EXTENSIONS OF HOLOMORPHIC MAPS

BY

PETER KIERNAN

ABSTRACT. Several generalizations of the big Picard theorem are obtained. We consider holomorphic maps f from $X-A$ into $M \subset Y$. Under various assumptions on X , A , and M we show that f can be extended to a holomorphic or meromorphic map of X into Y .

1. Introduction. The big Picard theorem says that any holomorphic map f from the punctured disk D^* into the Riemann sphere $P_1(\mathbb{C})$ which omits three points can be extended to a holomorphic map $f: D \rightarrow P_1(\mathbb{C})$. Kobayashi [3] has obtained the following generalization of this. Let A be a closed submanifold of the complex manifold X and let M be a hyperbolically imbedded⁽¹⁾ subspace (definition below) of the complex space Y . Then any holomorphic map $f: X-A \rightarrow M$ can be extended to a holomorphic map $f: X \rightarrow Y$. The purpose of this paper is to consider further generalizations of this result. In particular, it will be shown that if A has singularities, then in general f can only be extended to a meromorphic map. However, if the singularities of A are normal crossings, then f can be extended to a holomorphic map into Y .

2. Definitions. Our main tool for investigating the extension of holomorphic maps will be the Kobayashi pseudo-distance d_M which is associated with the complex space M . For completeness, we shall give the definition and a summary of some of the main properties of d_M which we use. The reader can consult [3] for further details.

Let p and q be points in the complex space M . By a *chain* α from p to q , we mean a sequence of points $p = p_0, p_1, \dots, p_k = q$ in M , points $a_1, \dots, a_k$ in the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and holomorphic maps $f_1, \dots, f_k$ of D into M with $f_i(0) = p_{i-1}$ and $f_i(a_i) = p_i$. The length $|\alpha|$ of α is defined by

$$|\alpha| = \sum_{i=1}^k d(0, a_i) = \sum_{i=1}^k \log \frac{1 + |a_i|}{1 - |a_i|}$$

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⁽¹⁾ Several characterizations of hyperbolically imbedded spaces and another application of Theorem 2 are given in a recent paper by this author. The paper is entitled *Hyperbolically imbedded spaces and the big Picard theorem* and has been submitted to *Mathematische Annalen*.

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where d is the Poincaré-Bergman distance on D . It is given by the metric

$$ds^2 = \frac{1}{(1 - |z|^2)^2} dz d\bar{z}.$$

We set $d_M(p, q) = \inf_{\alpha \in A} |\alpha|$, where A is the set of all chains from p to q . It is easy to see that d_M is a pseudo-distance on M . If d_M is a (complete) metric, we say that M is (complete) hyperbolic.

Some of the properties of this pseudo-distance which shall be useful to us are

(1) d_D is the Poincaré-Bergman distance on the unit disk D .

(2) Let $D^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Then d_{D^*} is a complete metric corresponding to the hermitian metric

$$ds^2 = \frac{1}{|z|^2 (\log |z|^2)^2} dz d\bar{z}.$$

Thus D^* is complete hyperbolic. However, if $\gamma_r = \{z = re^{it} \mid 0 \leq t \leq 2\pi\}$, then the diameter of γ_r converges to 0 as r converges to 0.

(3) If $f: M \rightarrow N$ is holomorphic, then f is distance decreasing. That is, if $p, q \in M$ then $d_N(f(p), f(q)) \leq d_M(p, q)$.

(4) M is (complete) hyperbolic iff the universal covering space of M is (complete) hyperbolic.

Let Y be a complex space and let M be a relatively compact, hyperbolic complex subspace of Y . We say that M is *hyperbolically imbedded* in Y if either of the following (equivalent) conditions are satisfied:

(i) If p and q are boundary points of M and if $\{p_n\}$ and $\{q_n\}$ are sequences in M such that $p_n \rightarrow p$, $q_n \rightarrow q$, and $d_M(p_n, q_n) \rightarrow 0$, then $p = q$.

