ON THE SEMISIMPLICITY OF GROUP RINGS OF SOLVABLE GROUPS

BY

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ABSTRACT. Let $K[G]$ denote the group ring of G over the field K of characteristic $p > 0$. An interesting unsolved problem is to find necessary and sufficient conditions on G for $K[G]$ to be semisimple. Even the special case in which G is assumed to be a solvable group is still open. In this paper we prove a number of theorems which may be of use in this special case.

We follow the notation of [3]. In addition let W be a subgroup of G . We say that W has locally finite index in G and write $[G : W] = \text{l.f.}$ if for all finitely generated subgroups L of G we have $[L : L \cap W] < \infty$. We say that G is a Δ -group if $G = \Delta(G)$. The main result of this paper is

Theorem. *Let K be a field of characteristic $p > 0$, let G be a solvable group and let H be a normal Δ -subgroup of G . Then $JK[G] \cap K[H] \neq 0$ if and only if H has an element h of order p with $[G : C_G(h)] = \text{l.f.}$*

We remark that this generalizes techniques found in [1], [2] and [5]. Moreover, recent results of A. E. Zalesskii in [5] guarantee that under certain circumstances if G is solvable and if $JK[G] \neq 0$ then such a subgroup H exists with $JK[G] \cap K[H] \neq 0$. In particular this solves the problem in the case of metanilpotent groups.

1. σ -nilpotent elements. Let $\langle \sigma \rangle$ be an infinite cyclic group and let Λ denote the ring of rational integers. For convenience set

$$\Lambda^+ = \{1, 2, 3, \dots, n, \dots\} \quad \text{and} \quad \Lambda! = \{1!, 2!, 3!, \dots, n!, \dots\}.$$

Suppose $\langle \sigma \rangle$ acts on a ring R . If $\alpha \in R$ we say that α is σ -nilpotent if there exists an infinite subset S of $\Lambda!$ such that for each $s \in S$ there exists an integer $r = r(s) \geq 1$ with

$$\alpha \alpha^{\sigma^s} \alpha^{\sigma^{2s}} \dots \alpha^{\sigma^{rs}} = 0.$$

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Let I be a right ideal of R . We say that I is σ -nil if every element of I is σ -nilpotent. Observe that if $\langle \sigma \rangle$ acts on a group H , then $\langle \sigma \rangle$ acts naturally on the group ring $K[H]$. In this section we study σ -nil right ideals in $K[H]$. No assumption will be made here on the characteristic of the field.

Suppose $\langle \sigma \rangle$ acts on an abelian group A . Then the polynomial ring $\Lambda[\sigma]$ acts in a natural way. If A is torsion (as a Λ -module) then we set

$$\text{moni } A = \{a \in A \mid a^{f(\sigma)} = 1 \text{ for some monic polynomial } f(\sigma) \in \Lambda[\sigma]\}.$$

If A is torsion free then we set

$$\text{poly } A = \{a \in A \mid a^{f(\sigma)} = 1 \text{ for some } f(\sigma) \in \Lambda[\sigma], f(\sigma) \neq 0\},$$

$$\text{nilp } A = \{a \in A \mid a^{(\sigma^n - 1)^m} = 1 \text{ for some } m, n \in \Lambda^+\}.$$

We have the following basic facts.

Lemma 1. *Let A be an abelian group acted upon by $\langle \sigma \rangle$.*

(i) *If A is torsion then $\text{moni } A$ is a $\langle \sigma \rangle$ -invariant subgroup of A and $\text{moni}(A/\text{moni } A) = \langle 1 \rangle$.*

(ii) *If A is torsion free then $\text{poly } A$ is a $\langle \sigma \rangle$ -invariant subgroup of A , $A/\text{poly } A$ is torsion free and $\text{poly}(A/\text{poly } A) = \langle 1 \rangle$.*

(iii) *If A is torsion free then $\text{nilp } A$ is a $\langle \sigma \rangle$ -invariant subgroup of A , $A/\text{nilp } A$ is torsion free and $\text{nilp}(A/\text{nilp } A) = \langle 1 \rangle$.*

Proof. (i) Let $f(\sigma)$ and $g(\sigma)$ be monic polynomials. Then so is $f(\sigma)g(\sigma)$. Suppose $a, b \in A$. If $a^{f(\sigma)} = 1$, $b^{g(\sigma)} = 1$ then $(ab^{-1})^{f(\sigma)g(\sigma)} = 1$. Thus clearly $\text{moni } A$ is a $\langle \sigma \rangle$ -invariant subgroup of A . If $b^{f(\sigma)} \in \text{moni } A$ then $b^{f(\sigma)g(\sigma)} = 1$ for some monic $g(\sigma)$ so $\text{moni}(A/\text{moni } A) = \langle 1 \rangle$.

(ii) Let $f(\sigma)$ and $g(\sigma)$ be nonzero polynomials. Then so is $f(\sigma)g(\sigma)$. Suppose $a, b \in A$. If $a^{f(\sigma)} = 1$, $b^{g(\sigma)} = 1$ then $(ab^{-1})^{f(\sigma)g(\sigma)} = 1$. Thus clearly $\text{poly } A$ is a $\langle \sigma \rangle$ -invariant subgroup of A . If $b^s \in \text{poly } A$, then $(b^{f(\sigma)})^s = (b^s)^{f(\sigma)} = 1$ for some nonzero $f(\sigma)$. Since A is torsion free, $b^{f(\sigma)} = 1$ and $b \in \text{poly } A$. Thus $A/\text{poly } A$ is torsion free. Finally if $b^{f(\sigma)} \in \text{poly } A$ then $b^{f(\sigma)g(\sigma)} = 1$ for some $g(\sigma) \neq 0$ so $\text{poly}(A/\text{poly } A) = \langle 1 \rangle$.

(iii) This follows as in (ii) with the additional observation that if $f(\sigma) = (\sigma^{n_1} - 1)^{m_1}$ and $g(\sigma) = (\sigma^{n_2} - 1)^{m_2}$ then $f(\sigma)g(\sigma)$ divides $(\sigma^{n_1 n_2} - 1)^{m_1 + m_2}$.

