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# THE EXISTENCE OF Irr($X$)

BY

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ABSTRACT. If $X$ is a compact totally ordered space, we obtain the existence of an irreducible semigroup with idempotents $X$, Irr($X$), with the property that any irreducible semigroup with idempotents $X$ is the idempotent separating surmorphic image of Irr($X$). Furthermore, it is shown that the Clifford-Miller endomorphism on Irr($X$) is an injection when restricted to each $\mathcal{H}$-class of Irr($X$). A construction technique for noncompact semigroups is given, and some results about the structure of such semigroups are obtained.

**Introduction.** A semigroup $S$ is *irreducible* if $S$ is a compact connected semigroup with identity 1 having no proper compact connected subsemigroup containing 1 and meeting $M(S)$, the minimal ideal of $S$. If $X$ is a compact totally ordered space, $(X, \min)$ will denote the semigroup $X$ under the operation

$$xy = \text{minimum}\{x, y\} \quad \text{for each } x, y \in X.$$

A semigroup $S$ *has idempotents* $X$ if $E(S) \simeq (X, \min)$, $E(S)$ being the set of idempotents of $S$. The main result of this paper obtains the existence of an irreducible semigroup with idempotents $X$, denoted Irr($X$), with the property that a compact semigroup $S$ is irreducible with idempotents $X$ if and only if $S$ is the idempotent separating surmorphic image of Irr($X$). Hofmann and Mostert attempted to construct Irr($X$) in Chapter B, §5 of [3], but there were errors in their construction. The first section of this paper is devoted to pointing out those errors. We next describe a technique which generalizes the construction of generalized hormoi to noncompact semigroups, and then establish some properties of the semigroups so constructed. In the main section of this work we establish the existence of Irr($X$), and in the last section we give a counterexample to another proposed structure for Irr($X$). The notation and terminology will be that of [3], and the reader is advised to review the definitions of a chainable collection and of a hormos on pp. 139–143 of that volume. The duality theory used will be Pontryagin Duality Theory for locally compact abelian groups, and standard references are [2] and [5]. This work forms part of the author's doctoral dissertation, and he wishes to express his deep gratitude to J. H. Carruth for his many helpful suggestions and his patient listening during its preparation. Thanks go also to A. D. Wallace for his

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helpful hints in the preparation of this manuscript.

**Some counterexamples.** When Hofmann and Mostert attempted to construct $\mathrm{Irr}(X)$, they proceeded as follows. First, they constructed a collection $(X, S'_x, m'_{xy})$ which they thought to be a chainable collection, and they let $\mathrm{Irr}(X) = \mathrm{Horm}(X, S'_x, m'_{xy})$. If $S$ is an irreducible semigroup with idempotents $X$, according to results in [3], $S = \mathrm{Horm}(X, S_x, m_{xy})$ for some chainable collection $(X, S_x, m_{xy})$, i.e. $S$ is a hormos. In order to construct the required idempotent separating surmorphism of $\mathrm{Irr}(X)$ onto $S$, they first constructed a surmorphism $\psi_x: H'_x \to H_x$, where $H'_x$ and $H_x$ are the groups of units of $S'_x$ and $S_x$, respectively, for each $x \in X$. They then extended $\psi_x$ to a surmorphism of $S'_x$ onto $S_x$ for each $x \in X$, and, using these extensions, they induced an idempotent separating surmorphism of $\mathrm{Irr}(X)$ onto $S$. The errors in this approach are two-fold. First, the collection $(X, S'_x, m'_{xy})$ failed to satisfy all the conditions for a chainable collection, and second, the theorem used to extend each $\psi_x$ to $S'_x$ is false. We treat these deficiences in the above order.

For $(X, S_x, m_{xy})$ to be a chainable collection, it must satisfy the following. If $x \in X$ is isolated from below in $X$, then $\phi_x: H_x \to \Pi_{y<x} H_y$ defined by $\phi_x(h) = (m_{yx}(h))_{y<x}$ is an isomorphism of $H_x$ onto $\varprojlim\{H_y, m_{yz}, y < x\}$, where, for each $w \in X$, $H_w$ is the group of units of $S_w$. In particular, this condition must be satisfied by the collection $(X, S'_x, m'_{xy})$ constructed by Hofmann and Mostert, but the following shows that it is not.

For each $x \in X$, Hofmann and Mostert let $H'_x = (\mathbf{R}_d^{[x,1]_{'X}})^\wedge$, where $\mathbf{R}_d$ is the discrete group of real numbers, ${}'X = \{x \in X \setminus \{1\}: x \text{ is isolated from above in } X\}$, $[x, 1]_{'X} = [x, 1] \cap {}'X$, and, for a locally compact abelian group $G$, $G^\wedge$ denotes the character group of $G$. If $x \le y \in X$, they defined $w'_{xy}: \mathbf{R}_d^{[x,1]_{'X}} \to \mathbf{R}_d^{[y,1]_{'X}}$ by $[w'_{xy}(f)][z] = f(z)$ for each $z \in [y, 1]_{'X}$ if $[y, 1]_{'X} \neq \square$, while $w'_{xy}(f) = 0$ if $[y, 1]_{'X} = \square$, and they let $m'_{xy}: H'_y \to H'_x$ be defined by $m'_{xy} = w'^{\phi}_{xy}$, the adjoint of $w'_{xy}$. Now, since inverse limits and direct limits are dual, the above condition on the groups is equivalent to the following. If $x \in X$ is not isolated from below in $X$, then

$$\phi'_x: \varinjlim\{\mathbf{R}_d^{[y,1]_{'X}}, w'_{yz}, y \le z < x\} \to \mathbf{R}_d^{[x,1]_{'X}}$$

defined by $\phi'_x(\eta_{y_0}(f)) = w'_{y_0 x}(f)$ is an isomorphism, where $f \in \mathbf{R}_d^{[y_0,1]_{'X}}$ and

$$\eta_{y_0}: \mathbf{R}_d^{[y_0,1]_{'X}} \to \varinjlim\{\mathbf{R}_d^{[y,1]_{'X}}, w'_{yz}, y \le z < x\}$$

is the natural map. We now give an example due to J. H. Carruth where $\phi'_x$ is not an isomorphism.

Let $X = \{1\} \cup \{1 - 1/n\}_{n \in \omega}$ under the natural order. Then 1 is not isolated from below in $X$, and so

$$\phi_1': \varinjlim\{\mathbf{R}_d^{[x,1]_{'X}}, w'_{xy}, x \le y < 1\} \to \mathbf{R}_d^{[1,1]_{'X}}$$

should be an isomorphism. But, $'X$ is the set of points $x \in X\backslash\{1\}$ with $x$ isolated from above in $X$, whence $'X = X\backslash\{1\}$. Thus, for $x \neq 1$, $[x, 1]_{'X}$ is infinite. If $f \in \mathbf{R}_d^{[0,1]_{'X}}$ with $f(z) = 1$ for each $z \in [0, 1]_{'X}$, then $w'_{0x}(f) \neq 0_x$ for any $x \in (0, 1]$, where $0_x$ is the identity of $\mathbf{R}_d^{[x,1]_{'X}}$. Therefore if

$$\eta_0: \mathbf{R}_d^{[0,1]_{'X}} \to \varinjlim\{\mathbf{R}_d^{[x,1]_{'X}}, w'_{xy}, x \le y < 1\}$$

is the natural map, $\eta_0(f) \neq \eta_0(0)$ and so $\varinjlim\{\mathbf{R}_d^{[x,1]_{'X}}, w'_{xy}, x \le y < 1\}$ is nontrivial. But $\mathbf{R}_d^{[1,1]_{'X}} = \mathbf{R}_d^{\square} = \{0\}$, and hence we clearly cannot have the required isomorphism, whence $(X, S'_x, m'_{xy})$ is not in general a chainable collection.

We now turn our attention to the other error in Hofmann and Mostert's attempted proof. As remarked above, if $S$ is an irreducible semigroup with idempotents $X$, then $S = \mathrm{Horm}(X, S_x, m_{xy})$ for some chainable collection $(X, S_x, m_{xy})$, and to obtain the desired surmorphism of $\mathrm{Irr}(X)$ onto $S$, they first constructed a surmorphism $\psi_x$ of $H'_x$ onto $H_x$ for each $x \in X$. They then attempted to extend $\psi_x$ to a surmorphism of $S'_x$ onto $S_x$, and the result they used to do this is the following.

**Statement.** Let $\phi: \Sigma \times G \to S$ be a surmorphism and suppose $\psi: M(\Sigma) \times G \to M(S)$ is a given surmorphism with $\psi(M(\Sigma) \times \{1\}) = \phi(M(\Sigma) \times \{1\})$ and $\psi((\infty, 0), g) = \phi((\infty, 0), g)$ for each $g \in G$. Then $\psi$ can be extended to a surmorphism of $\Sigma \times G$ onto $S$ with $\psi((0, 0), g) = \phi((0, 0), g)$ for each $g \in G$.

Here $\Sigma = \{(r, s(r)): r \in \mathbf{H}\}^* \subseteq \mathbf{H}^* \times \hat{\mathbf{R}_d}$ is the universal solenoidal semigroup (see [3, p. 71]), and $G$ is an arbitrary compact group. The following is a counterexample to this statement. Let $G = \{1\}$, $S = \Sigma$, $\phi: \Sigma \to \Sigma$ be the identity, and define $\psi: M(\Sigma) \to M(\Sigma)$ by $\psi((\infty, h)) = (\infty, h^{-1})$. Then $\phi(M(\Sigma)) = \psi(M(\Sigma))$ as required, but $\psi$ is not extendable to all of $\Sigma$. For suppose $\psi(p, s(p)) = (r, s(r))$ for $r \in \mathbf{H}\backslash\{0\}$. Then

$$\begin{aligned}(\infty, s(p)^{-1}) &= \psi(\infty, s(p)) = \psi[(\infty, 0)(p, s(p))]\\ &= \psi(\infty, 0)\,\psi(p, s(p)) = (\infty, 0)(r, s(r)) = (\infty, s(r)),\end{aligned}$$

whence $s(p)^{-1} = s(r)$, contradicting the fact that $s$, being the adjoint of the identity map of $\mathbf{R}_d$ into $\mathbf{R}$, is one to one. This completes the counterexample.

