NON-HOPFIAN GROUPS WITH FULLY INVARIANT KERNELS. I

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ABSTRACT. Let $\mathcal{L}$ consist of the groups $G(l, m) = \langle a, b; a^{-1}b^la = b^m \rangle$ where $|l| \neq 1 \neq |m|$, $lm \neq 0$ and l, m are coprime. We characterize the endomorphisms of these groups, compute the centralizers of special elements and show that the endomorphism $a \rightarrow a, b \rightarrow b^l$ is onto with a nontrivial fully invariant kernel. Hence $G(l, m)$ is non-Hopfian in the 'fully invariant sense.'

Our purpose is to prove the results announced in [1], concerning the endomorphisms of the non-Hopfian one-relator groups $\mathcal{L}$ found in [2] and isolated by G. Baumslag in [3]. $\mathcal{L}$ consists of the groups $G(l, m)$ presented by $\langle a, b; a^{-1}b^la = b^m \rangle$ where $|l| \neq 1 \neq |m|$, $lm \neq 0$ and l, m are coprime. Let G' denote the normal closure of b in $G(l, m)$ and N the kernel of the endomorphism $\eta: a \rightarrow a, b \rightarrow b^l$. We will prove

Theorem 1. *If $\tau: a \rightarrow A, b \rightarrow B \neq 1$ defines an endomorphism of $G(l, m)$ then*

(1) *B is in G' and can be written in the form $B = D^{-1}b^kD$ (D in $G(l, m)$).*

(2) *$DAD^{-1} = ca$ where c is in the centralizer of b^{lk} in G' .*

Theorem 2. *N is a proper fully invariant subgroup of $G(l, m)$ such that $G(l, m)/N$ is isomorphic to $G(l, m)$.*

A general reference for the proofs of these theorems is [4].

1. Basic lemmas.

Lemma 1. *$\eta: a \rightarrow a, b \rightarrow b^l$ defines an onto endomorphism of $G(l, m)$ with nontrivial kernel N where N is the normal closure of the subgroup generated by*

$$W(a, b) = ([b, a]^t b^s) b^{-1}$$

and

$$V(a, b) = a^{-1}ba([b, a]^t b^s)^{-m}$$

such that $(m - l)t + ls = 1$.

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Proof. That η is onto and $N \neq 1$ follow from [2]. Let H be presented by $(a, b; a^{-1}b^l a = b^m, W(a, b) = V(a, b) = 1)$. Define $\alpha: G \rightarrow H$ where $\alpha: a \rightarrow a, b \rightarrow b$ and $\beta: H \rightarrow G$ where $a \rightarrow a, b \rightarrow b^l$. Now observe that β is an isomorphism (with inverse $\bar{\beta}: H \rightarrow G, \bar{\beta}: a \rightarrow a, b \rightarrow [b, a]^l b^s$) and that $\alpha\beta = \eta$. Hence $\ker \eta = \ker \alpha\beta = N$.

The proofs of Lemmas 2 and 3 are immediate consequences of the Reidemeister-Schreier rewriting process [4].

Lemma 2. Let G' be the normal closure of b in $G(l, m)$. $G' = (\dots x_i \dots; x_i^l = x_{i+1}^m, \text{ where } i \text{ runs through the integers})$ and $x_i = a^i b a^{-i}$ in $G(l, m)$.

Lemma 3. In G' we have for $k > 0$,

$$(1) \quad (x_j)^{lk} = (x_{j+k})^{mk} \text{ for any } j,$$

$$(2) \quad (x_i)^{mk} = (x_{i-k})^{lk} \text{ for any } i.$$

Let G'' denote the commutator subgroup of G' and $Q(l, m)$, the additive subgroup of the rationals, generated by $(l/m)^i$ where i runs through the integers. G/G'' is isomorphic to the obvious split extension of $Q(l, m)$ by an infinite cyclic group. Explicitly

Lemma 4. G'/G'' is isomorphic to $Q(l, m)$ under the map $\gamma: x_i \rightarrow (l/m)^i$.