The following is a variant of a well-known result.

Lemma 2. *Let $\omega_1, \omega_2, \dots, \omega_t$ be a complete set of nonzero algebraic conjugates over the rational numbers $\mathcal{Q}$ and suppose that $|\omega_i| \leq 1$ for all i . Moreover, suppose there exist nonzero algebraic integers $c_1, c_2, \dots, c_t$ and an infinite subset $S \subseteq \Lambda^+$ such that $c_i \omega_i^s$ is an algebraic integer for all $i \leq t$ and $s \in S$. Then the ω_i 's are all roots of unity.*

Proof. Since the c_i 's are all algebraic integers it follows that if $c = c_1 c_2 \cdots c_t$ then $c\omega_i^s$ is an algebraic integer. Moreover, the product of all the algebraic conjugates of c has this same property so we may assume that c is a rational integer.

For each $s \in S$ let $f_s(\zeta) = \prod_{i=1}^t (\zeta - c\omega_i^s)$. Since c is a rational integer and since $\{\omega_i\}$ is a complete set of algebraic conjugates over $\mathcal{Q}$ it follows that $f_s(\zeta) \in \mathcal{A}[\zeta]$. Moreover, $c\omega_i^s$ is an algebraic integer so $f_s(\zeta) \in \Lambda[\zeta]$. Now $|\omega_i| \leq 1$ so $|c\omega_i^s| \leq c$ and it follows that there exists a fixed bound for the coefficients of $f_s(\zeta)$ independent of s . Since S is infinite, we conclude that there exists an infinite subset S_0 of S such that for all $s \in S_0$ the polynomials $f_s(\zeta)$ are identical.

Fix $s_0 \in S_0$. Then for each $s \in S_0$ there exists a permutation τ of the subscripts such that $c\omega_i^s = c\omega_{\tau(i)}^{s_0}$. Finally S_0 is infinite and the number of such permutations is finite so there exists two distinct elements $s, s' \in S_0$ having the same permutation. Then for all i , $c\omega_i^s = c\omega_i^{s'}$ and hence ω_i is a root of unity since $c \neq 0$ and $\omega_i \neq 0$.

Lemma 3. Let A be an abelian group acted upon by $\langle \sigma \rangle$. Let $a_0 = 1, a_1, a_2, \dots, a_k$ be a fixed finite set of distinct elements of A and let S be an infinite subset of Λ^+ . Suppose that for each $s \in S$ there exists an integer $r = r(s) \geq 1$ and a set of subscripts $i_0, i_1, \dots, i_r$ not all zero with

$$a_{i_0} a_{i_1}^{\sigma^s} a_{i_2}^{\sigma^{2s}} \cdots a_{i_r}^{\sigma^{rs}} = 1.$$

- (i) If A is torsion, then $\text{mon } A \neq \langle 1 \rangle$.
- (ii) If A is torsion free, then $\text{poly } A \neq \langle 1 \rangle$.
- (iii) If A is torsion free and $A = \text{poly } A$, then $\text{nilp } A \neq \langle 1 \rangle$.

Proof. It is easier to view this result additively. Thus in this proof we will assume that A is an additive abelian group and the action of $\Lambda[\sigma]$ on A will be indicated by right multiplication.

Let us consider the above situation. For each $s \in S$ let T_s denote the set of nonzero subscripts occurring in the s -equation. Since S is infinite and the number of subsets of $\{1, 2, \dots, k\}$ is finite, it follows that there exists an infinite subset S_0 of S such that $T_s = T$ is the same for all $s \in S_0$. We can then replace S by S_0 and $\{a_0, a_1, \dots, a_k\}$ by $\{a_i \mid i = 0 \text{ or } i \in T\}$ if necessary to obtain the irredundant situation in which each a_i for $i \geq 1$ occurs at least once in each equation.

Suppose first that A is a vector space over a field F . Then A is a right $F[\sigma]$ -module and $F[\sigma]$ is a principal ideal domain. Suppose that the above situation is irredundant and set

$$B = a_1 F[\sigma] + a_2 F[\sigma] + \cdots + a_k F[\sigma].$$

We show that there exists $b \in B$, $b \neq 0$ and $f(\sigma) \in F[\sigma]$, $f(\sigma) \neq 0$ with $bf(\sigma) = 0$. If not then B is a finitely generated $F[\sigma]$ -module which is torsion free as an $F[\sigma]$ -module. Then B is a free $F[\sigma]$ -module and hence there is a module homomorphism ϕ of B onto $F[\sigma]$. Now $\phi(a_0) = 0$, $\phi(a_1), \dots, \phi(a_k) \in F[\sigma]$ and these are not all zero since B is generated by $a_1, a_2, \dots, a_k$. Say $\phi(a_1) \neq 0$. Let t be the maximum degree of the polynomials $\phi(a_i)$. Since S is infinite we can choose $s \in S$ with $s > t$ and deduce that

$$\phi(a_{i_0}) + \phi(a_{i_1})\sigma^s + \phi(a_{i_2})\sigma^{2s} + \cdots + \phi(a_{i_r})\sigma^{rs} = 0.$$

By the irredundancy condition $\phi(a_1) \neq 0$ occurs here and since $s > \deg \phi(a_{i_j})$ we see that the above left-hand term is a nonzero polynomial in $F[\sigma]$, a contradiction. Thus for some $b \in B \subseteq A$, $b \neq 0$ there exists $f(\sigma) \in F[\sigma]$, $f(\sigma) \neq 0$ with $bf(\sigma) = 0$.

