THE ACCESS THEOREM FOR SUBHARMONIC FUNCTIONS

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ABSTRACT. A chain from a point z_0 of the open unit disk Δ to the boundary of Δ is a set $\Gamma = \bigcup \{\gamma_n \mid n = 0, 1, 2, \dots\}$ where the γ_n are compact, connected subsets of Δ , z_0 is in γ_0 , γ_n meets γ_{n+1} and the γ_n approach the boundary of Δ . The following "Access Theorem" is proved: *If u is subharmonic in Δ , z_0 is a point of Δ and $M < u(z_0)$, then there is a chain from z_0 to the boundary of Δ on which $u \geq M$ and on which u tends to a limit.* A refinement, in which the chain is a polygonal arc, is established, and an example is constructed to show that the theorem fails if $M = u(z_0)$ even for bounded, continuous subharmonic functions.

1. Introduction. In [5], the subharmonic analogues $\mathcal{Q}_s, \mathcal{B}_s, \mathcal{C}_s$ of MacLane's classes $\mathcal{Q}, \mathcal{B}, \mathcal{L}$ [6], were introduced and studied. It was shown that these classes $\mathcal{Q}_s, \mathcal{B}_s, \mathcal{C}_s$ were identical; a key step in the proof was Theorem 2, § 4 of [5] which, because of its topological content, we have labeled the Access Theorem. This result is of some interest in itself since it sheds light on both the topological and analytical behavior of subharmonic functions.

The original proof of the theorem was lengthy and rather technical. In this paper, we shall give a shorter and more conceptual proof, a polygonal version of the theorem, and, perhaps of equal importance, an example which shows that the natural sharpening of the theorem is false.

To state our results precisely we introduce some notation and terminology. The open unit disk is denoted Δ and its boundary $\partial\Delta$. The word *continuum* means compact connected set.

Definition 1. If z_0 is a point of Δ , a *chain from z_0 to $\partial\Delta$* is a subset Γ of Δ which can be written as the union of a sequence of continua $\{\gamma_n \mid n = 0, 1, 2, \dots\}$ in Δ with the properties that $z_0 \in \gamma_0$, $\gamma_n \cap \gamma_{n+1} \neq \emptyset$ for all n , and, for any compact set E in Δ , only finitely many γ_n meet E . Thus, a chain is a connected set consisting of a sequence of links, γ_n , which tend toward $\partial\Delta$ and the first of which contains z_0 .

Let μ be a function defined in Δ with values in the extended reals and let $\Gamma = \bigcup \gamma_n$ be a chain from $z_0 \in \Delta$ to $\partial\Delta$.

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Definition 2. We say that u tends to a limit, L , on Γ provided $\lim M_n = \lim m_n = L$ where $M_n = \max_{z \in \gamma_n} u(z)$, $m_n = \min_{z \in \gamma_n} u(z)$ and limits are taken in the extended sense.

We are now in a position to state the Access Theorem.

Theorem A. Let u be a subharmonic function defined in Δ and let $M < u(z_0)$ where z_0 is a point of Δ . There is a chain Γ from z_0 to $\partial\Delta$ on which $u \geq M$ and on which u tends to a limit.

The proof of this theorem and a refinement, Theorem B, in which the links of the chain are polygonal arcs, will be given in § 2; we remark that the result is easy to prove for harmonic functions using a modification of an argument of G. R. MacLane [6, p. 7]. In this case one can replace the condition $M < u(z_0)$ by $M \leq u(z_0)$; it is natural to ask whether this is true in general.

In § 3 we show that the answer is no by constructing a bounded, continuous, subharmonic function w in Δ such that the set where $w \geq w(0)$ does not even contain a chain from zero to $\partial\Delta$.

In the remainder of this section we list some properties of subharmonic functions. For details, the reader is referred to [3].

To begin, a function u is subharmonic in an open subset W of the complex plane provided the following three conditions are satisfied:

- (i) $u: W \rightarrow [-\infty, \infty)$,
- (ii) u is upper semicontinuous (u.s.c.) in W , and
- (iii) for all small enough r , u satisfies the mean value inequality

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta \quad \text{for all } z \in W,$$

where the integral is in the Lebesgue sense.