2. **Conjugacy in $G(l, m)$.** Here we examine certain subgroups of G' and their length functions to establish certain conjugacy properties.

For $j < k$ let

$$A(j, k) = (x_j, \dots, x_k; x_j^l = x_{j+1}^m, \dots, x_{k-1}^l = x_k^m),$$

$$A(j, j) = (x_j;) \text{ the infinite cyclic group,}$$

$$A(-\infty, k) = (\dots x_{k-1}, x_k; x_i^l = x_{i+1}^m, i < k),$$

$$A(j, +\infty) = (x_j, x_{j+1}, \dots; x_i^l = x_{i+1}^m, j \leq i).$$

The left factorization of $A(j, k)$ is given by the free product with amalgamation

$$A(j, k-1) * A(k, k), \quad x_{k-1}^l = x_k^m,$$

and the right factorization of $A(j, k)$ by

$$A(j, j) * A(j+1, k), \quad x_j^l = x_{j+1}^m.$$

For x in $A(j, k)$ let $\lambda_{jk}(x)$ denote the representative length w.r.t. the left factorization and $\rho_{jk}(x)$ the representative length w.r.t. the right factorization

[4]. We will refer to these functions respectively, as the left and right length functions of $A(j, k)$. Elements cyclically reduced w.r.t. these length functions will be referred to as left (resp. right) cyclically reduced. For $p \leq j$, $A(p, k)$ will be called a left extension of $A(j, k)$ and for $q \geq k$, $A(j, q)$ will be called a right extension of $A(j, k)$. For x in G' and $n \geq 0$, $a^{-n}xa^n$ (resp. a^nxa^{-n}) is called a left (resp. right) translate.

In proving results where a dual statement arises by substituting 'right' for 'left' we omit when obvious the proof of the dual. We will refer to Theorem 4.6 in [4] as Theorem 4.6.

The proofs of Corollaries 1 and 2 are immediate consequences of the properties of representative length functions [4].

Corollary 1. *If x is left (resp. right) cyclically reduced in $A(j, k)$ then x is left (resp. right) cyclically reduced in all left (resp. right) extensions of $A(j, k)$ and $\lambda_{j_k}(x) = \lambda_{p_k}(x)$ (resp. $\rho_{j_k}(x) = \rho_{j_q}(x)$).*

Corollary 2. *If x is left (resp. right) cyclically reduced in $A(j, k)$ and $\lambda_{j_k}(x) \geq 2$ (resp. $\rho_{j_k}(x) \geq 2$). Then for $|n| \geq 1$, x^n is left (resp. right) cyclically reduced in $A(j, k)$ and $\lambda_{j_k}(x^n) = |n|\lambda_{j_k}(x)$ (resp. $\rho_{j_k}(x^n) = |n|\rho_{j_k}(x)$).*

Corollary 3. *If x is left (resp. right) cyclically reduced in $A(j, k)$ and $\lambda_{j_k}(x) \geq 2$ (resp. $\rho_{j_k}(x) \geq 2$) then for all left (resp. right) translates y of x^l in a given left (resp. right) extension of $A(j, k)$, y is not conjugate to x^m in that extension.*

Proof. For $y = x^l$, by Corollaries 1 and 2, $\lambda_{p_k}(x^l) \neq \lambda_{p_k}(x^m)$ so, from Theorem 4.6, y and x^m are not conjugate in any left extension. For $y = a^{-n}x^la^n$, $\lambda_{p_k}(y) \leq 1$ and again by Theorem 4.6 the result follows.

An immediate consequence of Theorem 4.6 is

Lemma 5. *Let $G = A *_H B$, $H \subset \text{Center}(B)$. If g is cyclically reduced in G and g is conjugate to b in H then g is conjugate in A to b .*

Corollary 4. *If x is left (resp. right) cyclically reduced and $\lambda_{j_k}(x) \geq 2$ (resp. $\rho_{j_k}(x) \geq 2$) then x is not conjugate in any right (left) extension $A(j, q)$ (resp. $A(p, q)$) to an element in $A(q, q)$ (resp. $A(p, p)$).*

