THE GROWTH OF SUBUNIFORM ULTRAFILTERS

BY

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ABSTRACT. Some of the results on the topology of spaces of uniform ultra-
filters are applied to the space $\Omega(\alpha^+)$ of subuniform ultrafilters (i.e., the set
of ultrafilters which are α-uniform but not α^+-uniform) on α^+ when α is a regular car-
dinal. The main object is to find for infinite cardinals α, such that $\alpha = \alpha^{\underline{\alpha}}$, a topo-
logical property that separates the space $\beta(\Omega(\alpha^+))\backslash\Omega(\alpha^+)$ (the *growth* of $\Omega(\alpha^+)$)
from the space $U(\alpha^+)$ of uniform ultrafilters on α^+. Property Φ_α fulfils this rôle
defined for a zero-dimensional space X by the following condition: every non-
empty closed subset of X of type at most α is not contained in the uniform
closure of a family of α pairwise disjoint nonempty open-and-closed subsets of
X. The "infinitary" properties of $\Omega(\alpha^+)$, as they are measured by Φ_α, are more
closely related to those of $U(\alpha)$ than to those of $U(\alpha^+)$. A consequence of this
topological separation is that the growth of $\Omega(\alpha^+)$ is not homeomorphic to $U(\alpha^+)$
and, in particular, that $\Omega(\alpha^+)$ is not C^*-embedded in the space $\Sigma(\alpha^+)$ of α-uni-
form ultrafilters on α^+. These results are related to, and imply easily, the
Aronszajn-Specker theorem: *if $\alpha = \alpha^{\underline{\alpha}}$ then α^+ is not a ramifiable cardinal*. It
seems possible that similar questions on the C^*-embedding of certain spaces of
ultrafilters depend on (and imply) results in partition calculus.

Property Φ_α was introduced in [7], where it was proved by a diagonal argu-
ment that the space $U(\alpha)$ of uniform altrafilters on α has property Φ_α (Theorem
4.2 of [7]). It is easy to see that $U(\alpha)$ does not have property Φ_β for regular
cardinals β, $\omega \leq \beta < \alpha$ (Lemma 3.3). The main result of the present paper
(whose proof relies on Theorem 4.2 of [7]) is that if $\alpha = \alpha^{\underline{\alpha}}$ then the space
$\beta(\Omega(\alpha^+))\backslash\Omega(\alpha^+)$ has property Φ_α.

Informally we think of this result as showing the inability of the space $\Omega(\alpha^+)$
to break away from certain infinitary properties characteristic to α (such as Φ_α),
which are local after all, and to adopt the corresponding ones characteristic to
α^+ (such as Φ_{α^+}), in the process of "going to infinity". Instead, under the con-
straint of a pigeon-hole principle (contained in Lemma 2.1 and in the proof of

Theorem 3.2) the process of going to infinity is so "slow" that the local proper-
ties of $\Omega(\alpha^+)$ (such as Φ_α), normally of little importance in determining the pro-
perties of the growth of a space, become the dominant factor.

An immediate consequence of the main theorem is that if $\alpha = \alpha^{\underline{\alpha}}$ then $\Omega(\alpha^+)$
is not C^*-embedded in $\beta(\alpha^+)$ (Corollary 3.4). For $\alpha = \omega$ this (together with
Corollary 3.5, below, for $\alpha = \omega$) was a problem proposed in [1], [3], and proved by
Nancy M. Warren in [9], [10].