A function u satisfying (i) and (ii) is subharmonic in W if and only if the following condition holds:

- (iii)' If v is a harmonic function defined in an open subset W' of W , then $u - v$ is either constant or fails to have a maximum in W' .

Condition (iii)' yields the following *maximum principle*: if u is subharmonic in an open set W and u is not constant, then for each z in W , $u(z) < \max_{z \in W} u(z)$.

In the proof of the Access Theorem we shall need the following fact which is a consequence of Brelot's form of the Milloux-Schmidt inequality (see [1]) and the mean value inequality, (iii), given above.

Lemma. If u is subharmonic and bounded above in W and if $z \in W$, then, for any $\epsilon > 0$, there are circles of arbitrarily small radius centered at z on which $u \geq u(z_0) - \epsilon$.

2. **The theorem.** The proof of the theorem involves an inductive construction, each step of which uses two lemmas which we prove first. We need a preliminary definition.

Definition 3. Let u be a function defined in Δ , let z_0 be a point of Δ and let $M \leq u(z_0)$. We say that a point z of Δ belongs to the *compartment of z relative to M* provided there is a continuum in Δ containing both z_0 and z on which $u \geq M$. The compartment of z_0 relative to M is denoted $C(z_0, M)$.

Remarks. This definition was given in [5]. $C(z_0, M)$ need not be closed in Δ and, in particular, $C(z_0, M)$ does not necessarily coincide with the component of z_0 in $\{u \geq M\}$, as the example in § 3 shows.

Lemma 1. *If u is subharmonic in Δ and z_0 is a point of Δ then $C(z_0, u(z_0))$ has limit points in $\partial\Delta$.*

Proof. Fix R , with $|z_0| < R < 1$ and let $K = \{|z| \leq R\} \cap \{u \geq u(z_0)\}$. The set K is compact and we assert that $Q = K \cup \{|z| = R\}$ is connected. If not, then we may write $Q = A \cup B$ where A, B are compact, disjoint, nonempty and $\{|z| = R\}$ lies in B .

Clearly there is an open subset W of $\{|z| \leq R\}$ such that $A \subset W$ and the closure of W misses B . Then, in fact, W is open in Δ and its boundary lies in the set $\{u < M\}$. But then $\max_{z \in W} u(z)$ must be achieved in $A \subset W$ which contradicts the maximum principle for subharmonic functions.

Therefore, Q is connected. This implies that there is a continuum in Δ containing z_0 and meeting $|z| = R$ (namely, the component of Q which contains z_0). Since R may be chosen arbitrarily close to one this proves the lemma.

Lemma 2. *Let u be a nonconstant subharmonic function in Δ , z_0 a point of Δ and let $M < u(z_0)$. Then*

- (a) $\max u|_{C(z_0, M)} > u(z_0)$, and
- (b) if $|z_0| < R < 1$ and if $C(R) = C(z_0, M) \cap \{|z| \leq R\}$, then $\max u|_{C(z_0, M)} > \max u|_{C(R)}$.

Remark. Although certain steps in the proof have been stated as lemmas in [7] and [5] we shall, for completeness, fill in all details.

Proof of (a). Given R , with $|z_0| < R < 1$, define an auxiliary function v on the set $|z| \leq R$ by

$$v(z) = \begin{cases} u(z) & \text{if } z \in C(z_0, M), \\ M & \text{otherwise.} \end{cases}$$

We show that v is subharmonic in $|z| < R$.

Certainly v satisfies property (i), so we need to verify that properties (ii) and

(iii) hold at each point of $\{|z| < R\}$. We need only consider points which lie in the boundary of $C(z_0, M)$ relative to $\{|z| < R\}$, since at other points v is either locally constant or locally equal to u .