Proof. For $q = k$, we have from Theorem 4.6 and Corollary 2 that x^n , $|n| \geq 1$, is not conjugate in $A(j, k)$ to an element in $A(k, k)$. Suppose the result for $q \leq k + i$ and let $q = k + i + 1$ and note $\lambda_{j_q}(x) \leq 1$. If x is conjugate in $A(j, q)$ to an element y in $A(q, q)$, y not conjugate in $A(j, q)$ to an element of the amalgamated subgroup H generated by x_q^m then, by Theorem 4.6, x, y are in $A(q, q)$ and conjugate there. Thus $x = x_q^\mu$, $\mu \neq 0$. Now by Lemma 3,

$$x_q^{m^{i+1}} = x_{q-(i+1)}^{li+1} = x_k^{li+1}$$

so

$$x^{m^{i+1}} = (x_k^{li+1})^\mu$$

which is contrary to x^n not in $A(k, k)$ for $|n| \geq 1$. Thus if x is conjugate in $A(j, q)$ to an element y in $A(q, q)$ then y is conjugate in $A(j, q)$ to an element b in H . Since x is cyclically reduced in $A(j, q)$ and x is in $A(j, k+i)$, by Lemma 5, x would be conjugate to b in $A(j, k+i)$ contrary to our inductive hypothesis.

Corollary 5. *If x and y are left (resp. right) cyclically reduced in $A(j, k)$, $\lambda_{j,k}(x) \geq 2$ (resp. $\rho_{j,k}(x) \geq 2$) then x is conjugate to y in a right (resp. left) extension of $A(j, k)$ implies x conjugate to y in $A(j, k)$.*

Proof. For $q = k$ the result holds. Suppose the result holds for $q \leq k+i$ and let $q = k+i+1$. By Corollary 4, x is not conjugate in $A(j, q)$ to an element in $A(q, q)$. Hence, by Theorem 4.6, x is conjugate to y in $A(j, k+i)$. Applying the inductive hypothesis yields x conjugate to y in $A(j, k)$.

Corollary 6. *If x is not conjugate in G' to an element in $A(i, i)$ for any i and x is in $A(j, k)$ then $j < k$ and (1) there exists a j' and an element y such that $j \leq j' < k$, y is cyclically reduced in $A(j', k)$, $\rho_{j',k}(y) \geq 2$ and x is conjugate to y in $A(j, k)$; (2) there exists a k' and an element z such that $j < k' \leq k$, z is cyclically reduced in $A(j, k')$, $\lambda_{j,k'}(z) \geq 2$, x is conjugate to z in $A(j, k)$.*

Proof. Clearly $j < k$. For $k-j=1$ the result holds since x is not conjugate in G' to an element of either $A(j, j)$ or $A(j+1, j+1)$. Thus, cyclically reducing x in $A(j, j+1)$ yields y and setting $y = z$ yields our construction. Suppose the result is true for $k-j \leq n$ and let $k-j = n+1$. Now cyclically reduce x w.r.t. the right factorization of $A(j, k)$ to y' . If $\rho_{j,k}(y') \geq 2$ let $y = y'$. Otherwise y' is in the factor $A(j+1, k)$ and we can apply our inductive hypothesis, obtaining from y' an element y with the desired property. We obtain z similarly.

Corollary 7. *If x is not conjugate in G' to an element in $A(i, i)$ for any i then x^l is not conjugate in $G(l, m)$ to x^m .*

Proof. Let x be in $A(j, k)$ and y, z have the relation to x described in Corollary 6. If x^l is conjugate in $G(l, m)$ to x^m then there is an integer q and an element D in G' such that $a^{-q}x^la^q = D^{-1}x^mD$. Now there are u, v in $A(j, k)$ such that $x = u^{-1}yu$, $x = v^{-1}zv$. Hence

$$a^{-q}u^{-1}y^l u a^q = D^{-1}u^{-1}y^m u D, \quad a^{-q}v^{-1}z^l v a^q = D^{-1}v^{-1}z^m v D.$$

Transposing and conjugating by a^q yields

$$a^{-q}y^l a^q = (a^{-q}u a^q D^{-1}u^{-1})y^m (u D a^{-q}u^{-1}a^q),$$

$$a^{-q}z^l a^q = (a^{-q}v a^q D^{-1}v^{-1})z^m (v D a^{-q}v^{-1}a^q).$$

Now for some choice of integers e, f and g, h , $u D a^{-q}u^{-1}a^q$ is in $A(e, f)$, $v D a^{-q}v^{-1}a^q$ is in $A(g, h)$.