(i) Suppose A is torsion and assume that the situation is irredundant. Since each a_i has finite order we can choose an integer m and a prime q such that $a_i m q = 0$ for all i but $a_i m \neq 0$ for some i . Say $a_1 m \neq 0$. Let A_q denote the set of elements of A of order 1 or q . Then A_q is a $\text{GF}(q)[\sigma]$ -module and $a_i m \in A_q$ for all i . Now for each $s \in S$ we multiply the s -equation by m to obtain a corresponding equation in A_q . Since $a_1 m \neq 0$ occurs in each such equation we conclude from the above that there exists $b \in A_q$, $b \neq 0$ and $\bar{f}(\sigma) \in \text{GF}(q)[\sigma]$, $\bar{f}(\sigma) \neq 0$ with $b\bar{f}(\sigma) = 0$. Now $\text{GF}(q)$ is a field so we may suppose that $\bar{f}(\sigma)$ is monic and then we can choose $f(\sigma) \in \Lambda[\sigma]$ monic with $\bar{f}(\sigma) = f(\sigma) \bmod q$. Since $bq = 0$ we have $bf(\sigma) = 0$ so $b \in \text{moni } A$ and $\text{moni } A \neq \langle 0 \rangle$.

(ii) Since A is torsion free, A is naturally embedded in $A \otimes_{\Lambda} \mathfrak{Q}$ and $(A \otimes_{\Lambda} \mathfrak{Q})/A$ is torsion. Clearly $\mathfrak{Q}[\sigma]$ acts on $A \otimes_{\Lambda} \mathfrak{Q}$. Now each of the given equations can be viewed as an equation in $A \otimes_{\Lambda} \mathfrak{Q}$ so by our earlier observation there exists $\bar{f}(\sigma) \in \mathfrak{Q}[\sigma]$ and $\bar{b} \in A \otimes_{\Lambda} \mathfrak{Q}$ both not zero with $\bar{b}\bar{f}(\sigma) = 0$. Choose integers $m, n \neq 0$ with $b = \bar{b}m \in A$, $f(\sigma) = n\bar{f}(\sigma) \in \Lambda[\sigma]$. Then $b \neq 0$, $f(\sigma) \neq 0$ and $bf(\sigma) = 0$ so $b \in \text{poly } A$ and $\text{poly } A \neq \langle 0 \rangle$.

(iii) We suppose that the situation is irredundant and we may assume that

$$A = a_1 \Lambda[\sigma] + a_2 \Lambda[\sigma] + \cdots + a_k \Lambda[\sigma].$$

As above A is naturally embedded in $V = A \otimes_{\Lambda} \mathfrak{Q}$ and $\mathfrak{Q}[\sigma]$ acts on V . Since $A = \text{poly } A$ each a_i is annihilated by a nonzero polynomial in $\mathfrak{Q}[\sigma]$ and thus V is a finite dimensional $\mathfrak{Q}$ -vector space. Say $\dim_{\mathfrak{Q}} V = t$ and let $\{\omega_1, \omega_2, \dots, \omega_t\}$ be the set of eigenvalues of σ on V . Then this is a union of complete sets of algebraically conjugate elements over $\mathfrak{Q}$ and $\omega_i \neq 0$ since σ is one-to-one in its action on A .

Let $W = V \otimes_{\Omega} \Omega$ where $\Omega = \mathbb{Q}[\omega_1, \omega_2, \dots, \omega_t]$. Then $\Omega[\sigma]$ acts on W , $\dim_{\Omega} W = t$ and the eigenvalues of σ on W are precisely $\omega_1, \omega_2, \dots, \omega_t$. Let us study one such eigenvalue ω . From the Jordan canonical form for σ we see that there exists a nonzero linear functional $\phi: W \rightarrow \Omega$ with $\phi(b\sigma) = \phi(b)\omega$ for all $b \in W$. Multiplying ϕ by a rational integer if necessary we may suppose that $\phi(a_1), \phi(a_2), \dots, \phi(a_k)$ are algebraic integers. Moreover, since the a_i 's generate W as an $\Omega[\sigma]$ -module we cannot have $\phi(a_i) = 0$ for all i . Say $\phi(a_1) \neq 0$. Let c be the product of the nonzero $\phi(a_i)$'s so that $c \neq 0$ is an algebraic integer and let $m = \max\{|\phi(a_i)|/|\phi(a_j)| \mid \phi(a_j) \neq 0\}$.

Let $s \in S$. Then for some $r \geq 1$ we have

$$\phi(a_{i_0}) + \phi(a_{i_1})\omega^s + \phi(a_{i_2})\omega^{2s} + \dots + \phi(a_{i_r})\omega^{rs} = 0.$$

Since $\phi(a_1) \neq 0$ occurs in here there are at least two nonzero terms. Thus we may suppose that $r \geq 1$ is given above with $\phi(a_{i_r}) \neq 0$. Since the $\phi(a_{i_j})$ are algebraic integers this implies that $\phi(a_{i_r})\omega^{rs}$ is an algebraic integer and hence so is $c\omega^s$. Suppose now that $|\omega| > 1$. Then for all $s \in S$ the equation

$$\omega^{rs} = -\sum_{j=0}^{r-1} \left(\frac{\phi(a_{i_j})}{\phi(a_{i_r})} \right) \omega^{js}$$

yields

$$|\omega^{rs}| \leq m \sum_{j=0}^{r-1} |\omega^s|^j = \frac{m(|\omega|^{rs} - 1)}{(|\omega|^s - 1)} \leq \frac{m|\omega^{rs}|}{(|\omega|^s - 1)}.$$

Thus $|\omega|^s \leq m + 1$ for all $s \in S$, certainly a contradiction.

We have therefore shown that for each $i = 1, 2, \dots, t$ we have $|\omega_i| \leq 1$ and there exists a nonzero algebraic integer c_i such that $c_i \omega_i^s$ is an algebraic integer for all $s \in S$. By Lemma 2, each ω_i is a root of unity. Hence for some integer $n \geq 1$, $(\sigma^n - 1)$ has all eigenvalues 0 and thus $(\sigma^n - 1)^t$ annihilates all elements of $A \subseteq W$. Thus $\text{nilp } A \neq \langle 0 \rangle$ and the lemma is proved.