Later, we shall give a counterexample to another proposed structure for $\mathrm{Irr}(X)$, but this requires information which should properly appear during the proof of the existence of $\mathrm{Irr}(X)$.

**Linkable collections and $\mathcal{H}$-linked semigroups.** As noted above, the collection $(X, S'_x, m'_{xy})$ constructed by Hofmann and Mostert failed to satisfy all the properties of a chainable collection. However, using techniques identical to those for generating a hormos from a chainable collection, a semigroup can be constructed from a collection such as $(X, S'_x, m'_{xy})$ which shares all the properties of a hormos save compactness. Moreover, a semigroup so constructed plays a central role in our proof of the existence of $\mathrm{Irr}(X)$. In this section, we investigate such semigroups and we begin with the following definition.

**Definition 3.1.** $(X, S_x, m_{xy})$ is a *linkable collection* if

(a) $X$ is a compact totally ordered space with maximal element 1 and minimal element 0. $X' = \{x \in X\setminus\{0\}: x \text{ is isolated from below}\}$, and, if $x \in X'$, $x' = \sup[0, x)$.

(b) For each $x \in X$, $S_x$ is a compact semigroup with identity $1_x$ and minimal ideal $M_x$ satisfying:

(i) $M_x$ is a group with identity $e_x$.

(ii) If $x \notin X'$, then $S_x = M_x = H_x$ is a group, where $H_x$ is the group of units of $S_x$; if $x \in X'$, $H_x \cap M_x = \square$.

(iii) If $x \neq y$, then $S_x \cap S_y = \square$.

(c) For each pair $x, y \in X$ with $x \leq y$, there is a homomorphism $m_{xy}: S_y \to S_x$ satisfying:

(i) $m_{xx}$ is the identity.

(ii) If $x < y$, then $m_{xy}(S_y) \subset H_x$.

(iii) If $x \leq y \leq z$, then $m_{xy} \circ m_{yz} = m_{xz}$.

(iv) $M_{xy} \mid M_y$ is an injection if $y \in X'$ and $x = y'$.

(v) If $x \notin (X' \cup \{0\})$, then $\phi_x: H_x \to \Pi_{y<x} H_y$ defined by $\phi_x(h) = (m_{yx}(h))_{y<x}$ is an injection of $H_x$ into $\varprojlim\{H_y, m_{yz}, y \leq z < x\}$.

Note that any chainable collection is linkable.

**Lemma 3.2.** *Let $(X, S_x, m_{xy})$ be a linkable collection and let $S' = \bigcup_{x\in X} S_x$. If $s, t \in S'$ with $s \in S_x$, $t \in S_y$, and $z = x \wedge y$, define $st = m_{zx}(s)m_{zy}(t)$. With this multiplication $S'$ is an algebraic semigroup, and $S'$ is commutative if and only if each $S_x$ is commutative.*

**Proof.** This is the same as the proof of 5.2, p. 140 of [3].

**Lemma 3.3.** *Let $(X, S_x, m_{xy})$ be a linkable collection and let $S' = \bigcup_{x\in X} S_x$. Let $\mathfrak{B}$ be the basis of open intervals in $X$. If $U \in \mathfrak{B}$, $u = \inf U$, and $V \subseteq S_u$ is open, define $W(U, V) = \{s \in S': s \in S_x$ for some $x \in U$ and $m_{ux}(s) \in V\}$. Then $\mathcal{O} = \{W(U, V): U \in \mathfrak{B}$ and $V \subseteq S_{\inf U}$ is open$\}$ is a basis for a topology on $S'$ relative to which $S'$ is a topological semigroup when endowed with the multiplication of Lemma 3.2.*

**Proof.** Again, the proof is the same as that of 5.3, p. 140 of [3]. We should note two things, however. First the inclusion in $\mathcal{O}$ of the basis $\mathfrak{B}_x$ of $S_x\setminus H_x$ for each $x \in X'$ in [3] is superfluous. For, if $V \in \mathfrak{B}_x$, then $V = W(U, V)$ where $U = [x, 1]$. Secondly, although the fact that $H_x \simeq \varprojlim\{H_y, m_{yz}, y \leq z < x\}$ for $x$ not isolated from below is quoted at certain points in the proof in [3], condition (c(v)) of Definition 3.1 is easily seen to be sufficient.

**Lemma 3.4.** *Let* $(X, S_x, m_{xy})$ *be a linkable collection and let* $S' = \bigcup_{x \in X} S_x$ *be the topological semigroup constructed in Lemmas* 3.2 *and* 3.3. *If R is the relation on* $S'$ *whose cosets are* $\{s, m_{x'x}(s)\}$ *if* $s \in M_x$ *for some* $x \in X'$, *and* $\{s\}$ *otherwise, then R is a congruence on* $S'$ *and* $S'/R$ *is a topological semigroup when endowed with the quotient topology.*

**Proof.** The proof that $R$ is a congruence is the same as that for 5.5, p. 142 of [3]. To show $S'/R$ is a topological semigroup under the quotient topology, according to [1] it suffices to show $R$ forms an upper semicontinuous decomposition with compact cosets. As $R$ clearly has compact cosets, we show only that $R$ is upper semicontinuous. Moreover, it suffices to show $\bigcup\{R[s]\colon R[s] \subseteq V\}$ is open for each $V \in \mathcal{O}$, the basis for $S'$ defined in Lemma 3.3.

Let $W(U, V)$ be a basic open subset of $S'$, and let $s \in W(U, V)$. If $R[s] = \{s\}$, then $R[s] \subseteq W(U, V)$. Let $u = \inf U$, and suppose $R[s] = \{s, t\}$ with $s \in H_x$, $t \in M_y$, $y \in X'$, $x = y'$, and $m_{xy}(t) = s$. Then $m_{uy}(t) = m_{ux}(m_{xy}(t)) = m_{ux}(s) \in V$ as $s \in W(U, V)$, and so $t \in W(U, V)$ if $y \in U$. If $R[s] = \{s, m_{x'x}(s)\}$ where $s \in M_x$ and $x \in X'$, then $m_{x'x}(s) \in W(U, V)$ if $x' \in U$, since $m_{ux'}(m_{x'x}(s)) = m_{ux}(s) \in V$ as $s \in W(U, V)$, whence $R[s] \subseteq W(U, V)$ if $x' \in U$. Thus, the only possible cases under which $R[s] \not\subseteq W(U, V)$ for $s \in W(U, V)$ are

(a) $s \in M_x$ for $x \in X'$, $x = \inf U \in U$ and $x \neq 0$, or

(b) $s \in m_{x'x}(M_x)$ for $x \in X'$, $x' = \sup U \in U$ and $x \neq 1$.

Thus, $\bigcup\{R[s]\colon R[s] \subseteq W(U, V)\} = W(U, V)\backslash(A \cup B)$, where

$$A = \begin{cases} M_x & \text{if } x = \inf U \in U,\ x \neq 0, \text{ and} \\ \square & \text{otherwise;} \end{cases}$$

$$B = \begin{cases} m_{x'x}(M_x) & \text{if } x' = \sup u \in U,\ x' \neq 1, \\ \square & \text{otherwise;} \end{cases}$$

and, as $A$ and $B$ are closed, $\bigcup\{R[s]\colon R[s] \subseteq W(U, V)\}$ is open, whence $R$ is upper semicontinuous.

**Definition 3.5.** Let $(X, S_x, m_{xy})$ be a linkable collection and let $S' = \bigcup_{x \in X} S_x$ be the topological semigroup constructed in Lemmas 3.2 and 3.3. If $R$ is the congruence on $S'$ described in Lemma 3.4, then $S = S'/R$ is called a *linked semigroup* and is denoted by $S = \mathfrak{L}(X, S_x, m_{xy})$. If each $S_x$ is cylindrical, then $S$ is called an $\mathcal{H}$*-linked semigroup* and is denoted by $S = \mathcal{H}\mathfrak{L}(X, S_x, m_{xy})$.

Note that if $(X, S_x, m_{xy})$ is a linkable collection, then $R \cap (S_x \times S_x)$ is the diagonal of $S_x$ for each $x \in X$, whence we may identify $S_x$ with its image in $S = \mathfrak{L}(X, S_x, m_{xy})$. Furthermore, it is easy to see that the topology induced on $S_x$

from $\mathfrak{L}(X, S_x, m_{xy})$ is the same as the original topology on $S_x$. We conclude this section with the following.

**Proposition 3.6.** *Let $S = \mathfrak{L}(X, S_x, m_{xy})$ be a linked semigroup. If $A$ is any of Green's relations on $S$, then $A \cap (S_x \times S_x) = A_x$ for each $x \in X$, where $A_x$ is the relation $A$ on $S_x$. Moreover, if $S = \mathcal{H}\mathfrak{L}(X, S_x, m_{xy})$ and $0 = \inf X$, then $S$ is connected if and only if $S_0$ is connected.*

**Proof.** We prove the first part only for $A = \mathcal{R}$, the other proofs being similar. Since $S_x$ is a subsemigroup of $S$, we have $A_x \subseteq A \cap (S_x \times S_x)$. Conversely, suppose $(s, t) \in A \cap (S_x \times S_x)$. Then there are $a, b \in S$ with $s = ta$ and $t = sb$. If $a \in S_y$ and $b \in S_z$, then $s = ta = m_{wx}(t)m_{wy}(a)$ where $w = x \wedge y$. But, as $s \in S_x$, $w = x$, and so $x = x \wedge y$, whence $s = tm_{xy}(a)$ with $m_{xy}(a) \in S_x$. Similarly $x \leq z$ and $t = sm_{xz}(b)$ with $m_{xz}(b) \in S_x$, and so $(s, t) \in A_x$ completing the proof.