(ii) If p is a boundary point of M and U is a neighborhood of p in Y , there exists a neighborhood V of p in Y such that $\bar{V} \subset U$ and the distance between $M \cap (Y - U)$ and $M \cap V$ with respect to d_M is positive.

Intuitively, a relatively compact subspace M is hyperbolically imbedded in Y if M is hyperbolic and if the "distance" between any two distinct boundary points is greater than zero.

The following are some examples of hyperbolically imbedded spaces.

(5) Let M be a compact hyperbolic manifold. Then M is hyperbolically imbedded in $Y = M$.

(6) Let $Y = P_1(\mathbb{C})$ and $M = P_1(\mathbb{C}) - \{\text{three points}\}$. M is covered by the unit disk and therefore it is complete hyperbolic. Since M has only isolated boundary points, M is hyperbolically imbedded in Y .

(7) If M_1 and M_2 are hyperbolically imbedded in Y_1 and Y_2 respectively, then $M_1 \times M_2$ is hyperbolically imbedded in $Y_1 \times Y_2$. To see this let $\pi_j: M_1 \times M_2 \rightarrow M_j$ be the projection onto M_j and let $p = (p', p'')$, $q = (q', q'')$, $\{p_n = (p'_n, p''_n)\}$, and $\{q_n = (q'_n, q''_n)\}$ be as in (i) above. If $d_{M_1 \times M_2}(p_n, q_n) \rightarrow 0$, then $d_{M_1}(p'_n, q'_n) \rightarrow 0$ and $d_{M_2}(p''_n, q''_n) \rightarrow 0$ since π_1 and π_2 are distance de-

creasing. Thus $p' = q'$ and $p'' = q''$ and therefore $M_1 \times M_2$ is hyperbolically imbedded in $Y_1 \times Y_2$.

(8) Let $P_2(\mathbb{C})$ be the 2-dimensional complex projective space and let Q be a complete quadrilateral. That is, Q is the union of the six projective lines which pass through a set of four points in general position. Then $M = P_2(\mathbb{C}) - Q$ is hyperbolically imbedded in $P_2(\mathbb{C})$; see [3, p. 92].

Finally, we recall the definition of meromorphic map given by Remmert in [6]. A meromorphic mapping f from a complex space X into a complex space Y is a correspondence satisfying

- (i) For each point $x \in X$, $f(x)$ is a nonempty compact subset of Y .
- (ii) The graph $\Gamma_f = \{(x, y) \in X \times Y | y \in f(x)\}$ of f is a connected complex subspace of $X \times Y$ with $\dim \Gamma_f = \dim X$.
- (iii) There exists a dense subset X' of X such that $f(x)$ is a single point for each x in X' . Remmert has shown that the codimension of the set $X - X'$ is ≥ 2 . Thus if X is a normal space and $\dim X = 1$, then f is actually holomorphic.

3. Extension theorems. The following theorem is the basis for our extension theorems. The proof is essentially the same as Mrs. Kwack's proof of Theorem 3 in [5], although our statement is slightly more general. The winding number argument used below is due to Grauert and Reckziegel.

Theorem 1. *Let M be hyperbolically imbedded in Y and let $f_k: D^* \rightarrow M$ be a sequence of holomorphic maps where $D^* = \{z \in \mathbb{C} | 0 < |z| < 1\}$. Let $\{z_k\}$ and $\{z'_k\}$ be sequences in D^* converging to 0 and such that $f_k(z'_k) \rightarrow q \in Y$. Then*

- (i) $f_k(z_k) \rightarrow q$.
- (ii) Any holomorphic map $f: D^* \rightarrow M$ extends to a holomorphic map $f: D \rightarrow Y$.
- (iii) $f_k(0) \rightarrow q$.