If H is a subgroup of G we let π_H^G denote the natural projection $\pi_H^G: K[G] \rightarrow K[H]$. This is of course a module homomorphism if we view $K[G]$ and $K[H]$ as either right or left $K[H]$ -modules. Thus in particular if I is a right ideal of $K[G]$, then $\pi_H^G(I)$ is a right ideal of $K[H]$.

Lemma 4. *Let $N \triangleleft H$ be $\langle \sigma \rangle$ -invariant groups with $H/N = A$ abelian. Then $\Omega[\sigma]$ acts on A . Suppose that either*

- (i) *A is torsion and $\text{moni } A = \langle 1 \rangle$,*
- (ii) *A is torsion free and $\text{poly } A = \langle 1 \rangle$, or*
- (iii) *$A = \text{poly } A$ is torsion free and $\text{nilp } A = \langle 1 \rangle$.*

If $\alpha \in K[H]$ is σ -nilpotent, then $\pi_N^H(\alpha) \in K[N]$ is σ -nilpotent.

Proof. Write $\alpha = \alpha_0 + \alpha_1 y_1 + \cdots + \alpha_k y_k$ where $\alpha_i \in K[N]$, $\alpha_0 = \pi_N^H(\alpha)$ and $y_0 = 1, y_1, y_2, \dots, y_k$ are in distinct cosets of N in H and hence represent distinct elements in $H/N = A$. Say $a_0 = 1 \in A$ and $a_i = Ny_i \in A$ for $i = 1, 2, \dots, k$.

Suppose by way of contradiction that α_0 is not σ -nilpotent. Since α is σ -nilpotent there clearly exists an infinite subset S of $\Lambda!$ such that, for any $s \in S$ and for some $r = r(s) \geq 1$,

$$\alpha \alpha^{\sigma^s} \alpha^{\sigma^{2s}} \cdots \alpha^{\sigma^{rs}} = 0 \quad \text{and} \quad \alpha_0 \alpha_0^{\sigma^s} \alpha_0^{\sigma^{2s}} \cdots \alpha_0^{\sigma^{rs}} \neq 0.$$

Now writing α as above we have the ordered products

$$\prod_{i=0}^r (\alpha_0^{\sigma^{is}} + \alpha_1^{\sigma^{is}} y_1^{\sigma^{is}} + \cdots + \alpha_k^{\sigma^{is}} y_k^{\sigma^{is}}) = 0 \quad \text{and} \quad \prod_{i=0}^r (\alpha_0^{\sigma^{is}}) \neq 0.$$

Since $\alpha_j^{\sigma^{is}} \in K[N]$ this clearly implies that there exists subscripts i_j not all zero with $y_{i_0} y_{i_1}^{\sigma^s} y_{i_2}^{\sigma^{2s}} \cdots y_{i_r}^{\sigma^{rs}} \in N$. Thus for the same subscripts $a_{i_0}^{\sigma^s} a_{i_1}^{\sigma^{2s}} a_{i_2}^{\sigma^{3s}} \cdots a_{i_r}^{\sigma^{rs}} = 1$. Since our assumptions contradict the results of Lemma 3 we conclude that $\pi_N^H(\alpha) = \alpha_0$ is σ -nilpotent.

Lemma 5. Let A be an abelian group acted upon by $\langle \sigma \rangle$ and let $a_1, a_2, \dots, a_n$ be a finite number of elements of A . We let $\langle a_1, a_2, \dots, a_n \rangle^{(\sigma)}$ denote the subgroup of A generated by the elements a_i and their conjugates under $\langle \sigma \rangle$.

- (i) If A is torsion and $A = \text{mon } A$, then $\langle a_1, a_2, \dots, a_n \rangle^{(\sigma)}$ is finite.
- (ii) If A is torsion free and $A = \text{nilp } A$, then for some integer $m \geq 1$, $\langle a_1, a_2, \dots, a_n \rangle^{(\sigma^m)}$ has an ordering compatible with the action of σ^m .

Proof. (i) Since $\langle a_1, a_2, \dots, a_n \rangle^{(\sigma)} = \langle a_1 \rangle^{(\sigma)} \langle a_2 \rangle^{(\sigma)} \cdots \langle a_n \rangle^{(\sigma)}$ it suffices to show that, for any $a \in A$, $\langle a \rangle^{(\sigma)}$ is finite. Let $a \in A$. Then $a^m = 1$ for some $m \geq 1$ and $a^{f(\sigma)} = 1$ for some monic polynomial $f(\sigma) \in \Lambda[\sigma]$. It follows that $a^{\Lambda[\sigma]}$ is a homomorphic image of the additive abelian group $\Lambda[\sigma]/(m\Lambda[\sigma] + f(\sigma)\Lambda[\sigma])$.

Let $g(\sigma) \in \Lambda[\sigma]$. Since $f(\sigma)$ is monic we have $g(\sigma) = q(\sigma)f(\sigma) + r(\sigma)$ where $\deg r(\sigma) < \deg f(\sigma)$. Since there are only finitely many possibilities for $r(\sigma)$ modulo $m\Lambda[\sigma]$ it follows that $\Lambda[\sigma]/(m\Lambda[\sigma] + f(\sigma)\Lambda[\sigma])$ is finite. Thus $a^{\Lambda[\sigma]}$ is finite.

Finally $a^{\Lambda[\sigma]}$ is invariant under σ and σ acts in a one-to-one manner. Since $a^{\Lambda[\sigma]}$ is finite we see that σ maps this group onto itself so $a^{\Lambda[\sigma]}$ is also invariant under σ^{-1} . Thus $\langle a \rangle^{(\sigma)} = a^{\Lambda[\sigma]}$ is finite.