Case 1. $q \geq 0$; $a^{-q}z^l a^q$ is in $A(j - q, k')$. Since z is left cyclically reduced in $A(j, k')$ and $\lambda_{j, k'}(z) \geq 2$ by Corollary 3 $a^{-q}z^l a^q$ is not conjugate in any left extension of $A(j - q, k')$ to z^m . In particular choose $p = \min(g, j - q)$ and also let $r = \max(h, k')$. If x^l is conjugate in $G(l, m)$ to x^m then $a^{-q}z^l a^q$ is conjugate in $A(p, r)$ to z^m . Since z^m and $a^{-q}z^l a^q$ are left cyclically reduced in $A(p, k')$ and $\lambda_{p, k'}(z^m) \geq 2$, if z^m is conjugate in $A(p, r)$ to $a^{-q}z^l a^q$ then by Corollary 5, z^m is conjugate in $A(p, k')$ to $a^{-q}z^l a^q$. However, we have shown that $a^{-q}z^l a^q$ is not conjugate to z^m in $A(p, k')$ for $p \leq j - q$.

Case 2. $q < 0$; apply an analogous argument on y .

It is immediate from Corollary 7 that

Corollary 8. *If B is in G' then B^l is conjugate to B^m iff B is conjugate to x^k for some i and k .*

3. The endomorphisms of $G(l, m)$.

Proof of Theorem 1. Note that the defining relators have a -exponent sum zero so all relators have a -exponent sum zero which puts B in G' . In fact G' and hence G'' are fully invariant. Now $A^{-1}B^l A = B^m$, so by Corollary 8, B is conjugate in G' to x_c^k and hence in $G(l, m)$ to b^k . Dividing by G'' preserves the a -exponent on A . Every element in $G(l, m)/G''$ has the form ra^n where r is in $Q(l, m)$ of Lemma 4. Conjugation by a^n is seen to act on $Q(l, m)$ as multiplication by $(m/l)^n$. Letting $A \equiv ra^n \pmod{G''}$ and noting $b^k \equiv k \pmod{G''}$ reveals that when $A^{-1}B^l A = B^m$ is viewed mod G'' the consequence is $a^{-n} r l k r a^n = m k$ and hence $(m/l)^n k = m k$. Since $B \neq 1$, $k \neq 0$, so we have $n = 1$. Thus $D A D^{-1} = c a$ where c is in G' . Now $a^{-1} c^{-1} b^{lk} c a = b^{mk} = a^{-1} b^{lk} a$ so c is in the centralizer of b^{lk} .

4. Fully invariant kernels. The proofs of Lemmas 6 and 7 below follow from Lemma 3 by a straightforward computation.

Lemma 6. *For $n \neq 0$, $k \geq 0$,*

- (1) x_0^n is in $A(k, k)$ iff m^k/n ,
- (2) x_0^n is in $A(-k, -k)$ iff m^k/n .

Lemma 7. For each $n \neq 0$ there exist unique integers $p \leq 0 \leq k$ such that x_0^n is in $A(i, i)$ iff $p \leq i \leq k$.

Lemma 8. Let $G = A *_H B$. Suppose a is in $\text{Center}(A)$ and not in H . If $g^{-1}ag$ is in A , then g is in A .

Proof. If g has representative length zero g is in A . Suppose the result for g of length $< n$ and let $g = bg_1 \cdots g_n$. Note that g_1 is in A for otherwise $g^{-1}ag$ has length $1 + 2n$. Now a in $\text{Center}(A)$ and the inductive hypothesis yield the result.