1. **Notation and terminology.** The axiom of choice is assumed. Ordinal num-
bers will be denoted by ξ, ζ, η, σ. An ordinal is the set of all smaller ordinals,
i.e. $\xi < \zeta$ is equivalent to $\xi \in \zeta$. Nevertheless, we will make the notational
distinction between 0 (the first ordinal) and $\emptyset$ (the empty set). A cardinal number
is an initial ordinal. Cardinals will be denoted by α, β, γ. $0, 1, \cdots, n, \cdots$
denotes the sequence of natural numbers. The first infinite cardinal is ω. The
least cardinal greater than α is denoted by α^+. α is a limit cardinal if it is not
equal to β^+ for some β. A cardinal is regular if it is not equal to the sum of
fewer smaller cardinals. *In this paper α will always denote an infinite regular
cardinal.* A nonlimit cardinal is regular. β^α denotes the cardinal number of the
set of all mappings from α to β. We set $\alpha^{\underline{\beta}} = \Sigma\{\alpha^\gamma < \beta\}$. This cardinality of a
set A will be denoted by $|A|$. A set of cardinality α will usually be identified
with α itself. For a set A, let $S(A)$, $S_\alpha(A)$ be the set of all subsets of A, the
set of all subsets of A of cardinality less than α, respectively. Note that for
$\beta \leq \alpha$, $|S_\beta(\alpha)| = \alpha^{\underline{\beta}}$. The notion of an ultrafilter on α will be the usual one. An
ultrafilter containing a set with a single element is called principal; otherwise
nonprincipal. The set of all ultrafilters on α, topologized with the Stone topology,
and called the Stone-Čech compactification of α, will be denoted by $\beta(\alpha)$. The
set of principal ultrafilters in $\beta(\alpha)$ is identified with α in the natural way. An
ultrafilter on α is κ-uniform if each of its elements has cardinality at least κ. An
α-uniform ultrafilter on α is simply called uniform. The set of all uniform ultra-
filters on α, topologized as a subspace of $\beta(\alpha)$, is denoted by $U(\alpha)$. An ultra-
filter on α^+ will be called *subuniform* if each of its elements has cardinality at
least α, and there is an element of the ultrafilter with cardinality α. The set of
all subuniform ultrafilters on α^+, topologized as a subspace of $\beta(\alpha^+)$, is denoted
by $\Omega(\alpha^+)$. We set $\Sigma(\alpha) = U(\alpha^+) \cup \Omega(\alpha^+)$, again a subspace of $\beta(\alpha^+)$.

Let X be a (completely regular Hausdorff topological) space. For $Y \subset X$, we
set $cl_X Y$ for the closure of Y in X. A zero-set Z in X is a set of the form
$Z = \{x \in X: f(x) = 0\}$ for some real-valued continuous function f on X. The set of
all bounded real-valued continuous functions on X is denoted by $C^*(X)$. A subset
Y of X is C^*-embedded in X if for $f \in C^*(Y)$, there is $g \in C^*(X)$ such that
$g|Y = f$. As with sets, βX denotes the Stone-Čech compactification of X. A space

X is called totally disconnected if for any two distinct elements of X there is an open-and-closed subset of X containing one but not the other element; X is called zero-dimensional if for any two disjoint zero-sets of X there is an open-and-closed subset of X containing one and disjoint from the other zero-set. Let U be an open subset of a totally disconnected space X; the type $\tau(U)$ of U is the least cardinal β such that U is equal to the union of β open-and-closed subsets of X. If F is a closed subset of X, we set $\tau(F) = \tau(X \backslash F)$. We note that if F is a closed subset of a compact, zero-dimensional space, then F is a zero-set if and only if $\tau(F) \leq \omega$. A totally disconnected space X will be called an F_α-space if every open subset of X of type less than α is C^*-embedded in X. We remark that the usual notion of an F-space (meaning that every cozero set is C^*-embedded) is, in general, stronger than the notion of an F_{ω^+}-space, but that the two notions coincide for compact spaces. (The reason for choosing this definition for an F_α-space, rather than the exact analogue of an F-space to higher cardinals, is only one of convenience.)