- (ii) This is essentially the fact that a torsion free nilpotent group can be ordered. Since $A = \text{nilp } A$ for each i there exists $k_i, m_i \geq 1$ with $a_i^{(\sigma^{m_i-1})^{k_i}} = 1$. Since

$$\prod_{i=1}^n (\sigma^{m_i} - 1)^{k_i} \mid (\sigma^{m_1 m_2 \cdots m_n} - 1)^{k_1 + k_2 + \cdots + k_n}$$

we see that $(\sigma^m - 1)^k$ acts trivially on $\langle a_1, a_2, \dots, a_n \rangle$ where $m = m_1 m_2 \cdots m_n$ and $k = k_1 + k_2 + \cdots + k_n$. Thus clearly $(\sigma^m - 1)^k$ acts trivially on $\langle a_1, a_2, \dots, a_n \rangle^{(\sigma)} = B$.

Let $\tau = \sigma^m$ and for each $j \leq k$ set

$$B_j = \{b \in B \mid b^{(\tau-1)^j} = 1\}.$$

Then $\langle 1 \rangle = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_k = B$ and clearly B/B_j is torsion free. We now order each of the torsion free abelian quotients B_j/B_{j-1} for $j = 1, 2, \dots, k$. Observe that τ centralizes each such quotient so each of these orderings is compatible with the action of τ . Finally we order B . Let $b \in B$, $b \neq 1$ and choose j so that $b \in B_j - B_{j-1}$. Then we say $b > 1$ if and only if $bB_{j-1} > B_{j-1}$ under the ordering of B_j/B_{j-1} . It is now easy to check that this yields an ordering of B compatible with the action of $\tau = \sigma^m$.

We now come to the main result of this section.

Proposition 6. *Let $N \triangleleft H$ be $\langle \sigma \rangle$ -invariant groups with H/N abelian. Let $\alpha \in K[H]$, $\alpha \neq 0$ and suppose that $\alpha K[H]$ is a σ -nil right ideal. Then there exist $y \in \text{Supp } \alpha$ and a $\langle \sigma \rangle$ -invariant subgroup N^* of H with $[N^* : N] < \infty$ such that $\pi_{N^*}^H(\alpha y^{-1})K[N^*]$ is a σ -nil right ideal of $K[N^*]$.*

Proof. Set $A = A_0 = H/N$ and we define a series of $\langle \sigma \rangle$ -invariant subgroups of A . Let A_3 be the torsion subgroup of A . Then A/A_3 is torsion free and since $\Lambda[\sigma]$ acts on A/A_3 we set $A_1/A_3 = \text{poly}(A/A_3)$ and $A_2/A_3 = \text{nilp}(A/A_3)$. Moreover, A_3 is torsion and $\Lambda[\sigma]$ acts on A so we set $A_4 = \text{mon}_1 A_3$. We then have $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \langle 1 \rangle$. Define $\langle \sigma \rangle$ -invariant subgroups $H = N_0 \supseteq N_1 \supseteq N_2 \supseteq N_3 \supseteq N_4 \supseteq N$ by $N_i/N = A_i$.

Let $\beta \in \alpha K[H]$ so that β is σ -nilpotent. Since $H/N_1 \cong A/A_1$ is torsion free and $\text{poly } A/A_1 = \langle 1 \rangle$ by Lemma 1(ii) it follows from Lemma 4(ii) that $\pi_{N_1}^H(\beta)$ is σ -nilpotent. Since $N_1/N_2 \cong A_1/A_2$ is torsion free, $A_1/A_2 = \text{poly } A_1/A_2$ and $\text{nilp } A_1/A_2 = \langle 1 \rangle$ by Lemma 1(iii), it follows from Lemma 4(iii) that $\pi_{N_2}^H(\beta) = \pi_{N_2}^{N_1} \pi_{N_1}^H(\beta)$ is σ -nilpotent. Thus $\pi_{N_2}^H(\alpha K[H])$ is a σ -nil right ideal of $K[N_2]$.

Write $\pi_{N_2}^H(\alpha) = \sum_1^k \alpha_i y_i$ where $\alpha_i \in K[N_3]$, $1 \in \text{Supp } \alpha_i$ and $y_1, y_2, \dots, y_k$ are in distinct cosets of N_3 in N_2 . Thus $y_1, y_2, \dots, y_k$ represent distinct elements $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_k \in N_2/N_3$. Now $N_2/N_3 = \text{nilp } N_2/N_3$ so by Lemma 5 (ii) the group $\langle \bar{y}_1, \bar{y}_2, \dots, \bar{y}_k \rangle^{(\sigma)}$ has an ordering compatible with the action of σ^m for some fixed $m \geq 1$. Fix such an ordering and suppose that the above subscripts are so chosen that $\bar{y}_k > \bar{y}_i$ for $i \neq k$. Then $\bar{1} > \bar{y}_i \bar{y}_k^{-1}$ and $y_k \in \text{Supp } \alpha$ since $1 \in \text{Supp } \alpha_k$.

Now $\pi_{N_3}^H(\alpha y_k^{-1}) = \alpha_k$ and we show that $\alpha_k K[N_3]$ is a σ -nil right ideal of $K[N_3]$. Let $\gamma \in K[N_3]$. Then $\alpha y_k^{-1} \gamma \in \alpha K[H]$ is σ -nilpotent and hence so is $\pi_{N_2}^H(\alpha y_k^{-1} \gamma) = \pi_{N_2}^H(\alpha) y_k^{-1} \gamma$ since $y_k^{-1} \gamma \in K[N_2]$. Thus $\sum_{i=1}^k \alpha_i y_i y_k^{-1} \gamma$ is σ -nilpotent. Let S be the infinite subset of $\Lambda!$ associated with this property. Since there are at most finitely many elements of $\Lambda!$ not divisible by m we may assume that for all $s \in S$ we have $m \mid s$. Thus σ^s also preserves the ordering. Finally let $s \in S$. Then for some $r = r(s) \geq 1$ we have the ordered product

$$\prod_{j=0}^r \left(\sum_{i=1}^k \alpha_i^{\sigma^{js}} y_i^{\sigma^{js}} (y_k^{-1})^{\sigma^{js}} \gamma^{\sigma^{js}} \right) = 0.$$

Since $\alpha_i^{\sigma^{js}} \in K[N_3]$, $\gamma^{\sigma^{js}} \in K[N_3]$ and $\bar{y}_i^{\sigma^{js}} (\bar{y}_k^{-1})^{\sigma^{js}} < \bar{1}$ for $i < k$ we conclude that

$$\prod_{j=0}^r \alpha_k^{\sigma^{js}} \gamma^{\sigma^{js}} = 0.$$

Thus $\alpha_k \gamma = \pi_{N_3}^H(\alpha y_k^{-1}) \gamma$ is σ -nilpotent.