Fix such a point z_1 and suppose first that $u(z_1) = M$, whence $v(z_1) = M$. Then certainly v satisfies the mean value inequality, (iii), at z_1 since $v \geq M$ everywhere in $\{|z| < R\}$. Given $\epsilon > 0$, there is a neighborhood U of z_1 on which $u \leq (M + \epsilon)$. On this same neighborhood we also have $v \leq (M + \epsilon)$. Since ϵ is arbitrarily small, this shows that v is upper semicontinuous at z_1 .

Secondly, suppose $u(z_1) > M$. Choose $\epsilon > 0$ so that $(u(z_1) - \epsilon) > M$. By Brelot's form of the Milloux-Schmidt inequality there is a circle σ centered at z_1 , and lying in $\{|z| < R\}$ on which $u \geq (u(z_1) - \epsilon)$. Since z_1 is a limit point of $C(z_0, M)$, $C(z_0, M)$ must meet σ and this implies that $\sigma \subset C(z_0, M)$. From Lemma 1 it then follows that if z lies inside σ and $u(z) \geq M$ then $z \in C(z_0, M)$. In particular, $z_1 \in C(z_0, M)$ so that $v(z_1) = u(z_1)$. In addition, the definition of v implies that, inside σ , we have $v \geq u$. It is then immediate that v satisfies properties (ii) and (iii) at z_1 .

This completes the verification that v is subharmonic. Now suppose that $\max_{z \in C(z_0, M)} u(z) = u(z_0)$. Then $v(z) \leq u(z_0)$ for all z in $|z| \leq R$ and, since $v(z_0) = u(z_0)$ it follows from the maximum principle that v is constant, $v \equiv u(z_0)$, on $\{|z| < R\}$. This, in turn, implies that u is constant on $\{|z| < R\}$. Since R may be chosen arbitrarily close to one this contradicts our hypothesis that u is non-constant on Δ . Thus, we must have $\max_{z \in C(z_0, M)} u(z) > u(z_0)$.

Proof of (b). Suppose instead that $\max u|_{C(z_0, M)} = \max u|_{C(R)}$ for some R with $|z_0| < R < 1$. Since u is upper semicontinuous, there is a point z_1 in the closure of $C(R)$ with $u(z_1) \geq \max u|_{C(z_0, M)} > u(z_0)$. Using the modified Milloux-Schmidt inequality as we did above, we find that $z_1 \in C(z_0, M)$. Applying (a) to $C(z_1, u(z_1))$ we see that $\max u|_{C(z_1, u(z_1))} > u(z_1) \geq \max u|_{C(z_0, M)}$. But this is a contradiction since $C(z_1, u(z_1)) \subset C(z_0, M)$. This completes the proof of (b).

Proof of the Access Theorem (Theorem A of the Introduction). The theorem is trivial for constant functions, therefore, let u be a nonconstant subharmonic function in Δ , let z_0 be a point of Δ and let $M < u(z_0)$. We must construct a chain Γ from z_0 to $\partial\Delta$ lying in $C(z_0, M)$ on which u tends to a limit.

We shall first deal with the case in which u is bounded above on $C(z_0, M)$. The construction can be easily modified to cover the unbounded case and will be described afterwards.

To construct Γ we produce a sequence of compartments $A_n = C(z_n, p_n)$ ($n = 0, 1, 2, \dots$) and a sequence of continua γ_n , with the following properties:

1. $A_0 = C(z_0, M)$,
2. the A_n form a nested decreasing sequence, i.e. $A_0 \supset A_1 \supset A_2 \supset \dots$,

3. the A_n approach $\partial\Delta$, i.e. there is a sequence R_n increasing to one such that A_{n+1} lies in $\{|z| \geq R_n\}$,
4. γ_n lies in A_n , γ_0 contains z_0 and $\gamma_n \cap \gamma_{n+1} \neq \emptyset$ for all n ; and, finally,
5. for each $n \geq 1$ and each z in γ_n we have

$$m_{n-1} - 1/2^{n-1} \leq u(z) \leq m_n \quad \text{where } m_k = \max u|_{A_k}.$$

(For $n = 0$, we will have $M \leq u(z) \leq m_0$.)