2. In this section we prove some results that will be needed for the proof of the main theorem (given in §3).

2.1. Lemma. *Let F be a nonempty subset of $\beta(\Omega(\alpha^+))$, such that F is equal to the intersection of at most α open-and-closed subsets of $\beta(\Omega(\alpha^+))$. Then $F \cap \Omega(\alpha^+)$ is nonempty.*

Proof. Let $F = \bigcap_{\eta < \alpha} F_\eta$, where F_η is open-and-closed in $\beta(\Omega(\alpha^+))$. Set $U_\eta = \Omega(\alpha^+) \backslash F_\eta$, $\eta < \alpha$. If $F \cap \Omega(\alpha^+) = \varnothing$, then $\{ U_\eta : \eta < \alpha \}$ is an open cover of $\Omega(\alpha^+)$. Let $\Omega(\alpha^+)$ be equal to $\bigcup_{\xi < \alpha^+} S_\xi$, where

$S_\xi \subset S_\zeta$ for $\xi < \zeta < \alpha^+$,

S_ξ is open-and-closed in $\Omega(\alpha^+)$,

S_ξ is homeomorphic to $U(\alpha)$

for $\xi < \alpha^+$. Let $\phi : \alpha^+ \to S_\omega(\alpha)$ be any mapping such that for $\xi < \alpha^+$, $\bigcup \{ U_\eta : \eta \in \phi(\xi) \} \supset S_\xi$. Then, there is $A \in S_\omega(\alpha)$, such that $|\{ \xi < \alpha^+ : \phi(\xi) = A \}| = \alpha^+$. Clearly, then, $\bigcup \{ U_\eta : \eta \in A \} = \Omega(\alpha^+)$, contradicting the fact that F is nonempty.

2.2. Corollary. *Let F_η be open-and-closed sets in $\Omega(\alpha^+)$, $\eta < \alpha$. Then*

$$\mathrm{cl}_{\beta(\Omega(\alpha^+))} \left(\bigcap_{\eta < \alpha} F_\eta \right) = \bigcap_{\eta < \alpha} \mathrm{cl}_{\beta(\Omega(\alpha^+))} F_\eta.$$

Proof. We only need to prove that

$$\mathrm{cl}_{\beta(\Omega(\alpha^+))} \left(\bigcap_{\eta < \alpha} F_\eta \right) \supset \bigcap_{\eta < \alpha} \mathrm{cl}_{\beta(\Omega(\alpha^+))} F_\eta.$$

Let $p \in \bigcap_{\eta < \alpha} \mathrm{cl}_{\beta(\Omega(\alpha^+))} F_\eta \backslash \mathrm{cl}_{\beta(\Omega(\alpha^+))} (\bigcap_{\eta < \alpha} F_\eta)$. Then there is an open-and-closed set N of $\beta(\Omega(\alpha^+))$, such that $p \in N$ and $N \cap \mathrm{cl}_{\beta(\Omega(\alpha^+))} (\bigcap_{\eta < \alpha} F_\eta) = \varnothing$.

Then $F = N \cap \bigcap_{\eta < a} \mathrm{cl}_{\beta(\Omega(a^+))} F_\eta$ is the intersection of at most a open-and-closed subsets of $\beta(\Omega(a^+))$, with $p \in F$; however, $F \cap \Omega(a^+) \subset N \cap \bigcap_{\eta < a} F_\eta = \varnothing$, contradicting Lemma 2.1.

The following theorem has been proved as Theorem 3.1 in [5]. (We recall that in the present paper a is an infinite regular cardinal.)

2.3. Theorem. $U(a)$ *is a compact* F_{a^+}*-space.*

We will now prove a stronger result for the space $\Sigma(a^+)$ of all uniform and subuniform ultrafilters on a^+. The proof of the following lemma is left to the reader.

2.4. Lemma. *Let* $S = D \cup T$ *be a space, where* D *is discrete in* S *and* T *is compact. Then* S *is a paracompact (and hence normal) space.*

2.5. Theorem. $\Sigma(a^+)$ *is a compact* F_{a^+}*-space.*

Proof. It is clear that $\Sigma(a^+)$ is a compact, zero-dimensional space. The Boolean algebra of open-and-closed subsets of $\Sigma(a^+)$ is (isomorphic to) $S(a^+)/S_a(a^+)$, an a-complete Boolean algebra; thus, it is clear that the closure of any open subset of $\Sigma(a^+)$ of type less than a is open. It follows easily (e.g. as in Corollary 2.2 of [5]) that an open subset of $\Sigma(a^+)$ of type less than a is C^*-embedded in $\Sigma(a^+)$. Thus, we only have to prove that if G is an open subset of $\Sigma(a^+)$ of type (exactly) a, then G is C^*-embedded in $\Sigma(a^+)$.