We have therefore shown that there exists $y \in \text{Supp } \alpha$ with $\pi_{N_3}^H(\alpha y^{-1}) K[N_3] = I$ a σ -nil right ideal. Let $\beta \in I$. Since $N_3/N_4 \cong A_3/A_4$ is torsion and $\text{moni } A_3/A_4 = \langle 1 \rangle$ by Lemma 1(i), we conclude that $\pi_{N_4}^{N_3}(\beta)$ is σ -nilpotent by Lemma 4(i). Since $\pi_{N_4}^{N_3}(I) \supseteq \pi_{N_4}^H(\alpha y^{-1}) K[N_4]$ we see that the latter is a σ -nil right ideal of $K[N_4]$.

Now $N_4/N \cong A_4$ and $A_4 = \text{moni } A_4$. Since $(\text{Supp } \pi_{N_4}^H(\alpha y^{-1}))N/N$ is finite, there exists by Lemma 5(i) a finite $\langle \sigma \rangle$ -invariant subgroup N^*/N of A_4 with $\text{Supp } \pi_{N_4}^H(\alpha y^{-1}) \subseteq N^*$. Thus $\pi_{N^*}^H(\alpha y^{-1}) = \pi_{N_4}^H(\alpha y^{-1})$ and $\pi_{N^*}^H(\alpha y^{-1}) K[N_4] \supseteq \pi_{N^*}^H(\alpha y^{-1}) K[N^*]$ so the latter is a σ -nil right ideal of $K[N^*]$. This completes the proof.

2. The main theorems. In this section we apply the above proposition to the semisimplicity problem. We first generalize it slightly.

Lemma 7. Let $N \triangleleft H$ be $\langle \sigma \rangle$ -invariant groups with H/N a solvable Δ -group. Let $\alpha \in K[H]$, $\alpha \neq 0$ and suppose that $\alpha K[H]$ is a σ -nil right ideal. Then there exists $y \in \text{Supp } \alpha$ and a $\langle \sigma \rangle$ -invariant subgroup N^* of H with $[N^* : N] < \infty$ such that $\pi_{N^*}^H(\alpha y^{-1}) K[N^*]$ is a σ -nil right ideal of $K[N^*]$.

Proof. Let $H = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_k = N$ correspond to the derived series of solvable H/N . Then certainly each N_i is $\langle \sigma \rangle$ -invariant. We show by induction on i that there exists $\langle \sigma \rangle$ -invariant N_i^* with $H \supseteq N_i^* \supseteq N_i$, $[N_i^* : N_i] < \infty$ and $y_i \in \text{Supp } \alpha$ such that $\pi_{N_i^*}^H(\alpha y_i^{-1}) K[N_i^*]$ is a σ -nil right ideal of $K[N_i^*]$. For $i = 0$ we take $N_0^* = H = N_0$, and y_0 any element of $\text{Supp } \alpha$.

Let us suppose the result holds for some $i < k$. We consider the quotient N_i^*/N_{i+1} which has an abelian subgroup N_i/N_{i+1} of finite index. Since $H/N = \Delta(H/N)$ this implies easily that N_i^*/N_{i+1} has a center of finite index and hence (see Lemma 2.1 of [3]) N_i^*/N_{i+1} has a finite commutator subgroup M/N_{i+1} . Thus M is $\langle \sigma \rangle$ -invariant, $[M : N_{i+1}] < \infty$ and N_i^*/M is abelian.

By Proposition 6 applied to $M \triangleleft N_i^*$ there exists a $\langle \sigma \rangle$ -invariant subgroup N_{i+1}^* of N_i^* with $[N_{i+1}^* : M] < \infty$ and an element $z \in \text{Supp } \pi_{N_i^*}^H(\alpha y_i^{-1})$ such that $\pi_{N_{i+1}^*}^{N_i^*}(\pi_{N_i^*}^H(\alpha y_i^{-1}) \cdot z^{-1})K[N_{i+1}^*]$ is a σ -nil right ideal of $K[N_{i+1}^*]$. Since $[N_{i+1}^* : M] < \infty$ and $[M : N_{i+1}] < \infty$ we have $[N_{i+1}^* : N_{i+1}] < \infty$. Now $z \in N_i^*$ implies that $\pi_{N_i^*}^H(\alpha y_i^{-1}) \cdot z^{-1} = \pi_{N_i^*}^H(\alpha y_i^{-1} z^{-1})$. Also $z \in \text{Supp } \pi_{N_i^*}^H(\alpha y_i^{-1})$ implies that there exists $y_{i+1} \in \text{Supp } \alpha$ with $z = y_{i+1} y_i^{-1}$. Thus $\alpha y_i^{-1} z^{-1} = \alpha y_{i+1}^{-1}$ and by the above $\pi_{N_{i+1}^*}^H(\alpha y_{i+1}^{-1})K[N_{i+1}^*]$ is σ -nil. The induction step is proved and the $i = k$ case yields the result.