If $S = \mathcal{H}\mathfrak{L}(X, S_x, m_{xy})$ is connected, then $S_0$ is connected as $S_0 = M(S)$. Conversely, suppose $S_0$ is connected and $S = P \cup Q$. We suppose $S_0 \subseteq P$ and show $Q = \square$. If $A = \{x \in X: S_y \subseteq P \text{ for } 0 \leq y \leq x\}$, then $A \neq \square$ as $0 \in A$, and we let $z = \sup A$.

*Claim* 1. $z \in A$.

**Proof.** Either $z \in X'$ or not. If $z \in X'$, then $M_z \subseteq S_{z'} \subseteq P$ as $z' \in A$. Since $S_z$ is cylindrical, there is a surmorphism $f: \Sigma \times H_z \longrightarrow S_z$, and so, if $s \in S_z$, then $s \in f(\Sigma \times \{h\})$ for some $h \in H_z$. Since $f(\Sigma \times \{h\})$ is connected and meets $M_z$, we have $f(\Sigma \times \{h\}) \subseteq P$. Thus $S_z \subseteq P$.

If $z \notin X'$, then $S_z = H_z$ and $\{m_{yz}(h)\}_{y<z} \longrightarrow h$ for each $h \in H_z$. Moreover since $m_{yz}(h) \in P$ for each $y < x$, $h \in P^* = P$. Hence $S_z \subset P$ in this case also, which establishes the claim.

*Claim* 2. $z = 1$.

**Proof.** If $z \neq 1$, then we have two possibilities. Either $z = y'$ for some $y \in X'$ or $z \neq y'$ for any $y \in X'$.

If $z = y'$ for some $y \in X'$, then $M_y \subseteq S_z \subseteq P$, and an argument similar to that in the proof of Claim 1 shows $S_y \subseteq P$, whence $y \in A$, contradicting $z = \sup A$.

Thus, $z \neq y'$ for any $y \in X'$, and so there is a net $\{y_\alpha\}_{\alpha \in D} \subseteq X$ with $z < y_\alpha$ and $S_{y_\alpha} \cap Q \neq \square$ for each $\alpha \in D$. If $s_\alpha \in S_{y_\alpha} \cap Q$ for each $\alpha \in D$, then there are an $h \in S_z$ and a subnet $\{s_\beta\}_{\beta \in E}$ of $\{s_\alpha\}_{\alpha \in D}$ with $\{m_{zy_\beta}(s_\beta)\}_{\beta \in E} \longrightarrow h$ in $S_z$. It then follows that $\{s_\beta\}_{\beta \in E} \longrightarrow h$ in $S$, and so $h \in Q^* = Q$, contradicting $h \in S_z \subseteq P$. This contradiction proves Claim 2.

Thus $S \subseteq P$, and so $Q = \square$.

**The existence of** $\mathrm{Irr}(X)$. We now turn our attention to the main result of this paper. Our proof consists of three parts: We first construct a semigroup with idempotents $X$, which we denote $\mathrm{Irr}(X)_0$, with the property that any irreducible semigroup with idempotents $X$ is the image of $\mathrm{Irr}(X)_0$ under an idempotent separating surmorphism; then, using $\mathrm{Irr}(X)_0$, we construct a compact semigroup $T$ with the property that any irreducible semigroup with idempotents $X$ is the surmorphic image of $T$; finally, we map Irr$(X)_0$ into $T$ in a natural way, and find that any irreducible subsemigroup of the closure of the image of Irr$(X)_0$ in $T$ satisfies the properties desired for Irr($X$). In the last part of this section, we refine the construction of Irr$(X)_0$ and point out instances when Irr$(X)_0$ is compact, thus giving an explicit description of Irr($X$) in those instances. We begin by quoting some results we shall need in our proof.

**Theorem 4.1** [Koch [4]]. *Let $S$ be a compact connected semigroup with identity* 1 *and minimal ideal $M(S) \neq S$. If each subgroup of $S$ is totally disconnected, then there is an I-semigroup running from* 1 *to $M(S)$.*

Throughout this work we shall refer to Theorem 4.1 as Koch's Theorem.

**Theorem 4.2** [3, p. 150]. *Let $S = \mathrm{Horm}(X, S_x, m_{xy})$ be an abelian hormos. $S$ is irreducible if and only if each of the following is satisfied:*

(i) $H_1 = \{1\}$.

(ii) *If $x \neq y'$ for any $y \in X'$, then $H_x = \langle \bigcup_{x<y\in X} 1_x \cdot H_y\rangle^*$.*

(iii) *If $x \in X'$, then $M_x = H_{x'}$.*

**Theorem 4.3** [3, p. 151]. *Let $S = \mathrm{Horm}(X, S_x, m_{xy})$ be an irreducible semigroup with idempotents $X$. For each $y \in X'$, let $f_y: \Sigma \times H_y \to S_y$ be a surmorphism with $f_y((0, 0), h) = h$ for each $h \in H_y$. Then, for each $x \in X$ with $(x, 1] \cap X' \neq \square$,*

$$H_x = \left\langle \bigcup_{x<y\in X'} \{m_{xy}(f_y(M(\Sigma) \times \{1_y\}))\}\right\rangle^*.$$

**Theorem 4.4.** *If $X$ is a compact totally ordered space, there is a linkable collection $(X, S'_x, m'_{xy})$ with $S'_x = \Sigma \times H'_x$ for each $x \in X'$ so that $\mathrm{Irr}(X)_0 = \mathcal{H}\mathfrak{L}(X, S'_x, m'_{xy})$ is a connected semigroup. Moreover, if $S$ is a compact semigroup, then $S$ is irreducible with idempotents $X$ if and only if $S$ is the idempotent separating surmorphic image of* Irr$(X)_0$.

**Proof.** We break the proof into three parts:

(a) The construction of Irr$(X)_0$,

(b) the proof of the necessity in the last statement of theorem, and

(c) the proof of the sufficiency in the last statement of the theorem.

For part (a), the construction of Irr$(X)_0$, recall that if $X$ is a compact totally ordered space, $X' = \{x \in X\backslash\{0\}: x$ is isolated from below$\}$, where $0 = \inf X$, and if $x \in X'$, then $x' = \sup\{y \in X: y < x\}$. Also $1 = \sup X$.

We first construct the maximal subgroups of Irr$(X)_0$. For each $x \in X$, let $G'_x = \mathbf{R}_d^{(x,1]_{X'}}$, and, if $x \leq y \in X$, define $w'_{xy}: G'_x \to G'_y$ by $[w'_{xy}(f)][z] = f(z)$ if $z \in (y, 1]_{X'}$, while $w'_{xy}(f) = 0$ if $(y, 1]_{X'} = \square$ for each $f \in G'_x$. Clearly $G'_x$ is an abelian group for each $x \in X$ and $w'_{xy}$ is a surmorphism of $G'_x$ onto $G'_y$ if $x \leq y \in X$. Furthermore, that $w'_{xy} \circ w'_{yz} = w'_{xz}$ if $x \leq z \in X$ is also clear, and we note that $G'_{x'} \simeq \mathbf{R}_d \times G'_x$ for each $x \in X'$.

Now, let $H'_x = (G'_x)\hat{}$ for each $x \in X$, and, if $x \leq y \in X$, let $m'_{xy}: H'_y \to H'_x$ be defined by $m'_{xy} = w'^{\phi}_{xy}$. Since $w'_{xy}$ is a surmorphism, $m'_{xy}$ is an injection. Moreover, if $x \leq y \leq z \in X$,

$$m'_{xy} \circ m'_{yz} = w'^{\phi}_{xy} \circ w'^{\phi}_{yz} = w'^{\phi}_{xz} = m'_{xz},$$

and $H'_{x'} = (G'_{x'})\hat{} \simeq \hat{\mathbf{R}}_d \times H'_x$ if $x \in X$.

If $x \notin X'$, let $S'_x = H'_x$, while we let $S'_x = \Sigma \times H'_x$ if $x \in X'$. If $x \in X'$, extend $m'_{x'x}$ to $S'_x$ by noting that $M'_x \simeq H'_{x'}$ under $((\infty, g), h) \to (g, h)$, and that $m'_{x'x}$ is the core endomorphism of $S'_x$ followed by that isomorphism. Then, if $y < x$, let $m'_{yx} = m'_{yx'} \circ m'_{x'x}$. Clearly the collection $(X, S'_x, m'_{xy})$ satisfies all the properties of a linkable collection, the last following from the fact that $m'_{xy}$ is an injection for each pair $x, y \in X$ with $x \leq y$. If $\mathrm{Irr}(X)_0 = \mathfrak{HL}(X, S'_x, m'_{xy})$, then $S'_0 = H'_0 = (G'_0)\hat{}$, where $G'_0 = \mathbf{R}_d^{(0,1]_{X'}}$. Since $G'_0$ is torsion-free and discrete, $H'_0$ is divisible and compact, whence $H'_0 = S'_0$ is connected, and so $\mathrm{Irr}(X)_0$ is connected by Proposition 3.6. Clearly $\mathrm{Irr}(X)_0$ has idempotents $X$. This concludes the proof of part (a).

For part (b), the proof of the necessity in the last statement of the theorem, suppose $S = \mathrm{Horm}(X, S_x, m_{xy})$ is an irreducible semigroup with idempotents $X$. For each $y \in X'$, $S_y$ is cylindrical, and so there is a surmorphism $f_y: \Sigma \times H_y \to S_y$ with $f_y((0, 0), h) = h$ for each $h \in H_y$ [3, p. 88], and let $F_y = (M(\Sigma) \times \{1_y\})\hat{} \simeq \mathbf{R}_d$.