Let $G = \bigcup_{\xi < a} A'_\xi$, where $A_\xi \subset a^+$ and $A'_\xi = \mathrm{cl}_{\beta(a^+)} A_\xi \cap \Sigma(a^+)$ (see Lemma 2.8 below in this connection). We express G as the increasing union of open sets of type less than a, of a chain of length a, in the natural way: let $G_\xi = \bigcup_{\eta \le \xi} A'_\eta$ for $\xi < a$. Thus, G_ξ is open, $\tau(G_\xi) < a$, $G_\xi \subset G_\eta$ for $\xi < \eta < a$, $G = \bigcup_{\xi < a} G_\xi$. By the remarks above, $\overline{G}_\xi \ (= \mathrm{cl}_{\Sigma(a^+)} G_\xi)$ is open-and-closed in $\Sigma(a^+)$, and G_ξ is C^*-embedded in $\overline{G}_\xi$ for $\xi < a$. Let $H = \bigcup_{\xi < a} \overline{G}_\xi$; then H is open in $\Sigma(a^+)$, $\tau(H) \le a$ and G is dense in H.

We prove that G is C^*-embedded in H. Indeed, let $f \in C^*(G)$. Set $f_\xi = f|G_\xi$ for $\xi < a$. Let g_ξ be the unique continuous extension of f_ξ to $\overline{G}_\xi$. Further, for $\xi < \eta < a$, we have that $f_\xi \subset f_\eta$, and since the extensions are unique, that $g_\xi \subset g_\eta$. Let $g = \bigcup_{\xi < a} g_\xi$. Since $\overline{G}_\xi$ is open for $\xi < a$, g is continuous.

It is clear that $\overline{G}_\xi = \left(\bigcup_{\eta \le \xi} A_\xi \right)'$ for $\xi < a$. Set $B_\xi = \bigcup_{\eta \le \xi} A_\eta$, $B = \bigcup_{\xi < a} B_\xi = \bigcup_{\xi < a} A_\xi$. Thus $B_\xi \subset B_\eta$ for $\xi < \eta < a$. We now prove that H is C^*-embedded in $B \cup H$. Indeed, let $f \in C^*(H)$. Set $f_\xi = f|\overline{G}_\xi$ for $\xi < a$. We proceed by transfinite recursion. By 2.4, $B_0 \cup \overline{G}_0$ is a normal space, and $\overline{G}_0$ is a closed subset of $B_0 \cup \overline{G}_0$. Let $g_0 \in C^*(B_0 \cup \overline{G}_0)$ be an extension of f_0, such that

$$\sup\{|g_0(x)| : x \in B_0 \cup \overline{G}_0\} = \sup\{|f_0(x)| : x \in \overline{G}_0\}.$$

Let $\xi < \alpha$ and suppose that for every $\eta < \xi$ we have defined g_η such that $g_\eta \in C^*(B_\eta \cup \overline{G}_\eta)$, $f_\eta \subset g_\eta$,

$$\sup\{|g_\eta(x)| : x \in B_\eta \cup \overline{G}_\eta\} = \sup\{|f_\eta(x)| : x \in \overline{G}_\eta\},$$

$g_\zeta \subset g_\eta$ for $\zeta < \eta < \alpha$. We define

$$g_\xi^* = \left(\left(\bigcup_{\eta < \xi} g_\eta\right) \cup f_\xi\right) : \bigcup_{\eta < \xi}(B_\eta \cup \overline{G}_\eta) \cup \overline{G}_\xi \to \mathbf{R}$$

(where $\mathbf{R}$ denotes the space of real numbers).