Proof. (i) Assume that $f_k(z_k) \rightarrow p \neq q$ and that $|z_k| \leq |z'_k|$. Choose a coordinate neighborhood U of p such that U is an analytic subset of $W = \{(w_1, \dots, w_n) \in \mathbb{C}^n | |w_1|^2 + \dots + |w_n|^2 < 2\}$, $q \notin U$, and $p = (0, \dots, 0)$. Let $\rho_k(t) = z_k e^{it}$ for $0 \leq t \leq 2\pi$. Then the diameters of the sets ρ_k with respect to d_{D^*} are converging to 0. Since M is hyperbolically imbedded in Y and since f_k is distance decreasing, this implies $f_k(\rho_k)$ is converging to p . Thus for all sufficiently large k , there exists an annulus R'_k centered at 0 such that $\rho_k \subset R'_k$ and $f_k(R'_k) \subset B = \{(w_1, \dots, w_n) | |w_1|^2 + \dots + |w_n|^2 < 1\}$. Let R_k be the largest such annulus. We can assume that either R_k is not a punctured disk for any k or that R_k is a punctured disk for all k . We consider the former case first. Since $f_k(z'_k) \rightarrow q \notin U$, there exist a_k and b_k in $\overline{R_k}$ with $|a_k| < |b_k|$ and $|f_k(a_k)| = |f_k(b_k)| = 1$. By taking a subsequence and relabelling, we may assume that $f_k(a_k) \rightarrow q' \in \overline{B}$ and $f_k(b_k) \rightarrow q'' \in \overline{B}$. Let σ_k and τ_k be the curves defined by $\sigma_k(t) = a_k e^{it}$ and $\tau_k(t) = b_k e^{it}$ for $0 \leq t \leq 2\pi$. By the same reasoning used for ρ_k , we see that $f_k(\sigma_k) \rightarrow q'$ and $f_k(\tau_k) \rightarrow q''$.

Let $f_k = (f_k^1, \dots, f_k^m)$ in $f_k^{-1}(U)$. By rotating the coordinate ball W if necessary, we see that the winding number of the curves $f_k^1(\sigma_k)$ and $f_k^1(\tau_k)$ around the point $f_k^1(z_k)$ is 0 for all k sufficiently large. By Cauchy's theorem, we have

$$\int_{\sigma_k} \frac{f_k^{1'}(z)}{f_k^1(z) - f_k^1(z_k)} dz = \int_{f_k^1(\sigma_k)} \frac{dw_1}{w_1 - f_k^1(z_k)} = 0$$

and

$$\int_{\tau_k} \frac{f_k^{1'}(z)}{f_k^1(z) - f_k^1(z_k)} dz = \int_{f_k^1(\tau_k)} \frac{dw_1}{w_1 - f_k^1(z_k)} = 0.$$

Thus

$$0 = \int_{\sigma_k} \frac{f_k^{1'}(z)}{f_k^1(z) - f_k^1(z_k)} dz - \int_{\tau_k} \frac{f_k^{1'}(z)}{f_k^1(z) - f_k^1(z_k)} dz = 2\pi i(P - N)$$

where N and P are the number of zeros and poles of the function $f_k^1(z) - f_k^1(z_k)$ on the annulus R_k . This is a contradiction since $N > 0$ and $P = 0$.

If R_k is a punctured disk for all k , then the boundary of $\bar{R}_k$ is τ_k where b_k and τ_k are as above. Since $f_k(R_k) \subset B$, f_k extends holomorphically to $\bar{R}_k$ with $f_k(0)$ in B . Setting $\sigma_k = \emptyset$, the argument used in the preceding paragraph leads to a contradiction. This proves (i).

(ii) Let $\{z'_k\}$ be any sequence in D^* such that $f(z'_k) \rightarrow q \in Y$. Define $f(0) = q$. By (i), f is continuous. The Riemann extension theorem now implies that f is holomorphic at 0.