Let $x_1 = \inf C_X(1)$, where $C_X(1)$ is the component of $X$ containing 1. Then, by [3, 5.11, p. 145], $S' = (\bigcup\{S_y: x_1 < y\})^*$ is an irreducible subsemigroup of $S$, and since $\{1_y: x_1 \leq y\}$ is a compact connected subsemigroup of $S'$, we have $S' = \{1_y: x_1 \leq y\}$. Hence $H_y = \{1_y\}$ for each $y \in C_X(1)$.

Now if $x \in X$ with $(x, 1]_{X'} = \square$, Koch's Theorem implies there is an arc in $X$ from 1 to $x$, whence $x \in C_X(1)$. Thus, if $x \in X$ and $x \notin C_X(1)$, $(x, 1]_{X'} \neq \square$ and so, by Theorem 4.3, we have

$$H_x = \Big\langle \bigcup_{x<y\in X'} \{m_{xy}(f_y(M(\Sigma) \times \{1_y\}))\} \Big\rangle^*. \tag{1}$$

Now, if $x \in C_X(1)$, let $\psi_x: \hat{H}_x = G_x \to G'_x$ be the obvious map. If $x \in X$ with $(x, 1]_{X'} \neq \square$ define $\psi_x(g) = (f^{\phi}_y(m^{\phi}_{xy}(g)))_{x<y\in X'}$. $\psi_x$ is clearly a homomorphism and it follows easily from (1) that $\psi_x$ is one to one.

Now, if $x \leq z \in X$, $(z, 1]_{X'} \neq \square$, and $g \in G_x$, then

$$\begin{aligned} w'_{xz}(\psi_x(g)) &= w'_{xz}[(f^{\phi}_y(m^{\phi}_{xy}(g)))_{x<y\in X'}] = (f^{\phi}_y(m^{\phi}_{xy}(g)))_{z<y\in X'} \\ &= (f^{\phi}_y(m^{\phi}_{zy}(m^{\phi}_{xz}(g))))_{z<y\in X'} = \psi_x(m^{\phi}_{xz}(g)), \end{aligned}$$

whence the diagram

$$\begin{array}{ccc} G'_z & \xleftarrow{\ \psi_z\ } & G_z \\ {\scriptstyle w'_{xz}}\uparrow & & \uparrow{\scriptstyle m^{\phi}_{xz}} \\ G'_x & \xleftarrow[\ \psi_x\ ]{} & G_x \end{array}$$

commutes. If $z \in C_X(1)$, the above diagram is clear.

Now dualizing, if $\sigma_x = \psi_x^{\phi}$, then $\sigma_x\colon H'_x \to H_x$ is a surmorphism for each $x \in X$, and, if $x \leq z \in X$, the diagram

$$\begin{array}{ccc} H'_z & \xrightarrow{\ \sigma_z\ } & H_z \\ {\scriptstyle m'_{xz}}\downarrow & & \downarrow{\scriptstyle m_{xz}} \\ H'_x & \xrightarrow[\ \sigma_x\ ]{} & H_x \end{array}$$

commutes. In particular, if $z \in X'$ and $x = z'$, then $G'_x \simeq \mathbf{R}_d \times G'_x$ and $\psi_x = (f^{\phi}_z \circ m^{\phi}_{xz}) \times (\psi_z \circ m^{\phi}_{xz})$, and so $H'_x \simeq \hat{\mathbf{R}}_d \times H'_z$ and, if $(g, h) \in H'_x$, then

(2) $$\sigma_x(g, h) = m_{xz}(f_z((\infty, g), 1_z))m_{xz}(\sigma_z(h)) \quad \text{by duality.}$$

We now define $\bar{f}_S\colon \bigcup_{x \in X} S'_x \to \bigcup_{x \in X} S_x$ as follows: If $x \notin X$, then $S'_x = H'_x$ and we let $\bar{f}_S(h) = \sigma_x(h)$ for each $h \in S'_x$. If $x \in X'$, $S'_x = \Sigma \times H'_x$, and we let $\bar{f}_S((r, g), h) = f_x((\mathrm{id} \times \sigma_x)((r, g), h))$, where we recall that $f_x\colon \Sigma \times H_x \to S_x$ with $f_x((0, 0), h) = h$ for each $h \in H_x$ is the surmorphism guaranteed by the fact that $S_x$ is cylindrical. Hence, $\bar{f}_S((r, g), h) = f_x((\mathrm{id} \times \sigma_x)((r, g), h)) = f_x((r, g), \sigma_x(h)) = f_x((r, g), 1_x)\,\sigma_x(h)$.

Suppose now $x \in X'$ and let $h \in H'_x$. Then,

$$\bar{f}_S((0, 0), h) = f_x((0, 0), \sigma_x(h)) = \sigma_x(h),$$

whence $\bar{f}_S \mid H'_x = \sigma_x$. If $((\infty, g), h) \in M'_x$, then

$$\begin{aligned} m_{x'x}[\bar{f}_S((\infty, g), h)] &= m_{x'x}[f_x((\infty, g), \sigma_x(h))] \\ &= m_{x'x}[f_x(((\infty, g), 1_x)((0, 0), \sigma_x(h)))] = m_{x'x}[f_x((\infty, g), 1_x)f_x((0, 0), \sigma_x(h))] \\ &= m_{x'x}(f_x((\infty, g), 1_x))m_{x'x}(f_x(0, 0), \sigma_x(h)) = \sigma_{x'}(m'_{x'x}((\infty, g), h)), \end{aligned}$$

the last equality following from equation (2). We note that $f_S \mid S'_x$ is a surmorphism of $S'_x$ onto $S_x$ for each $x \in X$. Using these facts it is tedious but not hard to show $\bar{f}_S$ is a homomorphism.

To show $\bar{f}_S$ is continuous, let $s' \in S'_x$ with $\bar{f}_S(s') = s \in W(U, V)$. Then $s \in S_x$ and if $u = \inf U$, then $x \in U$ and $\bar{m}_{ux}(s) \in V$, and so, if $V' = \bar{f}_S^{-1}(V) \cap S_u$, then $V'$ is an open subset of $S'_u$ since $\bar{f}_S \mid S_u = \bar{f}_u \circ (\mathrm{id} \times \sigma_u)$ is continuous. Now, $s \in W(U, V')$ and it is easily shown that $\bar{f}_S(W(U, V')) \subseteq W(U, V)$, whence $\bar{f}_S$ is indeed continuous.

We wish to induce $f_S$: $\mathrm{Irr}(X)_0 \to S$ from $\bar{f}_S$: $\bigcup_{x \in X} S'_x \to \bigcup_{x \in X} S_x$. To do this, we show $\rho_{\eta'} \subseteq \rho_{\eta \circ \bar{f}_S}$, where, for a function $f$: $A \to B$, $\rho_f = \{(a, b) \in A \times A: f(a) = f(b)\}$, and $\eta'$ and $\eta$ are the natural maps from $\bigcup_{x \in X} S'_x$ to $\mathrm{Irr}(X)_0$ and from $\bigcup_{x \in X} S_x$ to $S$, respectively. If $\eta'(s) = \eta'(t)$ and $s \neq t$, then we may assume $s \in M'_x$ for some $x \in X'$ and $t = m'_{x'x}(s)$ by the symmetry of $\rho_{\eta'}$. Then $\bar{f}_S(s) \in M_x$, $\bar{f}_S(t) \in H'_x$, and $\bar{m}_{x'x}(\bar{f}_S(s)) = \bar{f}_S(m'_{x'x}(s)) = \bar{f}_S(t)$, whence $\eta(\bar{f}_S(s)) = \eta(\bar{f}_S(s))$, and so $f_S$ is indeed induced. Furthermore, in the proof of Lemma 3.4, it was shown that if $(Y, T_y, m_{yz})$ is a linkable collection, the congruence defined in that lemma is upper continuous, whence the natural map is closed. Thus $\eta'$ is closed, and since $\bar{f}_S$ and $\eta$ are continuous, $f_S$ is continuous. $f_S$ is clearly idempotent separating, and since $\bar{f}_S$ and $\eta$ are surmorphisms, $f_S$ is a surmorphism, thus completing the proof of part (b).

For part (c), the proof of the sufficiency in the last statement of the theorem, suppose $S$ is a compact semigroup and $f$: $\mathrm{Irr}(X)_0 \to S$ is an idempotent separating surmorphism. Then $f(X) \simeq X$ and $f(X) \subseteq E(S)$. Since $f$ is a surmorphism, if $e \in E(S)$, there is $x \in X$ with $e \in f(S'_x)$, and since $S'_x$ is compact, there is $e' \in E(S'_x)$ with $f(e') = e$. Thus, $e \in f(X)$, and so $E(S) = f(X) \simeq X$.

Since $\mathrm{Irr}(X)_0$ is a connected abelian $\mathcal{H}$-chain, $f(\mathrm{Irr}(X)_0) = S$ is a connected abelian $\mathcal{H}$-chain, and so $\mathcal{H}$ is a congruence on $S$ and $S/\mathcal{H}$ is an arc. Hence, by [3, p. 122, 11(b)], $S/\mathcal{H}$ is an $I$-semigroup, and therefore by [3, p. 143, 5.7], $S = \mathrm{Horm}(X, S_x, m_{xy})$, where $S_x = \eta^{-1}(x)$ if $x \notin X'$, while $S_x = \eta^{-1}(x', x]^*$ if $x \in X'$, $\eta$: $S \to S/\mathcal{H}$ being the natural map, and, of course, we assume the $S_x$'s are pairwise disjoint. We now show $f(S'_x) = S_x$ for each $x \in X$.

