \in W_K$, for all i. For, suppose for instance that $u_i^{-1} = w_i u_i'$ where $u_i' \in g_1^{-1}Hg_1 \cap U$ and $w_i \in W_H$. Then $u_i = (u_i')^{-1} w_i^{-1}$, and

$$h_i g_1 u_i = h_i g_1 (u_i')^{-1} g_1^{-1} \cdot g_1 w_i^{-1} = h_i' g_1 w_i^{-1},$$

where $h_i' \in H$.

Thus $h_i g_1 = y_i u_i^{-1}$ and $k_i g_2 = y_i v_i^{-1}$, where $u_i^{-1} \in W_H$ and $v_i^{-1} \in W_K$. By virtue of Lemma 3.1 all but finitely many of the u_i are equal, and the same is true for the v_i. Hence there exists a pair $j,\ l,\ j \neq l$, of positive integers, such that $u_j = u_l$ and $v_j = v_l$. But then

$$y_j y_l^{-1} = h_j h_l^{-1} = k_j k_l^{-1} \in H \cap K$$

which contradicts our choice of y_j, y_l as representatives of distinct $(H \cap K, U)$ cosets. This completes the proof.

Remark. An entirely similar proof shows that finite involvement of a subgroup U throughout a group A implies that for every pair of finitely generated subgroups H, K of A, $Ha_1U \cap Ka_2U$ contains only finitely many $(H \cap K, U)$ cosets, for all $a_1, a_2 \in A$. I have been unable to decide whether or not, conversely, the latter condition implies that U is finitely involved throughout A.

5. **Subnormal subgroups.** In this section we shall prove Theorem 2.3 (ii). For this we require the following lemma (cf. [8, Theorem 10]).

5.1. **Lemma.** *Let $G = (A*B;\ U)$ where $U \neq A$, $U \neq B$, and let H be a finitely generated subgroup containing a subnormal subgroup N of G, $N \nleq U$. Then H has finite index in G if and only if the intersection of each conjugate of H with U has finite index in U.*

The idea of the proof is the same as that of [8, Theorem 10] and [1, p. 679]. For convenience we formulate and prove as a separate lemma a result used in the proof.

5.2. **Lemma.** *Let $G = (A*B;\ U)$ and let T_A, T_B be left transversals, containing e, for U in A, B. Suppose that U has index > 2 in A and $U \neq B$. Then every subnormal subgroup N of G, $N \nleq U$, contains two elements whose initial and terminal syllables lie in $T_A \setminus \{e\}$, with distinct initial syllables, and a third element which begins and ends in elements of $T_B \setminus \{e\}$.*

Proof. It clearly suffices to show that whenever $N \trianglelefteq K < G$ where K contains such a triple of elements and $N \nleq U$, then N also contains such a triple. This is proved as follows. Suppose that $g_1, g_2 \in K$ have distinct initial syllables and both begin and end with elements of $T_A \setminus \{e\}$, and that $g_3 \in K$ begins and ends in elements of $T_B \setminus \{e\}$. Let $x \in N \setminus U$.

Suppose first that x begins and ends in elements of $T_B \setminus \{e\}$. Then $g_1 x g_1^{-1}$ and $g_2 x g_2^{-1}$ both belong to N, end and begin in elements of $T_A \setminus \{e\}$, and have distinct initial syllables since these are the same as those of g_1 and g_2 respectively. Then the three elements $g_1 x g_1^{-1}$, $g_2 x g_2^{-1}$, and x have the required properties.

Second, suppose that x begins in an element of $T_A \setminus \{e\}$ and ends in an element of $T_B \setminus \{e\}$. Since g_1 and g_2 have distinct initial syllables, at least one of these syllables is different from the initial syllable of x: suppose without loss of generality that g_1 has this property; then $g_1^{-1}xg_1$ belongs to N and both begins and ends in elements of $T_A \setminus \{e\}$. Hence the element $g_3^{-1}g_1^{-1}xg_1g_3$ belongs to N and begins and ends in elements of $T_B \setminus \{e\}$. This is the case first dealt with.

If x begins and ends in elements of $T_A \setminus \{e\}$ then $g_3^{-1}xg_3$ begins and ends in elements of $T_B \setminus \{e\}$ and again we are in the first case. Finally, if x begins with an element of $T_B \setminus \{e\}$ and ends with an element of $T_A \setminus \{e\}$, consider instead x^{-1} which falls into the second case above.

Proof of Lemma 5.1. Assume that U has index > 2 in A: The contrary case is, with minor modification, treated as in [8, Theorem 10].

Let

$$N = N_0 \,\Delta\, N_1 \,\Delta\, \cdots \,\Delta\, N_l = G \qquad (l \geq 0)$$

be a shortest subnormal chain connecting N and G. We first show that for all $g \in G \setminus U$ there exists an element $x \in G \setminus U$ such that (i) x begins and ends in syllables of type different from that of the final syllable of g (i.e. if g ends in an A-syllable then x begins and ends in B-syllables, and if g ends in a B-syllable then x begins and ends in A-syllables) and (ii) $gxg^{-1} \in N$.