We verify that g_ξ^* is well defined: our inductive assumption implies that $\bigcup_{\eta < \xi} g_\eta$ is well defined on $\bigcup_{\eta < \xi}(B_\eta \cup \overline{G}_\eta)$ and f_ξ is defined on $\overline{G}_\xi$; further,

$$\left(\bigcup_{\eta < \xi}(B_\eta \cup \overline{G}_\eta)\right) \cap \overline{G}_\xi = \bigcup_{\eta < \xi}\overline{G}_\eta \subset H$$

and both functions $\bigcup_{\eta < \xi} g_\eta$, f_ξ are equal to $f|(\bigcup_{\eta < \xi}(B_\eta \cup \overline{G}_\eta)) \cap G_\xi$ on the intersection. Thus g_ξ^* is well defined. Further, $\bigcup_{\eta < \xi} g_\eta$ is continuous, since the sets $B_\eta \cup \overline{G}_\eta$ are open-and-closed for $\eta < \xi$. Finally, g_ξ^* is continuous, because the sets

$$\bigcup_{\eta < \xi} B_\eta \cup \overline{\bigcup_{\eta < \xi} G_\eta} \quad \text{and} \quad \overline{G}_\xi \Big\backslash \overline{\bigcup_{\eta < \xi} G_\eta}$$

are open-and-closed in $\bigcup_{\eta < \xi}(B_\eta \cup \overline{G}_\eta) \cup \overline{G}_\xi$. We note that the space $B_\xi \cup \overline{G}_\xi$ is normal by Lemma 2.4, and that $\bigcup_{\eta < \xi}(B_\eta \cup \overline{G}_\eta) \cup \overline{G}_\xi$ is a closed subset in $B_\xi \cup \overline{G}_\xi$. Further, notice that our inductive assumption implies that

$$\sup\left\{|g_\xi^*(x)| : x \in \bigcup_{\eta < \xi}(B_\eta \cup \overline{G}_\eta) \cup \overline{G}_\xi\right\} = \sup\{|f_\xi(x)| : x \in \overline{G}_\xi\}.$$

Hence g_ξ^* is bounded; thus, we can find a continuous extension $g_\xi \in C^*(B_\xi \cup \overline{G}_\xi)$, such that

$$\sup\{|f_\xi(x)| : x \in \overline{G}_\xi\} = \sup\{|g_\xi(x)| : x \in B_\xi \cup \overline{G}_\xi\}.$$

This completes the recursive definition of g_ξ for $\xi < \alpha$. We let $g = \bigcup_{\xi < \alpha} g_\xi$. Since the sets $B_\xi \cup \overline{G}_\xi$, $\xi < \alpha$, are open-and-closed, g is continuous. Also,

$$\sup\{|g(x)| : x \in B \cup H\} = \sup\{|f(x)| : x \in B\},$$

and hence g is bounded. This completes the proof that H is C^*-embedded in $B \cup H$.

Finally, we note that $B \subset B \cup H \subset \text{cl}_{\beta(\alpha)^+}B = \beta(B)$. Hence, $B \cup H$ is C^*-embedded in $\text{cl}_{\beta(\alpha^+)}B$. Since $\text{cl}_{\beta(\alpha^+)}B$ is open-and-closed in $\beta(\alpha^+)$, it is C^*-embedded in $\beta(\alpha^+)$. Thus, by the transitivity of C^*-embedding, G is C^*-embedded in $\beta(\alpha^+)$, and since it is contained in $\Sigma(\alpha^+)$, it is C^*-embedded in $\Sigma(\alpha^+)$. This completes the proof of the theorem.

The following result is a generalization of Theorem 4.1 by Fine and Gillman

[3]; we remark that the reason for the additional restriction on the spaces S_ξ below is not significant, and only reflects our restrictive (but convenient) definition of an F_α-space.