If $x \notin X'$, then $S_x = H_x$ and clearly $f(S'_x) \subseteq S_x$. Moreover, if $s \in S_x$, then there is $t \in \mathrm{Irr}(X)_0$ with $f(t) = s$. There is $z \in X$ with $\{(1_x t)^{n_\alpha}\}_{\alpha \in D} \to 1_z$ for some subnet of $\{(1_x t)^n\}_{n \in \omega}$, and so $f(1_z) = \lim f(1_x t)^{n_\alpha} = \lim f(1_x)^{n_\alpha} f(t)^{n_\alpha} = \lim s^{n_\alpha} = 1_x$. Hence we have $1_x t \in S'_x$ and $f(1_x t) = s$, whence $S_x \subseteq f(S'_x)$. Thus $S_x = f(S'_x)$ if $x \notin X'$.

If $x \in X'$, then $\eta \circ f \mid S'_x$ is a surmorphism of $S'_x$ onto $[x', x]$, and recalling that $S'_x = \Sigma \times H'_x$, there is $p \geq 0$ with $(\eta \circ f)((r, s(r)), h) \in (x', x]$ if $r \leq p$. Moreover, $p > 0$ as $f$ is idempotent separating, and so, if $A = \{((r, s(r)), h): r \leq p, h \in H'_x\}$, then $f(A) \subseteq \eta^{-1}(x', x]$, whence $f(A) \subseteq S_x$. Furthermore, $\langle A \rangle^* = S'_x$, and so $f(S'_x) = f(\langle A \rangle^*) \subseteq f(\langle A \rangle)^* = \langle f(A) \rangle^* \subseteq S_x$, the last inequality following from the facts that $f(A) \subseteq S_x$ and $S_x$ is compact, whence $f(S'_x) \subseteq S_x$. Conversely, if $s \in S_x$, then $x' \leq \eta(s) \leq x$, and so $t \in \mathrm{Irr}(X)_0$ with $f(t) = s$ implies $f(1_x t) = s$ and $1_x t \in S'_x$

by an argument similar to that for $x \notin X'$. Thus $S_x \subseteq f(S'_x)$, and we have $S_x = f(S'_x)$ for each $x \in X$.

Now to show $S$ is irreducible, by Theorem 4.2 it suffices to show that $H_1 = \{1\}$, $H_x = \langle \bigcup \{1_x H_y : x < y\}\rangle^*$ if $x \neq y'$ for any $y \in X'$, and $M_x = H_{x'}$ if $x \in X'$. The first is clear as $H'_1 = \{1\}$ and $f(H'_1) = H_1$. If $x \neq y'$ for any $y \in X'$, then $H_x = f(H'_x)$, and since $H'_x = \langle \bigcup_{x<y\in X} 1_x \cdot H'_y\rangle^*$, clearly, we have $H_x = f(H'_x) = f(\langle \bigcup_{x<y} 1_x \cdot H'_y\rangle^*) \subseteq \langle \bigcup_{x<y} 1_x \cdot f(H'_y)\rangle^* = \langle \bigcup_{x<y} 1_x \cdot H_y\rangle^*$. Since the reverse containment is obvious, $H_x = \langle \bigcup_{x<y} 1_x \cdot H_y\rangle^*$. Finally, if $x \in X'$, then $M_x = f(M'_x) = f(H'_{x'}) = H_{x'}$, the middle equality following from the structure of $\mathrm{Irr}(X)_0$. Hence $S$ is indeed irreducible, thus concluding the proof of the theorem.

We now state and prove a lemma which we will later use to show that the Clifford-Miller endomorphism on Irr($X$) is an injection when restricted to each $\mathcal{H}$-class of Irr($X$).

**Lemma 4.5.** *Suppose $s \in H'_r \subseteq \mathrm{Irr}(X)_0$ with $s \neq 1_r$ for some $r \in X$, and suppose there are an irreducible semigroup $S$ with idempotents $X$ and an idempotent separating surmorphism $f$: $\mathrm{Irr}(X)_0 \to S$ with $f(s) \neq f(1_r)$. Then there are an irreducible semigroup $S'$ with idempotents $X$ and an idempotent separating surmorphism $f'$: $\mathrm{Irr}(X)_0 \to S'$ with $f'(m'_{yr}(s)) \neq f'(1_y)$ for each $y \in X$ with $y \leq r$.*

**Proof.** Since $S$ is irreducible with idempotents $X$, $S = \mathrm{Horm}(X, S_x, m_{xy})$ for some chainable collection $(X, S_x, m_{xy})$. From this collection we construct another chainable collection $(X, T_x, \overline{m}_{xy})$ with $S' = \mathrm{Horm}(X, T_x, \overline{m}_{xy})$ an irreducible semigroup with idempotents $X$.

If $r < y \in X$, let $T_y = S_y$, while if $y \leq r$, $y \notin X'$, let $T_y = H_r$. Finally, if $y \leq r$ and $y \in X'$, let $T_y = \mathbf{H}^* \times H_r$. If $r \leq x \leq y$, we let $\overline{m}_{xy} = m_{xy}$, while if $x \leq y \leq r$, $\overline{m}_{xy}$ is the identity if $y \notin X'$, and $\overline{m}_{xy}$ is the identity map composed with the core endomorphism of $T_y$ if $y \in X'$, where for $z \in X'$ with $z \leq r$, we identify $H_z$ with $\{0\} \times H_z \subseteq \mathbf{H}^* \times H_z$. Lastly, if $x \leq r \leq y$, let $\overline{m}_{xy} = \overline{m}_{xr} \circ \overline{m}_{ry}$. It is a simple task to show that $(X, T_x, \overline{m}_{xy})$ is a chainable collection, and we let $S = \mathrm{Horm}(X, T_x, \overline{m}_{xy})$.

We now construct an idempotent separating surmorphism $f'$: $\mathrm{Irr}(X)_0 \to S$. It clearly suffices to construct a surmorphism $\bar{f}$: $\bigcup_{x\in X} S'_x \to \bigcup_{x\in X} T_x$ with $f(S'_x) = T_x$ for each $x \in X$. If $r < x \in X$, define $\bar{f}|S'_x = \bar{f}_x = f|S'_x$. If $x \leq r$, then $H'_x = K'_x \oplus m'_{xr}(H'_r)$ since $G'_x \simeq \mathbf{R}_d^{(x,r]} X' \times G'_r$. If $x \notin X'$, then $S'_x = H'_x$, and we let $\bar{f}_x(k, h) = \overline{m}_{xr}(f(m'^{-1}_{xr}(h)))$. If $x \in X$, then $S'_x = \Sigma \times H'_x$, and we let $\bar{f}_x((r, g), (k, h)) = (r, 1_x)\, \overline{m}_{xr}(f(m'^{-1}_{xr}(h)))$. Clearly $\bar{f}_x$ is continuous and $\bar{f}_x(S'_x) = T_x$ for each $x \in X$. If $r < x$, then $\bar{f}_x$ is a homomorphism as $f$ is. If $x \leq r$ and $x \notin X'$, then $\bar{f}_x = \overline{m}_{xr} \circ f \circ m'^{-1}_{xr} \circ \pi_{2x}$ where $\pi_{2x}$: $H'_x \to m'_{xr}(H'_r)$ is the natural map, and so $\bar{f}_x$ is a homomorphism. If $x \leq r$ with $x \in X'$, then

$$\bar{f}_x(s) = [\pi_{1x}(s)]\,[\overline{m}_{xr} \circ f \circ m'^{-1}_{xr} \circ \pi_{2x}(s)],$$

where $\pi_{1x}: \Sigma \times H'_x \to \mathrm{H}^*$ and $\pi_{2x}: \Sigma \times H'_x \to m'_{xr}(H'_r)$ are the natural maps. Hence, since $T_x$ is abelian, $\bar{f}_x$ is a homomorphism. We define $\bar{f}: \bigcup_{x\in X} S'_x \to \bigcup_{x \in X} T_x$ by $\bar{f} \mid S'_x = \bar{f}_x$ for each $x \in X$. It is now tedious but not difficult to show that $\bar{f}$ is a homomorphism, and an argument similar to that in the proof of part (b) of Theorem 4.4 yields $\bar{f}$ is continuous.

It now follows from Theorem 4.4 that $S$ is irreducible with idempotents $X$ since $f'$: $\mathrm{Irr}(X)_0 \to S$ is clearly idempotent separating. Finally, if $y \leq r$, then $\bar{f}(m'_{yr}(s)) = \bar{f}(1_y \cdot s) = \bar{f}(1_y)\bar{f}(s) = \overline{m}_{yr}(f(s)) \neq 1_y$ as $\overline{m}_{yr}$ is the identity and $f(s) \neq 1_r$. This completes the proof of the lemma.

We now state and prove our main result.

**Main Theorem.** *Let $X$ be a compact totally ordered space. There is a irreducible semigroup with idempotents $X$, denoted by* $\mathrm{Irr}(X)$, *such that, for any compact semigroup $S$, $S$ is irreducible with idempotents $X$ if and only if $S$ is the idempotent separating surmorphic image of* $\mathrm{Irr}(X)$. *Moreover, the Clifford-Miller endomorphism on* $\mathrm{Irr}(X)$ *is an injection when restricted to each* $\mathcal{H}$*-class of* $\mathrm{Irr}(X)$.

**Proof.** As remarked at the beginning of this section, to obtain this result we first construct a compact semigroup $T$ which maps onto any irreducible semigroup with idempotents $X$. We then map $\mathrm{Irr}(X)_0$ into $T$ under a natural map, $\Phi$, and find that $\Phi(\mathrm{Irr}(X)_0)^*$ is a compact connected semigroup with idempotents $X$ which maps onto any irreducible semigroup with idempotents $X$ under an idempotent separating surmorphism. Finally we find that any irreducible subsemigroup of $\Phi(\mathrm{Irr}(X)_0)^*$ running from its identity to its minimal ideal satisfies the properties desired for $\mathrm{Irr}(X)$. In the course of the proof we shall also obtain results which show that the Clifford-Miller endomorphism on $\mathrm{Irr}(X)$ is an injection when restricted to each $\mathcal{H}$-class of $\mathrm{Irr}(X)$.