(iii) If $f_k(0)$ does not converge to q , then without loss of generality we may assume that $f_k(0) \rightarrow p \neq q$. Since each f_k is continuous, there exists a sequence $\{z_k\}$ in D^* such that $z_k \rightarrow 0$ and $f_k(z_k) \rightarrow p$. This contradicts (i). Q.E.D.

The following theorem was proven by Kwack in the case when M is compact and by Kobayashi in the case when A is nonsingular.

Theorem 2. *Let A be a closed complex subspace of a (nonsingular) complex manifold X . If the singularities of A are normal crossings and if M is hyperbolically imbedded in Y , then any holomorphic map $f: X - A \rightarrow M$ extends to a holomorphic map $f: X \rightarrow Y$.*

Proof. Since the singularities of A are normal crossings, we can assume that

$$X = D \times \dots \times D = D^n \times D^l = \{(w_1, \dots, w_n, t_1, \dots, t_l) \mid |w_i| < 1 \text{ and } |t_j| < 1\}$$

and that

$$X - A = (D^*)^n \times D^l = \{(w_1, \dots, w_n, t_1, \dots, t_l) \mid 0 < |w_i| < 1 \text{ and } |t_j| < 1\}.$$

The proof is by induction on n .

Case 1. $X - A = D^*$. This is part (ii) of Theorem 1.

Case 2. Assume we can extend f when $X - A = (D^*)^n$. We show that this implies we can extend f if $X - A = (D^*)^n \times D^l$.

Let $w = (w_1, \dots, w_n) \in D^n$ and $t = (t_1, \dots, t_l) \in D^l$. We denote the point $(w_1, \dots, w_n, t_1, \dots, t_l) \in D^n \times D^l$ by (w, t) . Let $f: (D^*)^n \times D^l \rightarrow M$ be holomorphic and define $f_t: (D^*)^n \rightarrow M$ by $f_t(w) = f(w, t)$. Since each map f_t extends to D^n , we can extend f to a map from $D^n \times D^l$ into Y by defining $f(w, t) = f_t(w)$. By the Riemann extension theorem, it suffices to show that the extended map f is continuous.

We assume that f is not continuous at some point, say $(w, 0)$ for simplicity, and then obtain a contradiction. If f is not continuous at $(w, 0)$, then there exists a sequence of points (w^k, t^k) in $(D^*)^n \times (D^*)^l$ with $(w^k, t^k) \rightarrow (w, 0)$ and such that the sequence $f(w^k, t^k) \rightarrow q \neq f(w, 0)$. Define $f_k: D^* \rightarrow M$ by $f_k(z) = f(w^k, z t^k / |t^k|)$. Let $z'_k = |t^k|$. Since $z'_k \rightarrow 0$ and $f_k(z'_k) = f(w^k, t^k) \rightarrow q$, part (iii) of Theorem 1 implies that $f_k(0) = f(w^k, 0) \rightarrow q$. But f_t is continuous for each t and therefore $f(w^k, 0) \rightarrow f(w, 0) \neq q$. Thus the assumption that f is not continuous is false.

Case 3. Assume that f can be extended if $X - A = (D^*)^n \times D^l$. We show that this implies that f can be extended if $X - A = (D^*)^{n+1}$. By the inductive hypothesis, we can extend f to $D^{n+1} - \{(0, \dots, 0)\}$. The map $g: D^* \rightarrow M$ defined by $g(z) = f(z, \dots, z)$ extends to D . Define $f(0, \dots, 0) = g(0)$. It suffices to show that f is continuous.

If f is not continuous, there exists a sequence $(w^k, t^k) = (w_1^k, \dots, w_n^k, t^k)$ in $(D^*)^{n+1}$ such that $f(w^k, t^k) \rightarrow q \neq f(0, \dots, 0)$. By applying Theorem 1 with $f_k(z) = f(z w^k / |w^k|, t^k)$ and $z'_k = |w^k|$, we see that $f(0, t^k) = f_k(0) \rightarrow q$. On the other hand, if we apply Theorem 1 with

$$f_k(z) = f(z t^k / |t^k|, \dots, z t^k / |t^k|, t^k) \quad \text{and} \quad z'_k = |t^k|,$$

we see that $f(0, t^k) \rightarrow f(0, \dots, 0) \neq q$. This is a contradiction and therefore f extends holomorphically. Q.E.D.