At this point the characteristic of the field comes into play. We assume throughout the remainder of this paper that K has characteristic $p > 0$. If $\alpha \in K[G]$ we define

$$\text{Quot } \alpha = \{xy^{-1} \mid x, y \in \text{Supp } \alpha\},$$

$$p\text{-Quot } \alpha = \{g \in \text{Quot } \alpha \mid g \neq 1 \text{ has order a power of } p\}.$$

Proposition 8. *Let H be a $\langle \sigma \rangle$ -invariant solvable Δ -group and let $\alpha \in K[H]$, $\alpha \neq 0$. If $\alpha K[H]$ is a σ -nil right ideal of $K[H]$ then there exists $g \in p\text{-Quot } \alpha$ and an integer $n \geq 1$ such that σ^n centralizes g .*

Proof: We apply Lemma 7 with $N = \langle 1 \rangle$. Then there exists a finite $\langle \sigma \rangle$ -invariant subgroup N^* of H and $y \in \text{Supp } \alpha$ such that $\pi_{N^*}^H(\alpha y^{-1})K[N^*]$ is a σ -nil right ideal of $K[N^*]$. Since N^* is finite there exists an integer n such that σ^n centralizes N^* . Now $\beta = \pi_{N^*}^H(\alpha y^{-1})$ is σ -nilpotent and since there are only finitely many elements of Λ ! not divisible by n it follows that for some s with $n \mid s$ we have $\beta^{\sigma^{s+1}} = \beta \beta^{\sigma^s} \beta^{\sigma^{2s}} \dots \beta^{\sigma^{rs}} = 0$. Thus β is nilpotent. Now $y \in \text{Supp } \alpha$ implies that $1 \in \text{Supp } \alpha y^{-1}$ so $1 \in \text{Supp } \beta$. Then Lemma 3.5 of [3] implies that there exists $g \in \text{Supp } \beta$, $g \neq 1$ such that g has order a power of p . Clearly, $g \in p\text{-Quot } \alpha$ and σ^n centralizes $g \in N^*$. The result follows.

We can now obtain our main results. If W is a subgroup of G we let

$$\sqrt{W} = \{x \in G \mid x^n \in W \text{ for some integer } n \geq 1\}.$$

Observe that $\sqrt{W}$ need not be a subgroup of G even if G is assumed to be solvable.

Theorem A. *Let K be a field of characteristic $p > 0$ and let H be a normal solvable Δ -subgroup of G . Suppose that $\alpha \in JK[G] \cap K[H]$ with $\alpha \neq 0$. Then*

$$G = \bigcup_{g \in p\text{-Quot } \alpha} \sqrt{C_G(g)}.$$

Moreover, if G is solvable then for some $g \in p\text{-Quot } \alpha$ we have $[G : C_G(g)] = 1$.

Proof: Let $x \in G$. We distinguish two cases according to whether the image of x in G/H has finite or infinite order.

Suppose first that $x^t \in H$ for some integer t . If $G_1 = \langle H, x \rangle$ then clearly $[G_1 : H] < \infty$. Now by Lemma 16.9 of [3], $\alpha \in JK[G] \cap K[H] \subseteq JK[H]$ and thus, by Lemma 19.6 of [3], $\alpha K[H]$ is nilpotent. Choose $y \in \text{Supp } \alpha$ so that $1 \in \text{Supp } \alpha y^{-1}$ and αy^{-1} is nilpotent. By Lemma 3.5 of [3] there exists an element $g \neq 1$ of order a power of p with $g \in \text{Supp } \alpha y^{-1}$. Thus $g \in p\text{-Quot } \alpha$. Finally $H = \Delta(H)$ so $[H : C_H(g)] < \infty$ and $[G_1 : C_{G_1}(g)] < \infty$. This implies that $x^n \in C_{G_1}(g)$ for some integer $n \geq 1$ and $x \in \sqrt{C_G(g)}$.

Now suppose that x has infinite order modulo H . Then x has infinite order and the map $\sigma^i \rightarrow x^i$ affords an isomorphism of the two infinite cyclic groups $\langle \sigma \rangle$ and $\langle x \rangle$. Since $\langle x \rangle$ acts on H we then obtain a natural action of $\langle \sigma \rangle$ on H . Observe that for all integers $s \in \mathbb{N}$, x^s has infinite order modulo H and thus Lemma 21.3 of [3] implies that $JK[G] \cap K[H]$ is a σ -nil ideal of $K[H]$. By Proposition 8, there exists $g \in p\text{-Quot } \alpha$ such that σ^n centralizes g for some $n \geq 1$. Since σ and x act the same way we have $x \in \sqrt{C_G(g)}$ and the first result follows.

Finally if G is solvable then the main result of [4] implies that for some $g \in p\text{-Quot } \alpha$ we have $[G : C_G(g)] = 1$.

Theorem B. Let K be a field of characteristic $p > 0$ and let G be a solvable group. Suppose H is a normal Δ -subgroup of G . Then $JK[G] \cap K[H] \neq 0$ if and only if H has an element b of order p with $[G : C_G(b)] = 1$.

Proof: Suppose $JK[G] \cap K[H] \neq 0$ and choose $\alpha \neq 0$ in this intersection. By Theorem A there exists $g \in p\text{-Quot } \alpha$ with $[G : C_G(g)] = 1$. Let $b \in \langle g \rangle$ have order p . Then $C_G(b) \supseteq C_G(g)$ so $[G : C_G(b)] = 1$.

Conversely suppose that there exists $b \in H$ of order p with $[G : C_G(b)] = 1$. Let $W = \langle b \rangle^H$. Then W is a finite normal subgroup of H whose order is divisible by p . We show that $JK[W] \subseteq JK[G]$. Since $JK[W] \neq 0$ and $JK[W] \subseteq K[H]$, this will yield $JK[G] \cap K[H] \neq 0$.

Since W is finite, it clearly suffices to show that if L is a finitely generated subgroup of G with $L \supseteq W$ then $JK[W] \subseteq JK[L]$. Now by definition $[L : C_L(b)] < \infty$ so since $C_L(b)$ clearly normalizes W we have $[L : N_L(W)] < \infty$. Observe that $N_L(W) \supseteq W$. Let N denote the core of $N_L(W)$ in L , that is, the intersection of all conjugates of $N_L(W)$. Then $[L : N] < \infty$ and $N \triangleleft L$. Since $W \triangleleft L \cap$

$H \triangleleft L$ and $W \subseteq N_L(W)$ we have $W \subseteq N$. Clearly $W \triangleleft N$. By Lemma 19.4 of [3], $JK[W] \subseteq JK[N]$ and, by Theorem 16.6 of [3], $JK[N] \subseteq JK[L]$. Thus $JK[W] \subseteq JK[L]$ and the result follows.