According to Theorem 4.4, if $S$ is an irreducible semigroup with idempotents $X$, then there is an idempotent separating surmorphism $f$: $\mathrm{Irr}(X)_0 \to S$, and so there is a congruence $\rho$ on $\mathrm{Irr}(X)_0$ so that the diagram

$$\begin{array}{ccc} \mathrm{Irr}(X)_0 & \xrightarrow{\;\;f\;\;} & S \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle\psi} & \\ & \mathrm{Irr}(X)_{0/\rho} & \end{array}$$

commutes (algebraically only), $\eta$ being the natural map and $\psi$ being the induced algebraic isomorphism. But if $U \subseteq S$ is open, then $\eta^{-1}(\psi^{-1}(U)) = f^{-1}(U)$, which is open in $\mathrm{Irr}(X)_0$ as $f$ is continuous. Hence $\psi^{-1}(U)$ is an open subset of $\mathrm{Irr}(X)_{0/\rho}$ under the quotient topology. Said another way, if $\mathcal{T}$ is the topology induced on $\mathrm{Irr}(X)_{0/\rho}$ from $S$ by $\psi$, then $\eta$: $\mathrm{Irr}(X)_0 \to (\mathrm{Irr}(X)_{0/\rho}, \mathcal{T})$ is continuous. Let $\mathcal{F}$ be the collection of all pairs $(\rho, \mathcal{T})$ such that

(a) $\rho$ is a congruence on $\mathrm{Irr}(X)_0$,

(b) $\mathcal{T}$ is a topology on $\mathrm{Irr}(X)_{0/\rho}$ so that $(\mathrm{Irr}(X)_{0/\rho}, \mathcal{T})$ is an irreducible semigroup with idempotents $X$, and

(c) $\eta: \mathrm{Irr}(X)_0 \to (\mathrm{Irr}(X)_{0/\rho}, \mathcal{T})$ is continuous and idempotent separating.

Then, by Theorem 4.4, if $S$ is any irreducible semigroup with idempotents $X$, there is $(\rho, \mathcal{T}) \in \mathcal{F}$ with $(\mathrm{Irr}(X)_{0/\rho}, \mathcal{T}) \approx S$.

If $T = \prod_{(\rho,\mathcal{T})\in\mathcal{F}} (\mathrm{Irr}(X)_{0/\rho}, \mathcal{T})$ under the Tychonoff topology and endowed with coordinatewise multiplication, then $T$ is a compact connected semigroup which maps onto any irreducible semigroup with idempotents $X$. Now define $\Phi: \mathrm{Irr}(X)_0 \to T$ by $\Phi(s) = (\eta(s))_{\eta\in\mathcal{F}}$, where we identify the pair $(\rho, \mathcal{T})$ and the natural map $\eta: \mathrm{Irr}(X)_0 \to (\mathrm{Irr}(X)_{0/\rho}, \mathcal{T})$. $\Phi$ is clearly a continuous homomorphism and $\Phi \mid X$ is one to one as $\eta \mid X$ is one to one for each $\eta \in \mathcal{F}$. Thus $\Phi(\mathrm{Irr}(X)_0)$ is a connected abelian subsemigroup of $T$ containing $(\eta(1))_{\eta\in\mathcal{F}} = 1_T$ and $(\eta(0))_{\eta\in\mathcal{F}} \in M(T)$, and an argument similar to that in the proof of part (c) of Theorem 4.4 shows $E(\Phi(\mathrm{Irr}(X)_0)) = \Phi(X)$.

**Sublemma 1.** *The Clifford-Miller endomorphism on $\Phi(\mathrm{Irr}(X)_0)$ is an injection when restricted to each $\mathcal{H}$-class of $\Phi(\mathrm{Irr}(X)_0)$.*

**Proof.** Let $1_0 \in E(\Phi(\mathrm{Irr}(X)_0)) \cap M(\Phi(\mathrm{Irr}(X)_0))$, and suppose $s \in \Phi(\mathrm{Irr}(X)_0)$. If $H(S)$, the $\mathcal{H}$-class of $s$ in $\Phi(\mathrm{Irr}(X)_0)$, is a group, then $H(s) = \Phi(H_y')$ for some $y \in X$ as $\Phi$ is idempotent separating. If $t \in H_y = H(s)$ with $t \neq 1_y$, then there is $u \in H_y'$, $u \neq 1_y$ with $\Phi(u) = t$. As $t \neq 1_y$, there is $(\rho, \mathcal{T}) \in \mathcal{F}$ with $\pi_{(\rho,\mathcal{T})}(t) \neq \pi_{(\rho,\mathcal{T})}(1_y)$, where $\pi_{(\rho,\mathcal{T})}: T \to (\mathrm{Irr}(X)_{0/\rho}, \mathcal{T})$ is the natural projection. Hence, if $\eta: \mathrm{Irr}(X)_0 \to (\mathrm{Irr}(X)_{0/\rho}, \mathcal{T})$ is the natural map, then $\eta(u) = \pi_{(\rho,\mathcal{T})}(t) \neq \pi_{(\rho,\mathcal{T})}(1_y) = \eta(1_y)$, and so by Lemma 4.5, there is $(\rho', \mathcal{T}') \in \mathcal{F}$ with $\eta'(1_x u) \neq \eta'(1_x)$ for $x \in X$ with $x \leq y$, where $\eta': \mathrm{Irr}(X)_0 \to (\mathrm{Irr}(X)_{0/\rho'}, \mathcal{T}')$ is the natural map. As $0 \leq y$, $\eta'(1_0 \cdot u) \neq \eta'(1_0)$, whence $1_0 \cdot t \neq 1_0$, and so the core endomorphism is an injection in this case.

Suppose now that $H(s)$ is not a group, and let $t \in H(s)$ with $s \neq t$. Then, there are $u, v \in \mathrm{Irr}(X)_0$ with $s = t\Phi(u)$ and $t = s\Phi(v)$, whence $s = s\Phi(vu)$, and so $s = s\Phi(vu)^n$ for each $n \in \omega$. Now, there is $y \in X$ and $\{\Phi(vu)^{n_\alpha}\}_{\alpha\in D} \to 1_y$ for some subnet of $\{\Phi(vu)^n\}_{n\in\omega}$, and so we have $s = s \cdot 1_y$. Moreover, since $uv = vu$, we also have $t = t \cdot 1_y$, and since $1_y \cdot u$, $1_y \cdot v \in H_y'$, $\Phi(1_y u)$, $\Phi(1_y v) \in H_y$, and $s = t\Phi(1_y u)$ and $t = s\Phi(1_y v)$. As $s \neq t$, $\Phi(1_y u) \neq \Phi(1_y v)$, and, by the first part of the sublemma, $1_0 \cdot \Phi(1_y u) \neq 1_0 \cdot \Phi(1_y v)$. Thus, $1_0 \cdot s = 1_0(t \cdot \Phi(1_y u)) = t \cdot 1_0 \Phi(1_y u) \neq t \cdot 1_0$, the inequality following from the fact that we are in the group $H_0$. This establishes the sublemma.

We now turn our attention to $\Phi(\mathrm{Irr}(X)_0)^*$. Suppose $s \in \Phi(\mathrm{Irr}(X)_0)^*$, and let $\{s_\alpha\}_{\alpha\in D} \subseteq \mathrm{Irr}(X)_0$ with $\{\Phi(s_\alpha)\}_{\alpha\in D} \to s$. For each $\alpha \in D$, pick $x_\alpha \in X$ with $s_\alpha \in S_{x_\alpha}$, and let $x \in X$ with $\{x_\alpha\}_{\alpha\in D} \to x$ by possibly picking a subnet. Again, by possibly picking a subnet, we may assume one of the following holds:

(a) $x_\alpha = x$ for each $\alpha \in D$. Then $s_\alpha \in S'_x$ for each $\alpha \in D$, and, as $S'_x$ is compact, $\Phi(S'_x)$ is closed, and so $s = \lim \Phi(S_\alpha) \in \Phi(S'_x) \subseteq \Phi(\mathrm{Irr}(X)_0)$.

(b) $x < x_\alpha$ for each $\alpha \in D$. Then $m'_{xx_\alpha}(s_\alpha) \in S'_x$ for each $\alpha \in D$, and, again as $S'_x$ is compact, $s = \lim \Phi(s_\alpha) = \lim \Phi(m'_{xx_\alpha}(s_\alpha)) \in \Phi(S'_x) \subseteq \Phi(\mathrm{Irr}(X)_0)$.

(c) $x_\alpha < x$ for each $\alpha \in D$. Suppose $x \neq \inf C_X(x)$, where $C_X(x)$ is the component of $X$ containing $x$. Then, as $\{x_\alpha\}_{\alpha \in D} \to x$, there is $\beta \in D$ with $x_\alpha \in C_X(x)$ for $\alpha \geq \beta$. Then $s_\alpha \in S'_{x_\alpha} = H'_{x_\alpha} = H'_x$ and $m'_{x_\alpha x}$ is the identity for $\alpha \geq \beta$. Hence, in this case, it is easily shown that there is $t \in H'_x$ with $\{s_\gamma\}_{\gamma \in E} \to t$ for some subnet of $\{s_\alpha\}_{\alpha \in D}$, and so $s = \lim \Phi(s_\gamma) = \Phi(t) \in \Phi(H'_x) \subseteq \Phi(\mathrm{Irr}(X)_0)$.

We can conclude from the above that, if $s \in \Phi(\mathrm{Irr}(X)_0)^* \setminus \Phi(\mathrm{Irr}(X)_0)$ and $\{\Phi(s_\alpha)\}_{\alpha \in D} \to s$ with $s_\alpha \in S'_{x_\alpha}$ and $\{x_\alpha\}_{\alpha \in D} \to x$, then $x_\alpha < x$ for each $\alpha \in D$, $x = \inf C_X(x)$, and $x \notin (X' \cup \{0\})$.