This is trivially so if $l = 0$ (i.e. if $N = G$). Suppose $l > 0$ and, as inductive hypothesis, that for each $g \in G \setminus U$ and each subnormal subgroup possessing a shorter subnormal chain (and not contained in U) there exists an element x satisfying (i) and (ii) above. Let x be such an element for g and N_1. Then $gxg^{-1} \in N_1$. By Lemma 5.2 there is an element $g_1 \in N$ which begins and ends in syllables of type different from that of the initial syllable of g. Then $gx^{-1}g^{-1}g_1gxg^{-1} \in N$, and $x_1 = x^{-1}g^{-1}g_1gx$ begins and ends in syllables different in type from the final syllable of g; also $gx_1g^{-1} \in N$. This completes the inductive step.

Suppose $g \in G \setminus U$ and let $x \in G \setminus U$ be an element satisfying (i) and (ii) above. Then the double coset HgU can be written $Hgxg^{-1}gU = HgxU$, since $H \geq N$. Hence every (H, U) coset (including HU, by Lemma 5.2) is double ended. Since H is finitely generated we infer from Lemma 3.6 that there are only finitely many (H, U) cosets in H. However since $(U : g^{-1}Hg \cap U)$ is finite for all $g \in G$ by hypothesis, this implies that H has finite index in G. The remark that the "only if" part of 5.1 is trivial completes the proof.

Theorem 2.3(ii) follows immediately from Lemma 5.1 and the following result.

5.3. **Lemma.** *Let $G = (A*B;\, U)$ where U $(\neq A, B)$ and A satisfy the hypothesis of Theorem* 2.3, *let T_A be a left transversal for U in A satisfying* (4) *and* (5), *and let T_B be any left transversal for U in B, containing e. Let H be a finitely generated subgroup containing a subnormal subgroup N of G, $N \nleq U$. Then the intersection of U with each conjugate of H has finite index in U.*

Proof. Suppose that for some $g \in G$, $g^{-1}Hg \cap U$ has infinite index in U. We may suppose $g = e$ by replacing H by $g^{-1}Hg$ and N by $g^{-1}Ng$. Let W be a left transversal for $H \cap U$ in U.

We shall show that every subnormal subgroup N, $N \nsubseteq U$, has the property that every element of W occurs as the representative of the U-syllable of some element of $N \backslash U$, beginning and ending in A-syllables.

This is trivially true for G itself. Suppose that

$$N = N_0 \triangle N_1 \triangle \cdots \triangle N_l = G \qquad (l > 0)$$

is a shortest subnormal chain beginning with N and, as inductive hypothesis, that the aforementioned property is possessed by subnormal subgroups with shorter subnormal chains. Thus N_1 is assumed to have the property. Suppose $g_1 \in N_1 \backslash U$ begins and ends in A-syllables, and has U-syllable u. By Lemma 5.2 there is an element $g_2 \in N \backslash U$ that ends and begins with B-syllables. Then $g_1^{-1} g_2 g_1 \in N$ has U-syllable u since T_A satisfies condition (4), and both ends and begins with A-syllables. Thus N (and hence H) has the property that every element of W occurs as the representative of the U-syllable of some element of N ending in an A-syllable. This contradicts (6) of Lemma 3.1, and the proof is complete.

6. **Isolated cyclic subgroups of free groups.** Theorem 1.1 is a special case of Theorem 2.3, established above. We have only to show that an isolated infinite cyclic subgroup of a free group satisfies the hypothesis of Theorem 2.3.

6.1. **Theorem.** *Let F be a free group freely generated by a set X and let U be an isolated nontrivial cyclic subgroup of F. Then U is malnormal in F. Further there is a left transversal T for U in F satisfying* (4), *such that for any element g of F there exists a finite subset $V \subset U$ such that*

$$Hg \subseteq TV\,(g^{-1}Hg \cap U); \tag{13}$$

i.e. U is finitely involved throughout F with respect to T.

Proof. It is a simple exercise to show that U is malnormal in F and we omit its proof.

We may assume that a generator u of U is cyclically reduced; for if not then some conjugate $f^{-1}uf$ is, and then we may consider fXf^{-1} in place of X.

Choose T as follows. Let T_1 be a complete set of representatives for cosets UfU $(f \in F)$ such that each representative is an element of smallest (reduced) length in its double coset. Then set $T = U(T_1 \backslash \{e\}) \cup \{e\}$. By the proof of Lemma 2.2, T is a left transversal for U in F.

If $g^{-1}Hg \cap U \neq \{e\}$, then it has finite index in U and (13) follows trivially. Assume therefore that $g^{-1}Hg \cap U = \{e\}$.