2.6. Theorem. *Let X be an F_α-space and let $S \subset X$ be the union of α open-and-closed subsets S_ξ, $\xi < \alpha$, of X, each S_ξ being equal to the union of less than α compact subsets. Then*

(a) *S is an F_α-space;*

(b) *if $G \subset S$ and $G \cap S_\xi$ is of type less than α in S for $\xi < \alpha$ then G is C^*-embedded in S;*

(c) *if X is zero-dimensional, then so is S.*

Proof. (b) We may assume that $S = \bigcup_{\xi < \alpha} S_\xi$ and that $S_\xi \subset S_\zeta$ for $\xi < \zeta < \alpha$. We note, since each S_ξ is an open subset of X of type less than α, that S_ξ is an F_α-space for $\xi < \alpha$. Let $g \in C^*(G)$, and set $g_\xi = g | G \cap S_\xi$ for $\xi < \alpha$. Let $\zeta < \alpha$ and suppose that g_ξ has been extended to $s_\xi \in C^*(S_\xi)$ for each $\xi < \zeta$, and that $s_\zeta \subset s_{\zeta'}$ for $\zeta < \zeta' < \xi$. The function $\bigcup_{\xi < \zeta} s_\xi \cup g_\zeta$ is well defined and continuous on the set $\bigcup_{\xi < \zeta} S_\xi \cup (G \cap S_\zeta)$, an open set of type less than α in the F_α-space S_ζ; hence it has an extension to a function $s_\zeta \in C^*(S_\zeta)$, such that

$$\sup\{|s_\zeta(x)| : x \in S_\zeta\} = \sup\left\{|s_\zeta(x)| : x \in \bigcup_{\xi < \zeta} S_\xi \cup (G \cap S_\zeta)\right\}.$$

Finally $\bigcup_{\xi < \alpha} s_\xi$ is a continuous extension of g to the space S.

(a) If G is an open subset of S of type less than α, then by (b), G is C^*-embedded in S.

(c) We assume that X is zero-dimensional. Let A, B be two disjoint zero-sets of S; we must prove that there is a $\{0, 1\}$-valued continuous function on S, which is equal to 0 on A and is equal to 1 on B. Let A_1, B_1 be open sets of S, with $\tau(A_1 \cap S_\xi) < \alpha$, $\tau(B_1 \cap S_\xi) < \alpha$ for $\xi < \alpha$, and such that $A \subset A_1$, $B \subset B_1$ $A_1 \cap B_1 = \emptyset$. (We can take A_1, B_1 to be disjoint cozero sets containing A, B, respectively; then $A_1 \cap S_\xi$, $B_1 \cap S_\xi$ are cozero sets in S_ξ; it is easy to see that the compactness condition on S_ξ implies that $\tau(A_1 \cap S_\xi^*) < \alpha$, $\tau(B_1 \cap S_\xi) < \alpha$.) Let $g \in C^*(A_1 \cup B_1)$ be equal to 0 on A_1 and to 1 on B_1. Note, since (by (a)) S_ξ is C^*-embedded in X, that S_ξ is zero-dimensional for $\xi < \alpha$. In the proof of (b), with $G = A_1 \cup B_1$, we add to the recursive assumption that $\bigcup_{\xi < \zeta} s_\xi$ is $\{0, 1\}$-valued; then s_ζ may be taken to be $\{0, 1\}$-valued.

2.7. Corollary. $\Omega(\alpha^+)$ *is a zero-dimensional (locally compact) F_{α^+}-space.*

Proof. Apply 2.6(a), (c) with $X = \Sigma(\alpha^+)$, an F_{α^+}-space by 2.5, $S = \Omega(\alpha^+)$, and with α^+ replacing α.

The following fact is also needed.

2.8. Lemma. *Let X be a locally compact, zero-dimensional space, and let F be an open-and-closed subset of $\beta X \backslash X$. Then there is an open-and-closed set N in X, such that $N' = \mathrm{cl}_{\beta X} N \backslash X = F$.*

Proof. Since X is zero-dimensional, βX is also zero-dimensional. The family $\{V \cap (\beta X \backslash X): V$ open-and-closed in $\beta X\}$ forms a base for the topology of $\beta X \backslash X$. By the local compactness of X, $\beta X \backslash X$ is compact, and thus F is equal to the union of a finite number of elements of the base. Thus $F = (V_1 \cup \cdots \cup V_n) \cap (\beta X \backslash X)$ where $n < \omega$ and V_i is open-and-closed in βX for $i \leq n$. We set $N = (V_1 \cup \cdots \cup V_n) \cap X$; it is clear that $N' = F$.