We now show $s \in H_x$.

Since $\mathrm{Irr}(X)_0$ is an $\mathcal{H}$-chain, $\Phi(\mathrm{Irr}(X)_0)$, and, hence, $\Phi(\mathrm{Irr}(X)_0)^*$ are $\mathcal{H}$-chains. If $\nu\colon \Phi(\mathrm{Irr}(X)_0)^* \to \Phi(\mathrm{Irr}(X)_0)^*/\mathcal{H}$ is the natural map, then $\{x_\alpha\}_{\alpha \in D} \to x$ implies $\{\nu(\Phi(s_\alpha))\}_{\alpha \in D} \to \nu(\Phi(1_x))$, and so $\nu(\Phi(s)) = \nu(\Phi(1_x))$. Thus $s \in H_x$, and so $\Phi(\mathrm{Irr}(X)_0)^* = (\bigcup_{x \in X'} \Phi(S'_x)) \cup (\bigcup_{x \notin X'} H_x)$, and in particular, $E(\Phi(\mathrm{Irr}(X)_0)^*) = E(\Phi(\mathrm{Irr}(X)_0))$, whence $\Phi(\mathrm{Irr}(X)_0)^*$ has idempotents $X$. Suppose now that $s, t \in \Phi(\mathrm{Irr}(X)_0)$ and that $s$ and $t$ are $\mathcal{H}$-related in $\Phi(\mathrm{Irr}(X)_0)^*$. Then, any idempotent that acts as an identity for one of $s$ or $t$ acts as an identity for the other, and so $s, t \in \Phi(S'_x)$ for some $x \in X$. If $s, t \in H_x$ for some $x \in X$, then $s$ and $t$ are $\mathcal{H}$-related in $\Phi(\mathrm{Irr}(X)_0)$. Suppose $s, t \in \Phi(S'_x)$ for some $x \in X'$. There are $a, b \in \Phi(\mathrm{Irr}(X)_0)^*$ with $s = ta$ and $t = sb$, and so $s = s(ab)^n$ for each $n \in \omega$. Thus $s = s1_z$ where $z \in X$ with some subnet of $\{(ab)^n\}_{n \in \omega}$ converging to $1_z$, and so $x \leq z$. Hence $1_x \cdot a$, $1_x \cdot b \in H_x$, and since $H_x \subseteq \Phi(\mathrm{Irr}(X)_0)$, $s$ and $t$ are $\mathcal{H}$-related in $\Phi(\mathrm{Irr}(X)_0)$ as $s = t \cdot a = t \cdot 1_x \cdot a$ and $t = sb = s \cdot 1_x \cdot b$. This shows that the $\mathcal{H}$-class of a point $s$ in $\Phi(\mathrm{Irr}(X)_0)^*$ is the same as its $\mathcal{H}$-class in $\Phi(\mathrm{Irr}(X)_0)$ unless $s$ is in some subgroup of $\Phi(\mathrm{Irr}(X)_0)$. We are now in a position to prove the following:

**Sublemma 2.** *The Clifford-Miller endomorphism on $\Phi(\mathrm{Irr}(X)_0)^*$ is an injection when restricted to each $\mathcal{H}$-class of $\Phi(\mathrm{Irr}(X)_0)^*$.*

**Proof.** According to the above and Sublemma 1, we have the result except in the case of a subgroup $H_x$ for $x \notin (X' \cup \{0\})$, and $x = \inf C_X(x)$. If $s \in H_x$ with $s \neq 1_x$, then $\Phi(\mathrm{Irr}(X)_0)^*$ compact implies $\phi_x\colon H_x \to \prod_{y<x} H_y$ by $\phi_x(h) = (1_y \cdot h)_{y<x}$ is an isomorphism of $H_x$ onto $\varprojlim\{H_y, m_{yz}, y \leq z < x\}$ where $m_{yz}$ is translation by $1_y$. Hence, there is $y < x$ and $1_w \cdot s \neq 1_w$ for $w \in [y, x)$. As $x = \inf C_X(x)$ and $x \notin (X' \cup \{0\})$, Koch's Theorem implies $[y, x) \cap X' \neq \square$. If $w \in [y, x) \cap X'$ then $H_w = \Phi(H'_w)$, and so the core endomorphism is an injection when restricted to $H_w$ by Sublemma 1. Now $1_w \cdot s \neq 1_w$ as $w \in [y, x)$, and so $1_0 \cdot s = 1_0 \cdot (1_w s) \neq 1_0$ by the above. Therefore the core endomorphism is an injection when restricted to $H_x$, and so the sublemma is established.

Now as $\mathrm{Irr}(X)_0$ is connected, $\Phi(\mathrm{Irr}(X)_0)$ and, hence $\Phi(\mathrm{Irr}(X)_0)^*$ are connected subsemigroups of $T$. Thus $\Phi(\mathrm{Irr}(X)_0)^*$ is a compact connected subsemigroup of $T$ containing $1_T$ and meeting $M(T)$. Let $\mathrm{Irr}(X)$ be any irreducible subsemigroup of $\Phi(\mathrm{Irr}(X)_0)^*$ containing $1_T$ and meeting $M(T)$. Then the Clifford-Miller endomorphism on $\mathrm{Irr}(X)$ is an injection when restricted to each $\mathcal{H}$-class of $\mathrm{Irr}(X)$ as this property is easily seen to be hereditary. We now show $\mathrm{Irr}(X)$ has idempotents $X$. If $S$ is the canonical $I$-semigroup through $X$, then $S$ is irreducible with idempotents $X$, and so there is $(\rho, \mathfrak{T}) \in \mathfrak{F}$ with $(\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T}) \simeq S$. If $\pi_{(\rho,\mathfrak{T})}: T \to (\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T})$ is the natural projection, then $\pi_{(\rho,\mathfrak{T})}(\mathrm{Irr}(X))$ is a compact connected subsemigroup of $(\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T})$ containing $\pi_{(\rho,\mathfrak{T})}(1_T)$, the identity of $(\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T})$, and meeting $\pi_{(\rho,\mathfrak{T})}(M(T))$, the minimal ideal of $(\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T})$. Hence, as $(\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T})$ is irreducible, $\pi_{(\rho,\mathfrak{T})}(\mathrm{Irr}(X)) = (\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T})$ and since $\mathrm{Irr}(X)$ is compact $\pi_{(\rho,\mathfrak{T})}(E(\mathrm{Irr}(X))) = E(\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T}) \simeq X$. Since $\mathrm{Irr}(X) \subseteq \Phi(\mathrm{Irr}(X)_0)^*$, $E(\mathrm{Irr}(X)) \subseteq \Phi(X)$, and, as $\pi_{(\rho,\mathfrak{T})}$ is clearly idempotent separating on $\Phi(X)$, we must have $E(\mathrm{Irr}(X)) = \Phi(X)$. Finally, if $S$ is an irreducible semigroup with idempotents $X$, there are $(\rho, \mathfrak{T}) \in \mathfrak{F}$ and an isomorphism $\psi: (\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T}) \to S$. If $\pi_{(\rho,\mathfrak{T})}: T \to (\mathrm{Irr}(X)_{0/\rho}, \mathfrak{T})$ is the natural projection, then $\psi \circ \pi_{(\rho,\mathfrak{T})} | \mathrm{Irr}(X)$ is the desired idempotent separating surmorphism of $\mathrm{Irr}(X)$ onto $S$.

The converse of the last statement of the theorem is simply [3, 5.31, p. 156]. This completes the proof.

We conclude this section with an outline of some refinements of the construction of $\mathrm{Irr}(X)_0$, and with some examples where $\mathrm{Irr}(X)_0$ is compact, thus fulfilling the role of $\mathrm{Irr}(X)$. The reader is advised to review the proofs of parts (a) and (b) of Theorem 4.4 before proceeding.

Let $X$ be a compact totally ordered space, and, for each $x \in X$, let

$$G'_x = \{f \in \mathbf{R}_d^{(x,1]_{X'}} : \exists\, z \in X' \backslash \{1\} \ni y \in (z, 1)_{X'} \Rightarrow f(y) = 0\}.$$

It is easily verified that $G'_x$ is a group for each $x \in X$, and that if $x \leq y \in X$, then $w'_{xy}(G'_x) = G'_y$, where $[w'_{xy}(f)][z] = f(z)$ for $z \in (y, 1]_{X'}$ and $f \in G'_x$. To see that the system $\{(G'_x)^\wedge, (w'_{xy})^\phi, x \leq y \in X\}$ satisfies the properties desired for the maximal subgroups of $\mathrm{Irr}(X)_0$, we show the following.