The following example shows that if M is not compact, restrictions on the singularities of A are necessary in order to make Theorem 2 remain true. Let $M = D \times (\mathbb{C} - \{-1, 1\}) \subset P_1(\mathbb{C}) \times P_1(\mathbb{C})$. Since both D and $\mathbb{C} - \{-1, 1\}$ are hyperbolically imbedded in $P_1(\mathbb{C})$, example (7) of § 2 implies that M is hyperbolically imbedded in $P_1(\mathbb{C}) \times P_1(\mathbb{C})$. Let $X = D \times D$ and $A = \{(z, w) | z = 0 \text{ or } z = \pm w\}$ and define $f: X - A \rightarrow M$ by $f(z, w) = (z, w/z)$. Clearly f does not extend to all of X , since $f(0, 0)$ cannot be defined in a continuous way.

This example is more or less typical of the general situation. To see this consider the following generalization. Let A be a closed complex subspace of $X = D^n = D \times \dots \times D$ with $(0, \dots, 0) \in A$. Let L denote the tautological line bundle over

$P_{n-1}(\mathbb{C})$ (i.e. L is the line bundle associated to the principal $\mathbb{C}^*$ bundle $\pi: \mathbb{C}^n - \{(0, \dots, 0)\} \rightarrow P_{n-1}(\mathbb{C})$ where π is the natural projection). Then $X - A$ can be considered as a subspace of L . If $X - A$ is hyperbolically imbedded in L , then the inclusion $i: X - A \rightarrow L$ is a counterexample to the singular version of Theorem 2. Note that in these examples the maps do extend to a meromorphic map. This is always the case.

Theorem 3. *Let A be a closed complex subspace of a complex space X and let M be hyperbolically imbedded in Y . If $f: X - A \rightarrow M$ is holomorphic, then f extends to a meromorphic map $f: X \rightarrow Y$.*

Proof. By a resolution of A in X we shall mean a triple (Z, B, ϕ) where

- (i) Z is a nonsingular complex manifold (not necessarily connected).
- (ii) B is a complex subspace of Z whose singularities are normal crossings.
- (iii) $\phi: Z \rightarrow X$ is a proper holomorphic map onto X with $\phi^{-1}(A) = B$ and such that ϕ^{-1} is meromorphic. By Hironaka [2], such a resolution always exists (at least locally). Define $\bar{f}: Z - B \rightarrow M$ by $\bar{f} = f \circ \phi$. By Theorem 2, $\bar{f}$ extends to a holomorphic map $\bar{f}: Z \rightarrow Y$. Thus f extends meromorphically to X by defining $f = \bar{f} \circ \phi^{-1}$.

Q.E.D.

We shall now weaken the restrictions on the singularities of A given in Theorem 2.

Theorem 4. *Let A be a closed complex subspace of a normal complex space X . Assume that (Z, B, ϕ) is a resolution of A in X (see proof above) such that for every pair of points p and q in B with $\phi(p) = \phi(q)$, there exist sequences $\{p_n\}$ and $\{q_n\}$ in $Z - B$ with $p_n \rightarrow p$, $q_n \rightarrow q$ and $d_{X-A}(\phi(p_n), \phi(q_n)) \rightarrow 0$. If M is hyperbolically imbedded in Y , then any holomorphic map $f: X - A \rightarrow M$ extends to a holomorphic map $f: X \rightarrow Y$.*

Proof. Let $\bar{f}: Z \rightarrow Y$ be defined as in the last proof. Since X is normal, it suffices to show that the meromorphic map $f = \bar{f} \circ \phi^{-1}$ is single valued. Since M is hyperbolically imbedded in Y and since $f|_{X-A}$ is distance decreasing, we have

$$\bar{f}(p) = \lim_{n \rightarrow \infty} f(\phi(p_n)) = \lim_{n \rightarrow \infty} f(\phi(q_n)) = \bar{f}(q).$$

This implies that f is single valued. Q.E.D.

