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ZERO POINTS OF KILLING VECTOR FIELDS, GEODESIC ORBITS, CURVATURE, AND CUT LOCUS

BY

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ABSTRACT. Let (M, g) be a compact, connected, Riemannian manifold. Let X be a Killing vector field on M. $f = g(X, X)$ is called the length function of X. Let D denote the minimum of the distances from points to their cut loci on M. We derive an inequality involving f which enables us to prove facts relating D, the zero ponts of X, orbits of X which are closed geodesics, and, applying theorems of Klingenberg, the curvature of M. Then we use these results together with a further analysis of f to describe the nature of a Killing vector field in a neighborhood of an isolated zero point.

1. An inequality.

Theorem 1. *Let X be a Killing vector field on M. Let q be a critical point of the length function $f = g(X, X)$ of X such that $f(q) \neq 0$. Assume the orbit γ of X through q is closed. Let a be another point of M and suppose the distance from q to a is ρ. Then we have*

$$(\sqrt{f(q)} - \sqrt{f(a)})\, D/\sqrt{f(q)} \leq 2\rho. \tag{1}$$

Proof. Denote by β the period of the orbit γ. We note that γ is a geodesic, since q is a critical point of f [2, p. 356]. Let τ be the orbit of X through a. Now assume $(\sqrt{f(q)} - \sqrt{f(a)})D/\sqrt{f(q)} > 2\rho$. Pick an integer m and a real number r so that

$$m\beta - r = D/\sqrt{f(q)} - \delta \tag{2}$$

where $\delta > 0$ is chosen sufficiently small so that $(\sqrt{f(q)} - \sqrt{f(a)})(D/\sqrt{f(q)} - \delta) > 2\rho$. Then we have

$$\sqrt{f(q)}\,(m\beta - r) - \sqrt{f(a)}\,(m\beta - r) > 2\rho. \tag{3}$$

Let $d(v, w)$ denote the distance between two points v and w in M. Let ϕ_t be the flow of X. We have $\phi_{m\beta-r}(q) = \phi_{m\beta}(\phi_{-r}(q)) = \phi_{-r}(q)$ since m is an integer and β is the period of γ. The length of the shortest segment of γ between q and $\phi_{-r}(q)$ is $\int_0^{m\beta-r} \sqrt{f(\phi_t(q))}\, dt$, which is equal to $\sqrt{f(q)}(m\beta - r)$ since f is constant along γ. This is the length of the shortest segment because

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by (2), $\sqrt{f(q)}(m\beta - r) = D - \delta\sqrt{f(q)} < D$ and clearly $D \leq \frac{1}{2}$ (length of γ). This is also the length of the shortest segment of γ between q and $\phi_r(q)$. Since γ is a geodesic and since the length of the shortest segment of γ between q and $\phi_r(q)$ is less than D, this segment is therefore length minimizing. Hence we have

$$d(q, \phi_r(q)) = \sqrt{f(q)}\,(m\beta - r). \tag{4}$$

Since f is constant along the orbit τ of X through a, we have that the length of a segment of τ between $\phi_{m\beta}(a)$ and $\phi_r(a)$ is $\sqrt{f(a)}(m\beta - r)$. Thus we have

$$d(\phi_{m\beta}(a), \phi_r(a)) \leq \sqrt{f(a)}\,(m\beta - r). \tag{5}$$

From (3), (4), and (5) we obtain $d(q, \phi_r(q)) > 2\rho + \sqrt{f(a)}(m\beta - r) \geq 2\rho + d(\phi_{m\beta}(a), \phi_r(a))$. Hence

$$d(q, \phi_r(q)) > 2\rho + d(\phi_{m\beta}(a), \phi_r(a)). \tag{6}$$

By the triangle inequality we obtain

$$d(q, \phi_r(a)) \leq d(q, \phi_{m\beta}(a)) + d(\phi_{m\beta}(a), \phi_r(a)).$$

But $d(q, \phi_{m\beta}(a)) = d(\phi_{m\beta}(q), \phi_{m\beta}(a)) = d(q, a) = \rho$ since $\phi_{m\beta}(q) = q$ and since $\phi_{m\beta}$ is an isometry. Thus

$$d(q, \phi_r(a)) \leq \rho + d(\phi_{m\beta}(a), \phi_r(a)). \tag{7}$$

Since ϕ_r is an isometry, we have

$$d(\phi_r(a), \phi_r(q)) = d(q, a) = \rho. \tag{8}$$

We observe that (7) implies $\phi_r(a)$ lies in a closed ball about $\phi_r(q)$ of radius $\rho + d(\phi_{m\beta}(a), \phi_r(a))$. And (8) implies that $\phi_r(a)$ lies in a closed ball about $\phi_r(q)$ of radius ρ. But by (6), these two balls have empty intersection. Hence we have that (1) is true.