Once these sets have been obtained we let $\Gamma = \bigcup \gamma_n$. By properties 1–4, Γ is a chain from z_0 to $\partial\Delta$. By property 2 the m_n form a nonincreasing sequence (bounded below by M) whose limit we denote by L . By the inequalities of condition 5, u tends to L on Γ .

We now show how to carry out the construction.

Let R_0 be chosen so that $1/2 < R_0$ and $|z_0| < R_0 < 1$, let $A_0 = C(z_0, M)$, $B_0 = A_0 \cap \{|z| \leq R_0\}$ and let $m_0 = \max u|_{A_0}$ and $n_0 = \max u|_{B_0}$. By Lemma 2 (a), $m_0 > n_0$ and therefore we can select a point z_1 in A_0 such that $u(z_1) = (m_0 - \epsilon_0)$ where ϵ_0 satisfies $\epsilon_0 < 1/2$ and $(m_0 - 2\epsilon_0) > n_0$. Let γ_0 be a continuum in A_0 containing z_0 and z_1 , and, finally, let $A_1 = C(z_1, p_1)$ where $p_1 = (m_0 - 2\epsilon)$. Figure 1 shows A_0 , γ_0 and A_1 .

Figure 1

We now verify properties 1–5. Note that A_0 contains A_1 because $z_1 \in A_0$ and $p_1 > n_0 \geq M$. Also since $n_0 = \max u|_{B_0}$ and since $p_1 > n_0$, the set A_1 must lie outside the circle $|z| = R_0$, by (b) of Lemma 2. Thus properties 2 and 3 have been checked, while properties 1, 4 and 5 are immediate for γ_0 and A_0 .

For the next step, we choose a number R_1 such that $3/4 < R_1$ and γ_0 lies in

$|z| < R_1$. Let $B_1 = A_1 \cap \{|z| \leq R_1\}$ and let $m_1 = \max u|_{A_1}$ and $n_1 = \max u|_{B_1}$. By Lemma 2 (a), $m_1 > n_1$ and therefore we can select a point z_2 in A_1 such that $u(z_2) = (m_1 - \epsilon_1)$ where ϵ_1 satisfies $\epsilon_1 < 1/4$ and $(m_1 - 2\epsilon_1) > n_1$. Let γ_1 be a continuum in A_1 containing z_1 and z_2 and, finally, let $A_2 = C(z_2, p_2)$ where $p_2 = (m_1 - 2\epsilon_1)$. Figure 2 illustrates the situation at this point.

Figure 2

To verify properties 1-5, note that as before $A_2 \subset A_1$ since $z_2 \in A_2$ and since $p_2 > n_1 = \max u|_{B_1} \geq u(z) > p_1$. Also since $p_2 > n_1$, A_2 lies outside the circle $|z| = R_1$ by Lemma 2 (b). Thus properties 2 and 3 hold for A_1 . Since z_1 belongs to both γ_0 and γ_1 and since γ_1 was chosen to lie in A_1 , property 4 holds. Finally, if $z_1 \in \gamma_1$ then since $\gamma_1 \subset A_1$ we have $u(z) \geq p_1 = (m_0 - 2\epsilon_0) > (m_0 - 1)$ and, trivially, $u(z) \leq m_1$. Thus 5 holds for $n = 1$.

The remaining steps are carried out just as above with increased subscripts. The only delicate part of the construction is to make sure that at the k th step z_k is chosen so that $u(z_{k+1}) = (m_k - \epsilon_k)$ where ϵ_k satisfies $\epsilon_k < 1/2^{k+1}$ and $(m_k - 2\epsilon_k) > n_k$. This insures that $A_{k+1} \subset A_k$ and A_{k+1} lies outside $|z| = R_k$ and it guarantees that for any z in γ_{k+1} one will have

$$u(z) \geq p_{k+1} = (m_k - 2\epsilon_k) > (m_k - 1/2^k).$$

This finishes the proof in the case where u is bounded on $C(z_0, M) = A_0$. If, instead, $\max u|_{A_0} = +\infty$, then we choose $R_0, A_0, B_0, m_0 = +\infty$ and $n_0 = \max u|_{B_0}$ as before. Now we must have $n_0 < +\infty$, since u is upper semicontinuous and does not assume the value $+\infty$ on Δ . Thus we can pick z_1 in A_0 such that $u(z_1) \geq (n_0 + 2)$. Let γ_0 be a continuum in A_0 containing z_0 and z_1 and let $A_1 = C(z_1, p_1)$ where $p_1 = (n_0 + 1)$.