Remark. If X is nonsingular and if A has only normal crossings, then A obviously satisfies the conditions of Theorem 4. Thus Theorem 4 is a generalization of Theorem 2, but in practice it is not a very satisfactory one because we are usually unable to actually construct a resolution. However, it does not seem very easy to give a condition which depends only on $X - A$ and which is not too restrictive.

In our final generalization of the Picard theorem, we consider the space $M_n = \{(w_1, \dots, w_n) \in \mathbb{C}^n \mid w_i \neq 0 \text{ and } w_i \neq 1 \text{ for } i = 1, \dots, n\}$. Let $[(w_1, v_1), \dots, (w_n, v_n)]$ be "homogeneous" coordinates for $P_1(\mathbb{C}) \times \dots \times P_1(\mathbb{C})$ and let $(w_0, w_1, \dots, w_n)$ be homogeneous coordinates for $P_n(\mathbb{C})$. Define $\phi: \mathbb{C}^n \rightarrow P_1(\mathbb{C}) \times \dots \times P_1(\mathbb{C})$ by $\phi(w_1, \dots, w_n) = [(w_1, 1), \dots, (w_n, 1)]$ and define $\psi: \mathbb{C}^n \rightarrow P_n(\mathbb{C})$ by $\psi(w_1, \dots, w_n) = (1, w_1, \dots, w_n)$. Thus ϕ imbeds M_n in $P_1(\mathbb{C}) \times \dots \times P_1(\mathbb{C})$ and ψ imbeds M_n in $P_n(\mathbb{C})$. By (7) of § 2, M_n is hyperbolically imbedded in $P_1(\mathbb{C}) \times \dots \times P_1(\mathbb{C})$ and therefore the results of this section apply in this case. On the other hand, an easy computation shows that the identity map $i: M_n \rightarrow M_n$ does not extend to a holomorphic map $i: P_1(\mathbb{C}) \times \dots \times P_1(\mathbb{C}) \rightarrow P_n(\mathbb{C})$. Since the singularities of $A = P_1(\mathbb{C}) \times \dots \times P_1(\mathbb{C}) - M_n$ are normal crossings, Theorem 2 implies that M_n is not hyperbolically imbedded in $P_n(\mathbb{C})$. However, we still have

Theorem 5. *Let A be a closed submanifold of a complex manifold X . Then every holomorphic map $f: X - A \rightarrow M_n$ extends to a holomorphic map $f: X \rightarrow P_n(\mathbb{C})$.*

Proof. By Theorem 2, f extends to a holomorphic map $f: X \rightarrow P_1(\mathbb{C}) \times \dots \times P_1(\mathbb{C})$. We can assume $X = D^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid |z_i| < 1 \text{ for } i = 1, \dots, m\}$ and that $A = \{(z_1, \dots, z_m) \in D^m \mid z_1 = 0\}$. Let $f = (f_1, \dots, f_n)$ where $f_i: X \rightarrow P_1(\mathbb{C})$. Without loss of generality, we can assume that f_i is given by two holomorphic functions, i.e. $f_i(z_1, \dots, z_n) = (g_i(z_1, \dots, z_n), b_i(z_1, \dots, z_n))$. Since $b_i|_{X-A} \neq 0$, we see that either $b_i|_A \equiv 0$ or $b_i|_A \neq 0$. This implies that $b_i(z_1, \dots, z_n) = z_1^{k_i} \hat{b}_i(z_1, \dots, z_n)$ where $k_i \geq 0$ and $\hat{b}_i(z_1, \dots, z_n) \neq 0$ in X . By the same reasoning, we have $g_i(z_1, \dots, z_n) = z_1^{l_i} \hat{g}_i(z_1, \dots, z_n)$ where $\hat{g}_i(z_1, \dots, z_n) \neq 0$ in X . This means that f (i.e. $\psi \circ f$) is given by