From Theorem 1 we obtain immediately the following

Theorem 2. *Suppose X is a Killing vector field on M and q is a critical point of $f = g(X, X)$ such that $f(q) \neq 0$. Suppose the orbit of X through q is closed. If p is a zero point of X, then $d(p, q) \geq D/2$.*

In particular, this theorem gives a lower bound for the distances between zero points of X and orbits of X which are nontrivial closed geodesics. Moreover, the lower bound depends only on the metric and not on the vector field X. To show that it cannot be improved, consider the following example: Let M be S^2 with the usual metric. Let X be the Killing vector field whose flow is a 1-parameter group of rotations about an axis through the north and south poles. Then any point q on the equator is a critical point of $f = g(X, X)$ such that $f(q) \neq 0$ and the north pole N is a zero point of X. D in this case is half of the circumference of S^2. Thus we have $d(N, q) = D/2$.

Under additional assumptions on M we can give a lower bound in terms of curvature:

Theorem 3. *Suppose M is a compact, connected Riemannian manifold of even dimension. Suppose the sectional curvatures K_σ satisfy $0 < K_\sigma \leq \lambda$ for all tangent planes σ. Let X be a Killing vector field on M. Then the distance from any zero point of X to any orbit of X which is a nontrivial closed geodesic is always $\geq \pi/4\sqrt{\lambda}$. If in addition we assume M is orientable or simply connected (which are equivalent assumptions), then this distance is $\geq \pi/2\sqrt{\lambda}$.*

Proof. By theorems of Klingenberg [1, pp. 227 and 230], $D \geq \pi/2\sqrt{\lambda}$ ($D \geq \pi/\sqrt{\lambda}$ if M is orientable). The result now follows immediately from Theorem 2.

As a corollary of Theorem 2 we have the following criterion for the zero set of a Killing vector field to be empty in terms of the distribution throughout M of orbits which are nontrivial closed geodesics.

Theorem 4. *Suppose X is a Killing vector field on M with the property that for any point a in M there exists an orbit γ_a of X which is a nontrivial closed geodesic such that the distance from a to γ_a is $< D/2$. Then the zero set of X is empty.*

2. **Isolated zero point of X.** Let X be a Killing vector field on M with an isolated zero point at p. Then p is a critical point of the function $f = g(X, X)$. We recall that to f there is associated a symmetric bilinear functional f_{**} on T_pM called the Hessian of f at p. The index of f at p is the maximal dimension of a subspace of T_pM on which f_{**} is negative definite. The critical point p is nondegenerate if the nullity of f_{**} on T_pM is zero.

Lemma. *p is a nondegenerate critical point of index zero of the function $f = g(X, X)$.*

Proof. Since X is Killing and p is an isolated zero point, there exists [3, p. 64] a coordinate neighborhood U of p with local coordinates $\{x^1, \cdots, x^n\}$ such that (i) $x^i(p) = 0$, $i = 1, \cdots, n$, (ii) if $X = \sum_{i=1}^n \xi^i(\partial/\partial x^i)$ on U, then $\xi^i(p) = 0$ and the matrix with (i, j) entry $(\partial\xi^i/\partial x^j)|_p$ is of the form

$$\begin{bmatrix} A_1 & & & & 0 \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & A_{n/2} \end{bmatrix}$$

where

$$A_k = \begin{bmatrix} 0 & \theta_k \\ -\theta_k & 0 \end{bmatrix}, \qquad \theta_k \neq 0,\ k = 1, \cdots, n/2,$$

and

(iii) if $g = \Sigma_{i,j} a_{ij} dx^i \otimes dx^j$, then $a_{ij}(p) = \delta_{ij}$. Thus we have $f = \Sigma_{i,j=1}^n \xi^i \xi^j a_{ij}$.

Let $v = \Sigma_i v^i(\partial/\partial x^i)|_p$ and $w = \Sigma_i w^i(\partial/\partial x^i)|_p$ be two vectors in T_pM. Extend these to vector fields $\widetilde{v} = \Sigma_i v^i(\partial/\partial x^i)$ and $\widetilde{w} = \Sigma_i w^i(\partial/\partial x^i)$ on U. Then by definition $f_{**}(v, w) = \widetilde{v}_p(\widetilde{w}(f))$. By a direct computation we see that

$$f_{**}(v, w) = 2\{\theta_1^2(v^1w^1 + v^2w^2) + \theta_2^2(v^3w^3 + v^4w^4) + \cdots + \theta_{n/2}^2(v^{n-1}w^{n-1} + v^nw^n)\}.$$

From this it is obvious that f_{**} is positive definite on all of T_pM and the nullity of f_{**} is zero. Hence we have the result.

Theorem 5. *There exists a local coordinate neighborhood U of p with local coordinates $\{x^1, \cdots, x^n\}$ such that with respect to these coordinates, $f = (x^1)^2 + \cdots + (x^n)^2$ on U.*

Proof. This is an immediate consequence of the previous lemma and the Morse lemma [4, p. 6].

We have that the following situation prevails near p:

Theorem 6. *There exists a connected open neighborhood U of p satisfying*

(i) *$U - \{p\}$ is a disjoint union of hypersurfaces of M, each of them diffeomorphic to S^{n-1}. (n (even) is the dimension of M.) We call these hyperspheres.*

(ii) *The function f is constant on each hypersphere.*

(iii) *X is tangent to each hypersphere.*

(iv) *Restricted to each hypersphere, the length of X is constant.*

Proof. (i) and (ii) follow from Theorem 5. (iii) follows from the fact that f is constant along each integral curve of X, and hence any integral curve of X which meets $U - \{p\}$ lies in one of the hyperspheres. (iv) comes from (ii) and (iii).