Let t be any element of $T_1 \backslash \{e\}$. Then the meets (i.e. largest common initial segments) of u with t and t^{-1} have lengths at most half the length of u, by the choice of t as a shortest element of UtU. It follows that, if l, k are nonzero integers of signs ϵ, δ $(=\pm 1)$ respectively and v is the reduced form of $u^{\epsilon}tu^{\delta}$, then $w = u^{l-\epsilon}vu^{k-\delta}$ is reduced as written, i.e. no cancellation occurs.

Now suppose that (13) does not hold. By the choice of T this means that there is an infinite set $\{w_1, w_2, \cdots\} \subseteq Hg$ such that, in reduced form,

$$w_i = u^{l_i} v_i u^{-k_i} \qquad (i = 1, 2, \cdots)$$

where (by replacing u by u^{-1} if necessary) k_i is a positive integer, $k_i \to \infty$, and l_i is an integer. Since $w_i \in Hg$, we have $gw_i^{-1} \in H$ $(i = 1, 2, \cdots)$. Hence there exists a positive integer n such that, for all $i > n$, gu^{k_i} is an initial segment of an element of H. For $i > n$ write $gu^{k_i} = g_1 u^{n_{i-n}}$ in reduced form, where $n_j \geq 0$ $(j = 1, 2, \cdots)$ and $n_j \to \infty$ as $j \to \infty$.

Let S be a right Schreier transversal for H in F (i.e. a complete set of right coset representatives, containing e, and closed under taking initial segments). Let $a_1, \cdots, a_r$ be Schreier free generators of H constructed from S and X (i.e. the nontrivial elements of the form $sx((sx)\phi)^{-1}$, where $s \in S$, $x \in X$ and $(sx)\phi$ is the representative in S of Hsx). For each $j = 1, 2, \cdots$, let $b_j \in H$ be a shortest element of H in the generators $a_1, \cdots, a_r$, such that $g_1 u^{n_j}$ is an initial segment of b_j. Suppose b_j ends in $a_{m(j)}^{\delta(j)}$ where $\delta(j) = \pm 1$, and write

$$b_j' a_{m(j)}^{\delta(j)} = b_j, \qquad a_{m(j)}^{\delta(j)} = s_j x_j^{\delta(j)} ((s_j x_j^{\delta(j)})\phi)^{-1}.$$

(Note that if $sx((sx)\phi)^{-1} \neq e$, then it is reduced as written [4, Lemma 7.23]. We shall also use the fact that in the product

$$sx^{\epsilon}((sx^{\epsilon})\phi)^{-1} \cdot s'(x')^{\delta}((s'(x')^{\delta})\phi)^{-1} \qquad (s, s' \in S; x, x' \in X)$$

where neither factor is e and the factors are not inverse to each other, neither x^{ϵ} nor $(x')^{\delta}$ $(\epsilon, \delta = \pm 1)$ is cancelled in reducing [4, Lemma 7.2.4].) We now distinguish two cases. Firstly suppose $g_1 u^{n_j}$ is an initial segment of $b_j' s_j$; say $g_1 u^{n_j} = b_j' p_j$, where p_j is an initial segment of s_j. Then $(g_1 u^{n_j})\phi = p_j$.

Secondly suppose $g_1 u^{n_j}$ is not an initial segment of $b_j' s_j$. Thus $g_1 u^{n_j} = b_j' s_j x_j^{\delta(j)} q_j^{-1}$, where q_j is a terminal segment of $(s_j x_j^{\delta(j)})\phi$. Hence

$$(g_1 u^{n_j})\phi = (s_j x_j^{\delta(j)} q_j^{-1})\phi = ((s_j x_j^{\delta(j)})\phi) q_j^{-1},$$

which is an initial segment of $(s_j x_j^{\delta(j)})\phi$.

In either case $(g_1 u^{n_j})\phi$ occurs as an initial segment of some free generator from $\{a_1, \cdots, a_r\}$ or its inverse. However, the elements $g_1 u^{n_j}$ $(j = 1, 2, \cdots)$ belong to different cosets, since, if $j \neq i$, then

$$e \neq g_1 u^{n_i} u^{-n_j} g_1^{-1} \in gUg^{-1},$$

which intersects H trivially. This is impossible since r is finite, i.e. since H is finitely generated.

Remark. It seems likely that (13) can be established for noncyclic malnormal subgroups of F.

Note added in proof. The author has found the following more general version of Theorem 2.3, which includes all previous results in the same direction. Its proof requires only slight changes in the above.

Theorem. *Let* $G = (A * B;\ U)$ *where there exists a left transversal* T *for* U *in* A, *containing* e, *and satisfying*

$$U(T\setminus\{e\}) = (T\setminus\{e\})V_1$$

for some finite subset V_1 *of* U; *and, for every coset* Hg *of every finitely generated subgroup* $H \leq A$,

$$Hg \subseteq TV_2(g^{-1}Hg \cap U),$$

where V_2 *is a finite subset of* U *depending on* Hg. *Then conclusions* (i) *and* (ii) *of Theorem* 2.3 *hold.*

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