Lemma 9. Let $G = C *_K (A *_H B) = (C *_K A) *_H B$. If a is in $\text{Center}(A)$ and not in K, H then $g^{-1}ag = a$ implies g in A .

Proof. View G with respect to the factors C and $A *_H B$. If g has length zero g is in A . Suppose $g = kg_1 \cdots g_n$ and assume the result for lengths $< n$. Note g_1 is in $A *_H B$ since $g^{-1}ag$ has length 1. If $n > 1$ then g_1 is in K for otherwise $g^{-1}ag$ has length ≥ 3 and so by Lemma 8 and the inductive hypothesis the result holds. If $n = 1$, g is in $A *_H B$ and Lemma 8 yields the result.

Lemma 10. Let H_i be the subgroup generated by x_i^l . The $\text{Center}(A(p, k))$, $p \leq 0 \leq k$, is $\bigcap_{i=p}^{k-1} H_i$ if $p \neq k$ and $A(p, p)$ for $p = k$.

Proof. Take induction on $k - p$ applying [4, Corollary 4.5].

Lemma 11. The centralizer of $(x_0^l)^j$, $j \neq 0$, in G' is $A(p, k)$ where p, k are given in Lemma 7 for $n = lj$.

Proof. View G' as

$$A(-\infty, p^{-1}) {}_{H_{p-1}}^* (A(p, k) {}_{H_k}^* A(k+1, +\infty))$$

and note x_0^n is not in H_k, H_{p-1} . Hence if x_0^n is in $\text{Center}(A(p, k))$ then $A(p, k)$ is the centralizer of x_0^n in G' , by Lemma 9. Now x_0^n is in $A(i, i)$ for $p \leq i \leq k$ and so in H_i for $p \leq i \leq k-1$. By Lemma 10, x_0^n is in $\text{Center}(A(p, k))$.

Lemma 12. Let η be as in Lemma 1. If $n = lj$, $j \neq 0$, and c in G' commutes with b^n then $c\eta$ commutes with b^n .

Proof. Let $A(p, k)$ be the centralizer of b^n in G' as in Lemma 11 and observe that $A(p, k)\eta \subseteq A(p, k)$.

Proof of Theorem 2. We show for any endomorphism τ , $N\tau \subseteq N$. For $\tau: a \rightarrow a, b \rightarrow 1$, we have $N\tau = 1$. Suppose τ is as in Theorem 1. Note that $N\tau \subseteq N$ iff $DW(a, b)\tau D^{-1}$ and $DV(a, b)\tau D^{-1}$ are in N . Now

$$DW(a, b)\tau D^{-1} = W(ca, b^k), \quad DV(a, b)\tau D^{-1} = V(ca, b^k).$$

It follows directly from Lemma 12 that $W(ca, b^k)\eta = 1 = V(ca, b^k)\eta$.

5. **The generalized Hopfian problem.** Let P be a property. G is said to be non-Hopfian in the P -sense iff there is a proper normal subgroup N possessing property P such that G/N is isomorphic to G . Thus the groups in $\mathfrak{L}$ are non-Hopfian in the fully invariant sense. It is pointed out in [1] that reduced free groups are Hopfian in the fully invariant sense. In fact it follows from [4, Theorem 3.3] that free groups of arbitrary rank are Hopfian in the characteristic sense.

Problem. Are there groups Hopfian in the fully invariant sense but non-Hopfian in the characteristic sense?

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REFERENCES

1. M. Anshel, *The endomorphisms of certain one-relator groups and the generalized Hopfian problem*, Bull. Amer. Math. Soc. 77 (1971), 348–350. MR 42 #7757.
- 1a. Michael Anshel (Orleck), *Non-Hopfian groups with fully invariant kernels*, Ph.D. Thesis, Adelphi University, Garden City, N.Y., 1967.
2. G. Baumslag and D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. 68 (1962), 199–201. MR 26 #204.
3. G. Baumslag, *Residually finite one-relator groups*, Bull. Amer. Math. Soc. 73 (1967), 618–620. MR 35 #2953.
4. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Pure and Appl. Math., vol. 13, Interscience, New York, 1966. MR 34 #7617.

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