2.9. Theorem. *Let F be a closed subset of $\beta(\Omega(a^+))\backslash\Omega(a^+)$ of type at most α. Then there is a closed subset W of $\Omega(a^+)$, of type at most α, and such that $W' = \mathrm{cl}_{\beta(\Omega(a^+))} W \backslash \Omega(a^+) = F$.*

Proof. Let $F = \bigcap_{\eta < \alpha} F_\eta$, where F_η is open-and-closed in $\beta(\Omega(a^+))\backslash\Omega(a^+)$ for $\eta < \alpha$. Since, by 2.7, $\Omega(a^+)$ is locally compact and zero-dimensional, there is, by 2.8, an open-and-closed set N_η in $\Omega(a^+)$, such that $N'_\eta = F_\eta$ for all $\eta < \alpha$. We set $W = \bigcap_{\eta < \alpha} N_\eta$; thus W is a closed subset of $\Omega(a^+)$ of type at most α. By 2.2,

$$W' = \mathrm{cl}_{\beta(\Omega(a^+))} W \backslash \Omega(a^+) = \mathrm{cl}_{\beta(\Omega(a^+))}\left(\bigcap_{\eta < \alpha} N_\eta\right)\backslash\Omega(a^+)$$

$$= \bigcap_{\eta < \alpha} (\mathrm{cl}_{\beta(\Omega(a^+))} N_\eta)\backslash\Omega(a^+) = \bigcap_{\eta < \alpha} N'_\eta$$

$$= \bigcap_{\eta < \alpha} F_\eta = F.$$

3. The main results. The following result, mentioned in the introduction, will be used.

3.1. Theorem. *$U(a)$ has property Φ_α. In detail, this means the following: Let F be a closed, nonempty subset of $U(a)$ of type at most α, and let $\{V_\eta, \eta < \alpha\}$ be a family of pairwise disjoint, nonempty subsets of $U(a)$; then, there is $p \in F$ and an open-and-closed set N in $U(a)$ such that $p \in N$ and $|\{\eta < \alpha: V_\beta \cap N \neq \varnothing\}| < \alpha$ (this last condition is expressed by saying that p is not in the uniform closure of $\{V_\eta, \eta < \alpha\}$).*

The proof of this theorem is given in Theorem 4.2(i) (together with 4.1(ii)) of [7].

We now state and prove the main theorem of this paper.

3.2 Theorem. *If $\alpha = a^\omega$ then $\beta(\Omega(a^+))\backslash\Omega(a^+)$ has property Φ_α.*

Proof. Let F be a closed, nonempty subset of $\beta(\Omega(a^+))\backslash\Omega(a^+)$, such that $F = \bigcap_{\eta < \alpha} F_\eta$, where F_η is open-and-closed in $\beta(\Omega(a^+))\backslash\Omega(a^+)$ for $\eta < \alpha$.

Let $\{V_\eta,\ \eta < \alpha\}$ be a family of pairwise disjoint, nonempty open-and-closed subsets of $\beta(\Omega(\alpha^+))\backslash\Omega(\alpha^+)$. By 2.7 and 2.8, there are open-and-closed sets N_η in $\Omega(\alpha^+)$ such that $N'_\eta = \text{cl}_{\beta(\Omega(\alpha^+))}N_\eta \backslash\Omega(\alpha^+) = V_\eta$ for $\eta < \alpha$. Let $\Omega(\alpha^+)$ be equal to $\bigcup_{\xi < \alpha^+} S_\xi$, where

$\quad S_\xi \subset S_\zeta$ for $\xi < \zeta < \alpha^+$,

$\quad S_\xi$ is open-and-closed in $\Omega(\alpha^+)$ for $\xi < \alpha^+$,

$\quad S_\xi$ is homeomorphic to $U(\alpha)$ for $\xi < \alpha^+$.