Next we consider the question: How far can the neighborhood U be extended so that the conditions of Theorem 6 continue to hold?

Theorem 7. *There exists a connected open neighborhood V of p which contains the neighborhood U of Theorem 6 and which also satisfies conditions (i)–(iv). Moreover, $\overline{V}$ meets an orbit of X which is a nontrivial geodesic.*

Proof. Let Y be the vector field grad $f/\|\text{grad } f\|^2$ defined on the open submanifold $M - B$ of M, where B denotes the set of critical points of f. Y is of course nowhere zero on $M - B$. We remark that the orbits (as point sets) of Y are the same as the orbits of grad $f|_{M-B}$. These orbits are perpendicular to the level surfaces of f. Let $r > 0$ be sufficiently small so that the level surface $f = r$ has a nonempty intersection S with the neighborhood U of Theorem 5. Then S is a hypersphere. Let $t \to \alpha_t(v)$ be the integral curve of Y through an arbitrary point v of S with $\alpha_0(v) = v$. Then $d\alpha_t(v)/dt = \text{grad } f/\|\text{grad } f\|^2$. Hence we have

$$\frac{df(\alpha_t(v))}{dt} = g\left(\text{grad } f, \frac{\text{grad } f}{\|\text{grad } f\|^2}\right) = 1.$$

Thus we have that

$$f(\alpha_t(v)) = f(\alpha_0(v)) + t = r + t \tag{1}$$

and $\alpha_t(v)$ is defined for t in an interval $(-r, \epsilon_v)$. Let $\epsilon = \inf_{v \in S} \epsilon_v$. Then there exists a family of local diffeomorphisms $\{\phi_t\}_{t \in (-r,\epsilon)}$, each defined on a neighborhood of S. For v in S, $\phi_t(v) = \alpha_t(v)$. Thus $f(\phi_t(v)) = r + t$ for any v in S by (1). Hence f has the constant value $r + t$ on $\phi_t(s)$ and, since ϕ_t is a diffeomorphism, $\phi_t(S)$ is a hypersphere. Now let $V = \{p\} \cup \bigcup_{t \in (-r,\epsilon)} \phi_t(S)$. Then $V \supseteq U$ and V satisfies conditions (i)–(iv).

Now we show that $\overline{V} - \{p\} \cap B \neq \emptyset$. For suppose the contrary. Let A be the open ball about p whose boundary is S. Let v be an arbitrary point of S. Consider the curve $[0, \epsilon) \to M$: $t \to \alpha_t(v)$. This curve is contained in the compact set $\overline{V} - A \subset M - B$. We have that $\|Y\|$ is bounded from above on this compact set. This means that the length of the curve $[0, \epsilon) \to M$: $t \to \alpha_t(v)$ is finite, where the length of this curve is defined by $\lim_{t \to \epsilon} \int_0^t \|Y(\alpha_s(v))\| \, ds$. It then follows easily that $\lim_{t \to \epsilon} \alpha_t(v)$ exists. Call this limit p_v. Then p_v is in $\overline{V} - A$.

Let ρ be the distance from B to $\overline{V} - A$. It is clear that the integral curve of Y starting at p_v goes on for a distance (measured along the integral curve) of at least $\rho/2$. Now let $C = \{q \in M \mid d(q, \overline{V} - A) \leq \rho/2\}$. Then C is compact and $C \subset M - B$. Thus $\|Y\|$ is bounded from above on C. This implies there exists $\eta > 0$ such that the curve $[0, \epsilon) \to M$: $t \to \alpha_t(v)$ may be extended to $[0, \epsilon + \eta) \to M$: $t \to \alpha_t(v)$ and η is independent of v in S. This contradicts the definition of ϵ. Thus $\overline{V} - \{p\} \cap B \neq \emptyset$. From the construction of V it is obvious that if q is a point in $\overline{V} - \{p\} \cap B$, then $f(q) \neq 0$. Thus the orbit of X through q is a nontrivial geodesic meeting V. This concludes the proof.

To show that in general the neighborhood V cannot be further extended in such a way that conditions (i)–(iv) still hold, we need only consider the projective plane with the Killing vector field induced by the vector field X on S^2 given in the example after Theorem 2.

According to Theorem 7, it is reasonable to regard the distance d from p to the nearest orbit of X which is a nontrivial geodesic as a measure of the "size" of the neighborhood V. In case all the orbits of X are closed, we have by Theorem 2 a lower bound for d which is independent of X: namely, $d \geq D/2$. Moreover, a manifold having a Killing vector field with an isolated zero point is even dimensional [3, p. 63]. Thus if the sectional curvatures K_σ of M satisfy $0 < K_\sigma \leq \lambda$, we have by Theorem 3 that $d \geq \pi/4\sqrt{\lambda}$, again assuming all orbits of X are closed.

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