**Lemma 4.6.** *Let $S$ be an irreducible semigroup with idempotents $X$, and let $\{G_x, m^\phi_{xy}, x \leq y \in X\}$ be the dual of the system of maximal subgroups and bonding homomorphisms of $S$. If $\psi_x: G_x \to \mathbf{R}_d^{(x,1]_{X'}}$ is the injection defined in the proof of Theorem 4.4, part (b), then $\psi_x(G_x) \subseteq G'_x$ for each $x \in X$.*

**Proof.** If $x \in C_X(1)$, this is clear. If $x_1 = \inf C_X(1)$ and $x_1 \in X'$, then $G'_x = \mathbf{R}_d^{(x,1]_{X'}}$ for each $x \in X$ and so there is nothing to prove. Suppose $x_1 \notin X'$. Since

$S$ is compact, $\phi_{x_1}: H_{x_1} \to \Pi_{y<x_1} H_y$ with $\phi_{x_1}(h) = (m_{yx_1}(h))_{y<x_1}$ is an isomorphism of $H_{x_1}$ onto $\varprojlim\{H_y, m_{yz}, y \le z < x_1\}$. Hence, by duality, $\eta_{x_1}$: $\varinjlim\{G_x, m^{\phi}_{xy}, x \le y < x_1\} \to G_{x_1}$ with $\eta_{x_1}(\eta(g)) = m^{\phi}_{xx_1}(g)$ is an isomorphism, where $g \in G_x$ and $\eta: \bigvee_{x<x_1} G_x \to \varinjlim\{G_x, m^{\phi}_{xy}, x \le y < x_1\}$ is the natural map. Thus, as $\psi_x$ is an injection for each $x \in X$, $\eta'_{x_1}: \varinjlim\{\psi_x(G_x), w'_{xy}, x \le y < x_1\} \to \psi_{x_1}(G_{x_1})$ with $\eta'_{x_1}(\eta'(g)) = w'_{xx_1}(g)$ is an isomorphism, where $g \in \psi_x(G_x)$ and $\eta'$: $\bigvee_{x<x_1} \psi_x(G_x) \to \varinjlim\{\psi_x(G_x), w'_{xy}, x \le y < x_1\}$ is the natural map. As $S$ is irreducible, $H_{x_1} = \{1_{x_1}\}$, and so $G_{x_1} = \{0\} = \psi_{x_1}(G_{x_1})$. Now, let $x \in X$ with $(x, 1]_{X'} \neq \square$ and fix $g \in G_x$. Then $x < x_1$ and $\eta'(\psi_x(g)) = 0$. Hence there is $r < x_1$ with $w'_{xr}(\psi_x(g)) = 0$, and by Koch's Theorem, there is $z \in [r, x_1) \cap X'$ as $x_1 = \inf C_X(1)$. Then, $w'_{xz}(\psi_x(g)) = w'_{rz}(w'_{xr}(\psi_x(g))) = w'_{rz}(0) = 0$, whence $[\psi_x(g)][y] = 0$ for $z \le y \in X'$. Hence $\psi_x(g) \in G'_x$, and so $\psi_x(G_x) \subseteq G'_x$, completing the proof of the lemma.

Suppose $X \backslash (X' \cup \{0\}) \subseteq C_X(1)$, and let $x \in C_X(1)$. If $x_1 < x$, then $G'_x = \{0\} = G'_y$ for each $y \in [x_1, x]$, whence $G'_x \simeq \varinjlim\{G'_y, w'_{yz}, y \le z < x\}$. If $x = x_1$, then $G'_{x_1} = \{0\} = \varinjlim\{G'_y, w'_{yz}, y \le z < x_1\}$ since the $G'_y$ are "locally zero" near $x_1$. Hence, in this case, $H'_x \simeq \varprojlim\{H'_y, m'_{yz}, y \le z < x\}$ if $x \notin (X' \cup \{0\})$, whence $(X, S'_x, m'_{xy})$ is a chainable collection. Since this collection satisfies the hypotheses of Theorem 4.2, $\mathrm{Irr}(X)_0 = \mathrm{Horm}(X, S'_x, m'_{xy})$ is irreducible with idempotents $X$. In particular, $\mathrm{Irr}(X)_0$ is irreducible if $X = X' \cup \{0\}$.

*Another counterexample.* We now present a counterexample to another proposed structure for $\mathrm{Irr}(X)$. When it was discovered that the full direct product $\mathbf{R}_d^{(x,1]_{X'}}$ would not work as the dual to $H'_x$ in $\mathrm{Irr}(X)$, the next logical group to try was the weak direct product, or direct sum, which we denote by ${}^{(x,1]_{X'}}\mathbf{R}_d$. For each $x \in X$, let $G'_x = {}^{(x,1]_{X'}}\mathbf{R}_d$, and if $x \le y \in X$, let $w'_{xy}: G'_x \to G'_y$ be defined by $[w'_{xy}(f)][z] = f(z)$ for $y < z \in X'$ and $f \in G'_x$. For each $x \in X$, let $S'_x = H'_x$ if $x \notin X'$, while, if $x \in X'$, let $S'_x = \Sigma \times H'_x$, where $H'_x = (G'_x)^{\wedge}$. Let $S' = \mathrm{Horm}(X, S'_x, m'_{xy})$, where $m'_{xy}: S'_y \to S'_x$ is the natural extension of $(w'_{xy})^{\phi}$. Then $S'$ is an irreducible semigroup with idempotents $X$ (we do not prove this as $S'$ does not satisfy the properties of $\mathrm{Irr}(X)$ as the following example shows).

Let $X = \{1, e, \{x_n\}_{n\in\omega}\}$ where $1 > e > x_{n+1} > x_n$ for each $n \in \omega$, and $\{x_n\}_{n\in\omega} \to e$.

Let $T' = \Sigma \times \hat{\mathbf{R}}_d$, and let $\phi: M(\Sigma) \times \hat{\mathbf{R}}_d \to \hat{\mathbf{R}}_d$ by $\phi((\infty, g)h) = gh$. Note that $\phi \mid \{(\infty, 0)\} \times \hat{\mathbf{R}}_d$ is essentially the identity. Let $\rho = \rho_{\phi} \cup \Delta$, note that $\rho$ is a closed congruence on $T'$, and let $T = T'/\rho$. It is clear that the Clifford-Miller endomorphism on $T$ when restricted to the group of units acts like the identity map.

We now form a chainable collection $(X, S_x, m_{xy})$. For each $n \in \omega$, let $S_{x_n} = T$, and note that $M(S_{x_{n+1}}) \simeq H_{x_n}$ under $h \to ((0, 0), h)$. Define $m_{x_n x_{n+1}}: S_{x_{n+1}} \to$

$S_{x_n}$ to be the core endomorphism of $S_{x_{n+1}}$ composed with the above isomorphism and note that $m_{x_n x_{n+1}} | H_{x_{n+1}}$ is essentially the identity map. If $k < n$, let $m_{kn} = m_{k\,k+1} \circ \circ \circ m_{n-1\,n}$. Let $S_e = H_e = \mathbf{R}_d^{\wedge}$, and let $S_1 = \Sigma$. Note that $H_e \simeq \varprojlim\{H_{x_n}, m_{x_n x_k}, n \leq k \in \omega\}$, and define $m_{x_n e}: H_e \to S_{x_n}$ by $m_{x_n e}(h) = ((0, 0), h)$. Finally, let $m_{e1}: S_1 \to S_e$ be the core endomorphism on $S_1$ followed by the isomorphism of $M(\Sigma)$ onto $H_e = \mathbf{R}_d^{\wedge}$ given by $(\infty, g) \to g$, and let $m_{x_n 1} = m_{x_n e} \cdot m_{e1}$ for each $n \in \omega$. Then $(X, S_x, m_{xy})$ is a chainable collection and, if $S =$ Horm$(X, S_x, m_{xy})$, $S$ is easily seen to be irreducible with idempotents $X$ by using Theorem 4.2.

Now suppose $f: S' \to S$ is an idempotent separating surmorphism. Then, for each $x \in X$, $f | S_x': S_x' \to S_x$ is a surmorphism, and so we can obtain $f_x: \Sigma \times H_x \to S_x$ with $f_x((0, 0), h) = h$ for each $h \in H_x$. As in the proof of Theorem 4.4, part (b), we can then obtain $\psi_x: G_x = H_x^{\wedge} \to G_x'$ where $\psi_x(g) = (f_y^{\phi}(m_{xy}^{\phi}(g)))_{x<y\in X'}$. We now show $\psi_{x_0}(G_{x_0}) \not\subseteq G_{x_0}'$.

Since $H_{x_0} = \mathbf{R}_d^{\wedge}$, $G_{x_0} = H_{x_0}^{\wedge} = \mathbf{R}_d$. Let $g \in G_{x_0}$ with $g \neq 0$. For each $n \in \omega$, $m_{x_0 x_n}$ is an isomorphism of $M(S_{x_n})$ onto $H_{x_0}$. Now, $f_{x_n}(\Sigma \times \{1_{x_n}\})$ is the closure of a one-parameter semigroup in $S_{x_n}$ running from $1_{x_n}$ to $M(S_{x_n})$, and since any such contains $M(S_{x_n})$, $f_{x_n}(M(\Sigma) \times \{1_{x_n}\}) = M(S_{x_n})$. Hence, $f_{x_n}^{\phi}: M(S_{x_n})^{\wedge} \to \mathbf{R}_d$ is an injection, and so $f_{x_n}^{\phi}(m_{x_0 x_n}^{\phi}(g)) \neq 0$. As $n$ is arbitrary, $\psi_{x_0}(g) \notin G_{x_0}'$, thus completing the counterexample.

**Conclusion.** Our main result establishes the existence of a generator in the category of irreducible semigroups with idempotents $X$. From a semigroup standpoint, however, the whole point in finding such an object is to gain more information about the objects in the category from the the generator. Thus, one would ideally desire a complete description of Irr$(X)$, as Hofmann and Mostert attempted to obtain. If an explicit description of the system of maximal subgroups and bonding homomorphisms could be obtained, then it is not hard to see that a complete description of Irr$(X)$ would be forthcoming. Moreover, since Irr$(X)$ is irreducible with idempotents $X$, Irr$(X)$ is the idempotent separating surmorphic image of Irr$(X)_0$, and so the dual of the system of maximal subgroups is contained in the dual of the system for Irr$(X)_0$. We have seen that the extremes, the full direct product and the weak direct product, will not suffice. Another candidate proposed by Karl Hofmann is the following:

For each $x \in X$, let

$$G_x' = \{f \in \mathbf{R}_d^{(x,1]_{X'}} : z \in (x, 1] \ni z \notin X' \Rightarrow \exists\, y \in (x, z) \cap X' \text{ so that } f(w) = f(y) \text{ for } w \in (y, z)_{X'}\}.$$

This conjecture is still unsettled.

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