$$f(z_1, \dots, z_n) = \left(1, z_1^{l_1} \frac{\hat{g}_1(z_1, \dots, z_n)}{\hat{b}_1(z_1, \dots, z_n)}, \dots, z_1^{l_n} \frac{\hat{g}_n(z_1, \dots, z_n)}{\hat{b}_n(z_1, \dots, z_n)} \right)$$

where $l_i = k_i' - k_i$. This implies that f extends to a holomorphic map $f: X \rightarrow P_n(\mathbb{C})$. Q.E.D.

4. Applications. Let $\mathcal{D}$ be a bounded symmetric domain in $\mathbb{C}^n$ and let Γ be an arithmetically defined, discrete subgroup of the largest connected group of holomorphic automorphisms of $\mathcal{D}$. Assume Γ acts freely on $\mathcal{D}$. There are two compactifications of $\mathcal{D}/\Gamma$. The Satake compactification $(\mathcal{D}/\Gamma)_{B.B.}^*$ is the same as that of Borel-Baily as a topological space. The other compactification $(\mathcal{D}/\Gamma)_{P.S.}^*$ is that of Pyateckiy-Šapiro. In an unpublished work, Borel proved Corollary 1 for $(\mathcal{D}/\Gamma)_{B.B.}^*$. His proof shows that $\mathcal{D}/\Gamma$ is hyperbolically imbedded in $(\mathcal{D}/\Gamma)_{B.B.}^*$. In [4], Kobayashi and Ochiai showed that $\mathcal{D}/\Gamma$ is hyperbolically imbedded in $(\mathcal{D}/\Gamma)_{P.S.}^*$ and proved Corollary 1 for this case when A is nonsingular. Baily has shown that there is a one-one continuous map $\psi: (\mathcal{D}/\Gamma)_{B.B.}^* \rightarrow (\mathcal{D}/\Gamma)_{P.S.}^*$,

but it is not known whether $(\mathcal{D}/\Gamma)_{P.S.}^*$ is even Hausdorff.⁽²⁾ Using these remarks and letting M be either $(\mathcal{D}/\Gamma)_{B.B.}^*$ or $(\mathcal{D}/\Gamma)_{P.S.}^*$ we have

Corollary 1. *Let X and A be as in Theorem 4 (or 2). Then any holomorphic map $f: X - A \rightarrow \mathcal{D}/\Gamma$ extends to a holomorphic map $f: X \rightarrow M$. If A is a complex subspace of a complex space X , then any holomorphic map $f: X - A \rightarrow \mathcal{D}/\Gamma$ extends to a meromorphic map $f: X \rightarrow M$. In particular, if $f: \mathcal{D}/\Gamma \rightarrow \mathcal{D}'/\Gamma'$ is holomorphic, then f extends to a meromorphic map $f: M \rightarrow M'$.*

The next corollary can be interpreted as a "generalization" of Liouville's theorem.

Corollary 2. *Let X be a complex manifold with $d_X \equiv 0$ and let A be a closed complex subspace of X . If M is a compact hyperbolic space and $f: X - A \rightarrow M$ is holomorphic, then f is a constant map. (For example, any map from $\mathbb{C} - \{0, 1\}$ into a compact Riemann surface of genus ≥ 2 is a constant map.) If $X - A$ is hyperbolic, and Γ is a group of automorphisms of $X - A$ such that $X - A$ is a covering space of $(X - A)/\Gamma$, then $(X - A)/\Gamma$ is not compact.*

Proof. Since M is compact hyperbolic, f extends to a map $f: X \rightarrow M$. Since $d_X \equiv 0$ and d_M is a proper distance, the distance decreasing property of f implies that f is a constant map. Q.E.D.