Properties 1-4 hold, as they did in the bounded case.

We next choose R_1 , B_1 , $m_1 = \max u|_{A_1}$, $n_1 = \max_{z \in A_1} u(z) = \max_{z \in B_1} u(z)$ as before. If $m_1 < +\infty$ then we proceed from here as in the bounded case.

If $m_1 = +\infty$, then $n_1 < +\infty$ and we choose z_2 in A_1 , such that $u(z_2) \geq n_1 + 2$. Let γ_1 be a continuum in A_1 containing z_1 and z_2 and let $A_2 = C(z_2, p_2)$ where $p_1 = (n_1 + 1)$.

Continue the construction in this way. If at some step $\max_{z \in A_k} u(z) < +\infty$, then proceed as in the bounded case. If at every step $\max_{z \in A_k} u(z) = +\infty$, then choose z_{k+1} in A_k such that $u(z_{k+1}) \geq (n_k + 2)$, let γ_k be a continuum in A_k from z_k to z_{k+1} and let $A_{k+1} = C(z_{k+1}, p_{k+1})$ where $p_{k+1} = (n_k + 1)$. In this case the behavior u on $\Gamma = \bigcup \gamma_n$ can be determined as follows. For z in γ_{k+2} we have $u(z) \geq p_{k+z} = (n_{k+1} + 1)$ and $n_{k+1} = \max_{z \in B_{k+1}} u(z) \geq u(z_1) = (n_k + 2)$. Thus, for z in γ_{k+2} , we have $u(z) \geq (n_k + 3) \geq (n_0 + 2k + 3)$, and therefore u tends to $+\infty$ on Γ .

This completes the proof of Theorem A.

Remark. This proof does not go through in higher dimensions, although at least one of us thinks the theorem still holds. The problem is to find an analogue of Lemma 2, which guarantees that one can force nonconstant functions to increase as z moves towards the boundary.

The chain obtained in the proof of this theorem need not be "nice;" indeed, the links $\{\gamma_n\}$ may be complicated continua and they may intersect each other in a messy fashion. Such pathology can be completely eliminated. Let us call a subset A of Δ a *polygonal arc from z_0 to $\partial\Delta$* if there is a homeomorphism ϕ of $[0, 1)$ onto A such that $\phi(0) = z_0$, $|\phi(x)|$ approaches 1 as x approaches 1 and A is polygonal, i.e. A is the union of a collection (necessarily infinite) of straight line segments.

Clearly, a polygonal arc can be regarded as a chain (in many ways) with polygonal arcs as links; moreover these arcs meet "nicely." If $A = \phi([0, 1))$ is a polygonal arc from z_0 to $\partial\Delta$ then we say a function u defined in Δ *tends to a limit on A* if $u(\phi(x))$ tends to a limit (in the extended sense) as x approaches 1. This, of course, coincides with the corresponding definition for arbitrary chains.

Theorem B. *Let u , z_0 and M be as in Theorem A. There is a polygonal arc A from z_0 to $\partial\Delta$ on which $u \geq M$ and on which u tends to a limit.*

Aside from a simple topology argument, the proof involves a lemma of Talpur which we state as follows:

Lemma [7, § 1]. *Suppose u is subharmonic in a neighborhood U of a continuum γ and that $u \geq M$ on γ . Given two points w_1 and w_2 of γ and an $L < M$, there is a polygonal arc with endpoints w_1 and w_2 , lying in U , on which $u \geq L$.*

We shall omit the proof of the lemma, except to point out that the polygonal arc can be obtained by piecing together short polygonal arcs whose existence is guaranteed by a deep result of Hayman [2, Theorem 4, p. 193].