It is interesting to consider an example now. Let $A = Z_p$ be the cyclic group of order p and let $B = \prod Z_p$ be an infinite direct product of copies of Z_p . If $G = A \wr B$ and if K has characteristic $p > 0$ then, by Theorem 21.6 of [3], $JK[G]$ is the augmentation ideal of $K[G]$. Now G has a normal abelian subgroup $H \neq \{1\}$ so clearly $JK[G] \cap K[H] \neq 0$. By Lemma 21.5(ii) of [3] we see that H has no element b of order p with $[G : C_G(b)] < \infty$. Thus the condition $[G : C_G(b)] = 1.f.$ in the above theorem cannot be replaced by the condition $[G : C_G(b)] < \infty$.

Theorem C. *Let K be a field of characteristic $p > 0$ and let $H \triangleleft G$ with $\Delta(H)$ solvable. Let I be an ideal of $K[G]$ with $I \cap K[H]$ nilpotent. Suppose $\alpha \in I \cap K[H]$ with $\alpha \neq 0$. Then*

$$G = \bigcup_{g \in (p\text{-Quot } \alpha) \cap \Delta(H)} \sqrt{C_G(g)}.$$

Moreover, if G is solvable then there exists $g \in (p\text{-Quot } \alpha) \cap \Delta(H)$ with $[G : C_G(g)] = 1.f.$

Proof. Clearly $I \cap K[H]$ is G -invariant and hence so is $\theta(I \cap K[H])$ where θ is the projection map $\theta: K[H] \rightarrow K[\Delta(H)]$. By Lemma 20.1(ii) of [3], $\theta(I \cap K[H])$ is a nilpotent ideal of $K[\Delta(H)]$. Observe that $\Delta(H)$ is a normal solvable Δ -subgroup of G .

Let $x \in G$ and let $\beta \in \theta(I \cap K[H])$. Then $\beta^{x^{is}} \in \theta(I \cap K[H])$ since $\theta(I \cap K[H])$ is G -invariant and hence $\beta\beta^{x^s}\beta^{x^{2s}} \dots \beta^{x^{rs}} = 0$ for some $r \geq 1$ since the ideal is nilpotent. If $\langle \alpha \rangle$ is an infinite cyclic group we let $\langle \sigma \rangle$ act on $\Delta(H)$ by way of the homomorphism $\sigma^j \rightarrow x^j$. By the above $\theta(I \cap K[H])$ is a σ -nil ideal of $K[\Delta(H)]$.

Let $\alpha \in I \cap K[H]$ with $\alpha \neq 0$ and let $z \in \text{Supp } \alpha$. Then $1 \in \text{Supp } \alpha z^{-1}$ so $\theta(\alpha z^{-1}) \neq 0$. By Proposition 8 there exists $g \in p\text{-Quot } \theta(\alpha z^{-1})$ and an integer $n \geq 1$ such that σ^n centralizes g . Thus $x \in \sqrt{C_G(g)}$. Since

$$g \in p\text{-Quot } \theta(\alpha z^{-1}) \subseteq p\text{-Quot } \alpha z^{-1} = p\text{-Quot } \alpha$$

the first part follows.

Finally if G is solvable then the main result of [4] implies that for some $g \in p\text{-Quot } \alpha$ we have $[G : C_G(g)] = 1.f.$

At this point it is apparent that a strengthening of some results of A. E. Zalesskii in [5] will solve the semisimplicity problem for solvable groups. As an indication we offer the following

Proposition 9. *Let K be a field of characteristic $p > 0$ and let G be metanilpotent. Say N is a normal nilpotent subgroup of G with G/N nilpotent and set $H = \Delta(N\Delta(G))$. Then $JK[G] \neq 0$ if and only if there exists an element $b \in H$ of order p with $[G : C_G(b)] = l.f.$*

Proof. If such an element $b \in H$ exists then by Theorem B we have $JK[G] \cap K[H] \neq 0$ so $JK[G] \neq 0$.

Conversely let us assume that $JK[G] \neq 0$. Suppose $K[G]$ has a nonzero nilpotent ideal. Then by Theorem 20.2 of [3], $JK[G] \cap K[\Delta(G)] \neq 0$ so certainly $JK[G] \cap K[H] \neq 0$. On the other hand, if $K[G]$ has no nonzero nilpotent ideal, then by Theorem 2 of [5] we again have $JK[G] \cap K[H] \neq 0$. The result follows from Theorem B.

Added in proof. The semisimplicity problem for group rings of solvable groups has been solved. Indeed A. E. Zalesskiĭ has proved the following lovely result.

Theorem. *Let G be a solvable group. Then G has a normal Δ -subgroup H with the following property: if I is a nonzero ideal of $K[G]$ then $I \cap K[H] \neq 0$.*

We remark that the subgroup H above is a characteristic subgroup of G given by a specific construction. It seems reasonable to call it the *Zalesskiĭ subgroup* and denote it by $\mathfrak{Z}(G)$. Then the above combines with our Theorem B to prove

Theorem. *Let K be a field of characteristic $p > 0$ and let G be a solvable group. Then $JK[G] \neq 0$ if and only if $\mathfrak{Z}(G)$ contains an element b of order p with $[G : C(b)] = l.f.$*

More recently Professor Zalesskiĭ has extended these arguments slightly to show that if G is finitely generated and solvable then $JK[G] = NK[G]$. From this one obtains easily

Theorem. *Let K be a field of characteristic $p > 0$ and let G be a solvable group. Then*

$$JK[G] = JK[H] \cdot K[G]$$

where H is the characteristic locally finite subgroup of G given by

$$H = \{x \in G \mid [G : C(x)] = l.f. \text{ and } x \text{ has finite order}\}.$$

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