Corollary 3. *Let $V \subset P_n(\mathbb{C})$ be a complex manifold and let $M_n \subset P_n(\mathbb{C})$ be as in Theorem 5. Let $M = V \cap M_n$. Let A be a closed complex subspace of a complex space X . Then any holomorphic map $f: X - A \rightarrow M$ extends to a meromorphic map $f: X \rightarrow V$. If X and A are both nonsingular, then $f: X \rightarrow V$ is holomorphic.*

Proof. Since M is hyperbolically imbedded in $P_1(\mathbb{C}) \times \cdots \times P_1(\mathbb{C})$, it is clear that f extends meromorphically. The second statement follows directly from Theorem 5. Q.E.D.

The previous corollary generalizes Corollary A of a recent paper of Griffiths [1]. Griffiths has shown that under the proper restrictions on the imbedding of V in $P_n(\mathbb{C})$, M will have the property that it is covered by a bounded domain in $\mathbb{C}^m$ which is topologically a cell.

5. Conclusion. In this paper we have considered generalizations of the big Picard theorem. Except for Theorem 5, we have replaced the range space $P_1(\mathbb{C}) - \{3 \text{ points}\}$ by a hyperbolically imbedded complex space, and then we have tried to replace the domain space D^* by more general spaces. Now we comment on the relevancy of the assumption "hyperbolically imbedded".

To do this consider the following examples.

- (i) Let $M' = \{(z_0, z_1, z_2) \in P_2(\mathbb{C}) \mid z_0 \neq 0, z_1 \neq 0, z_1 \neq z_0, z_2 \neq \pm z_0 e^{z_0/z_1}\}$.

(2) Borel has recently shown that the two compactifications of $\mathcal{D}/\Gamma$ are equivalent.

Define $\phi: M' \rightarrow (C - \{0, 1\}) \times (C - \{-1, 1\})$ by

$$\phi(z_0, z_1, z_2) = (z_1/z_0, (z_2/z_0)e^{-z_0/z_1}).$$

Clearly ϕ is a biholomorphism and thus M' is complete hyperbolic. However, the map $g: D^* \rightarrow M'$ defined by $g(z) = (1, z, 2e^{1/z})$ does not extend to a meromorphic map $g: D \rightarrow P_2(C)$.

(ii) Let $M'' = \{(z_0, z_1, z_2) \in P_2(C) | z_0 \neq 0, z_1 \neq 0, z_1 \neq z_0, z_2 z_1 \neq \pm z_0^2\}$. Since the map $b: D^* \times D \rightarrow M''$ defined by $b(z, w) = (1, z, w/z)$ does not extend to a holomorphic map $b: D \times D \rightarrow P_2(C)$, M'' is not hyperbolically imbedded in $P_2(C)$. Define $\psi: M'' \rightarrow (C - \{0, 1\}) \times (C - \{-1, 1\})$ by $\psi(z_0, z_1, z_2) = (z_1/z_0, z_1 z_2/z_0^2)$. It is easy to see that ψ is a biholomorphism which extends to a bimeromorphism $\psi: P_2(C) \rightarrow P_1(C) \times P_1(C)$. Thus Theorem 3 holds for maps into M'' .

The first example shows that some assumptions on the way in which M is contained in Y are necessary if any extension theorem is going to work. The second example indicates that it is not necessary for M to be hyperbolically imbedded in Y in order for Theorem 3 to be true. In fact, it is possible that it may be sufficient for M to be a (complete) hyperbolic Zariski open set of a closed algebraic manifold Y . It also shows that these last conditions are not strong enough to imply that you can extend holomorphically over a nonsingular subspace. Theorem 5 implies that "hyperbolically imbedded" is not a necessary condition for the nonsingular version of Theorem 2. Finally, the discussion preceding Theorem 5 indicates that strong assumptions such as "hyperbolically imbedded" are probably necessary if Theorem 2 is to hold.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER 8, B.C., CANADA