Now to prove Theorem B, suppose u , z_0 and $M < u(z_0)$ are given. Let $M' = \frac{1}{2}(M + u(z_0))$ and, using Theorem A, select a chain $\Gamma = \bigcup \gamma_n$ from z_0 to $\partial\Delta$ on which $u \geq M'$ and on which u tends to a limit.

Let p_n and q_n denote the minimum and maximum of u on γ_n , $n \geq 0$, and note that $p_n \geq M'$ for all n and that $q_n < +\infty$ for all n . For $n \geq 1$, let z_n be a point of $\gamma_n \cap \gamma_{n+1}$. For $n \geq 0$, choose an open neighborhood U_n of γ_n lying in Δ such that:

- (1) U_n lies in the $1/2^n$ neighborhood of γ_n , and
- (2) the maximum of u on U_n is at most $(q_n + 1/2^n)$.

(The existence of such neighborhoods is guaranteed by the upper semicontinuity of u .)

Now we apply Talpur's lemma. For each $n \geq 0$, choose a polygonal arc β_n lying in U_n with endpoints z_n and z_{n+1} on which the following inequalities hold:

(3) If z is in β_0 , $u(z) \geq M$, and if z is in β_n , $n \geq 1$, $u(z) \geq p_n - \epsilon_n$, where the ϵ_n are chosen so that $p_n - \epsilon_n \geq M$ and $\epsilon_n \rightarrow 0$.

The set $B = \bigcup \beta_n$ forms a chain from z_0 to $\partial\Delta$ (because of condition (1)) and on this chain $u \geq M$ and u tends to the same limit as it did on the original chain (by (2) and (3)).

Now consider the set $(B \cup \partial\Delta)$, a closed connected subset of the closed disk $(\Delta \cup \partial\Delta)$. Collapse $\partial\Delta$ to a point; the quotient space is homeomorphic to S^2 . Denoting the quotient map by π , $\pi: (\Delta \cup \partial\Delta) \rightarrow S^2$, we have that $\pi(\partial\Delta)$ is a point p of S^2 and that $\pi|_\Delta$ is a homeomorphism of Δ onto $S^2 - p$.

The image of $B \cup \partial\Delta$ under π is a compact connected set C containing $\pi(z_0)$ and p . Moreover, because of the way B was constructed, it is easy to see that C is locally connected. Thus C is a Peano continuum [4] and, as is well known, C contains an arc with endpoints $\pi(z_0)$ and p . To be specific let ψ be a homeomorphism of $[0, 1]$ into C such that $\psi(0) = \pi(z_0)$ and $\psi(1) = p$.

Now let $\phi: [0, 1] \rightarrow \Delta$ be defined by $\phi(x) = \pi^{-1}(\psi(x))$. Then ϕ is a homeomorphism of $[0, 1)$, $\phi(0) = z_0$ and $|\phi(x)| \rightarrow \partial\Delta$ as $x \rightarrow 1$ (because $\pi(\phi(x)) \rightarrow p$ as $x \rightarrow 1$). Moreover $\phi([0, 1))$ lies in B so that it must be polygonal. Finally, it is clear that on this polygonal arc, u tends to the same limit as it did on B .

Remark. This proof also fails in higher dimensions. The problem here is that the result of Hayman no longer holds and the key result of Talpur is not established.

3. The example. In this section we shall construct a bounded, continuous function w , subharmonic in Δ , taking on the value 0 at the origin z_0 , but such that $C(z_0, 0)$ contains no chain from z_0 to $\partial\Delta$.

Let S be a copy of the real line which is smoothly embedded in Δ so that z_0 is a point of S and the closure of S consists of S and two disjoint vertical segments spanning Δ . For definiteness, let us take

$$S = \left\{ (x, y) \in \Delta \mid -\frac{3}{4} \leq x \leq \frac{3}{4} \text{ and } y = \frac{\sqrt{7}}{4} \sin g(x) \text{ where } g(x) = \frac{16}{9-16x^2} - \frac{16}{9} \right\}.$$

Then the closure of S in Δ consists of S and the segments $\{(\pm 3/4, y) \mid -\sqrt{7}/4 < y < \sqrt{7}/4\}$ (see Figure 3).

Figure 3

Since S is smoothly embedded in Δ we can choose a sequence $\{U_n\}$ of wavy open strips with disjoint closures which lie above S , between the lines $x = \pm(3/4)$ and which converge monotonically to S . Figure 4, in which the oscillations of S have been flattened, illustrates the desired configuration.

Figure 4

The boundary of each wavy strip consists of four arcs; the two arcs which lie in Δ except for the endpoints will be called the top and bottom edges and the two lying in $\partial\Delta$ will be called the left and right edges. Each strip U_n is to be chosen so that there is a conformal map ϕ_n defined in a neighborhood of the closure $\bar{U}_n$ of U_n which takes $\bar{U}_n$ homeomorphically onto a closed rectangle $R_n = \{(x, y) \mid -K_n \leq x \leq K_n, 0 \leq y \leq 3\pi\}$ with the edges of $\bar{U}_n$ going onto the corresponding edges of R_n . (Observe that since the U_n 's must have "width" tending to 0 and "length" tending to $+\infty$, the numbers K_n will tend rapidly to $+\infty$.)

Let v_n be defined on R_n by $v_n(z) = \text{Im}(M_n e^z)$ where $M_n = e^{-K_n/n}$. Thus v_n , being the imaginary part of a holomorphic function, is harmonic (and *a fortiori* subharmonic) in the interior of R_n . Moreover $|v_n| < 1/n$ on R_n , v_n has the value zero on the top and bottom edges and has negative values in the subrectangle of R_n given by $\pi < y < 2\pi$. Qualitatively, the graph of v_n is a wave form of amplitude at most $1/n$ having two crests and one trough.

We now pull this wave back to $\bar{U}_n$; for z in $\bar{U}_n$ let $u_n(z) = v_n(\phi_n(z))$. As is well known [3], the composition of a subharmonic function and a holomorphic function is subharmonic, thus each u_n is subharmonic on U_n , continuous on $\bar{U}_n$, vanishes on the top and bottom edges of $\bar{U}_n$ and $|u_n| \leq 1/n$ on $\bar{U}_n$.

In a similar manner we may construct a sequence of strips $\{U'_n\}$ converging to S from below and on each of these a subharmonic wave form u'_n satisfying properties analogous to the u_n .

We now define a sequence of functions w_n on Δ as follows:

$$w_n = \begin{cases} u_k & \text{on } U_k, k = 1, \dots, n, \\ u'_k & \text{on } U'_k, k = 1, \dots, n, \\ 0 & \text{elsewhere in } \Delta. \end{cases}$$

Each function w_n is subharmonic on Δ . To see this observe that the u_k and u'_k vanish on the top and bottom edges of their domains and are positive near these edges. Hence properties (ii) and (iii) surely hold along the edges of the appropriate U_k and U'_k . Elsewhere w_n is either locally constant ($w_n \equiv 0$) or else locally agrees with one of the subharmonic functions u_k or u'_k ($k \leq n$).

The sequence w_n converges uniformly to a function w defined on Δ . Because each w_n is continuous and the convergence is uniform, w itself is continuous on Δ . Also w satisfies the mean value inequality, property (iii), since the convergence is uniform and all the w_n and w are Riemann integrable on any circle contained in Δ . To summarize, the function w is bounded, continuous and subharmonic on Δ .

Clearly w vanishes on S and on the two "flaps" of Δ given by $|x| \geq 3/4$, while in each U_n and U'_n there is a trough where w takes on negative values.

Thus, $C(z_0, 0) = S$; the two vertical boundary segments of the closure of S are not in $C(z_0, 0)$, because, by construction, a continuum in Δ which contains z_0 and meets either segment would have to cut across infinitely many troughs' and therefore could not lie in $C(z_0, 0)$.

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