Note that a subset of $\Omega(\alpha^+)$ is relatively compact (i.e. it has compact closure) in $\Omega(\alpha^+)$ if and only if it is a subset of S_ξ for some $\xi < \alpha^+$. Thus $N_\eta \cap N_{\eta'} \subset S_{\xi(\eta,\eta')}$ for some $\xi(\eta,\eta') < \alpha^+$, if $\eta < \eta' < \alpha$. By taking ξ, such that $\xi(\eta,\eta') < \xi < \alpha^+$ for all $\eta < \eta' < \alpha$, and by replacing N_η by $N_\eta\backslash S_\xi$, we may assume that the family $\{N_\eta,\ \eta < \alpha\}$ consists of pairwise disjoint sets. Further, we note that we may assume that $V_\eta \cap F = \varnothing$ for all $\eta < \alpha$; otherwise, any $p \in V_\eta \cap F$ (for some $\eta < \alpha$) and $N = V_\eta$ will satisfy the conclusion of the theorem.

By Theorem 2.9, there is a closed subset W of $\Omega(\alpha^+)$, of type at most α, such that $W' = \text{cl}_{\beta(\Omega(\alpha^+))}W\backslash\Omega(\alpha^+) = F$. Then $(W \cap N_\eta)' = W' \cap N'_\eta = F \cap V_\eta = \varnothing$ (using Corollary 2.2). Thus $W \cap N_\eta \subset S_{\xi(\eta)}$ for some $\xi(\eta) < \alpha^+$; in the same way, as above, we may assume that $W \cap N_\eta = \varnothing$ for $\eta < \alpha$. We set

$\quad B = \{x \in W: x$ is not in the uniform closure of the family $\{N_\eta,\ \eta < \alpha\}\}$.

Thus $x \in B$ if and only if there is some open-and-closed set $N(x)$ containing x, such that $|\{\eta < \alpha: N(x) \cap N_\eta \neq \varnothing\}| < \alpha$. By Theorem 3.1, we conclude that B is not relatively compact in $\Omega(\alpha^+)$. Indeed, if B is relatively compact, then $B \subset S_\xi$ for some $\xi < \alpha^+$. Since W is not relatively compact (because $W' = F \neq \varnothing$), there is ζ, $\xi < \zeta < \alpha^+$, such that $(W\backslash B) \cap (S_\zeta\backslash S_\xi) \neq \varnothing$. Thus $S_\zeta\backslash S_\xi$ is homeomorphic to $U(\alpha)$, and $W\backslash B$ is a nonempty closed subset of it, of type not exceeding α, every element of which is in the uniform closure of the family $\{N_\eta \cap (S_\zeta\backslash S_\xi): \eta < \alpha\}$, in contradiction to 3.1. By transfinite recursion, we can find a set $C = \{c_\xi,\ \xi < \alpha^+\}$ and open-and-compact sets $N(c_\xi)$ in $\Omega(\alpha^+)$ for $\xi < \alpha^+$, such that $C \subset B$, $|C| = \alpha^+$, if $D \subset C$, then D is relatively compact if and only if $|D| \leq \alpha$, $c_\xi \in N(c_\xi)$ for $\xi < \alpha^+$, $N(c_\xi) \cap N(c_\zeta) = \varnothing$ for $\xi < \zeta < \alpha^+$, and $|\{\eta < \alpha: N(c_\xi) \cap N_\eta \neq \varnothing\}| < \alpha$ for $\xi < \alpha^+$. We set $A_\xi = \{\eta < \alpha: N(c_\xi) \cap N_\eta \neq \varnothing\}$, and let $\phi: \alpha^+ \to S_\alpha(\alpha)$ be defined by $\phi(\xi) = A_\xi$. Since $|S_\alpha(\alpha)| = \alpha^\alpha = \alpha$, there is $A \in S_\alpha(\alpha)$, such that $A = A_\xi$ for α^+ many $\xi < \alpha^+$. We set $D_A = \{\xi < \alpha^+: A = A_\xi\}$. We consider the set

$$G = \bigcup_{\eta < \alpha}\{N_\eta: \eta \in \alpha\backslash A\} \cup \bigcup\{N(c_\xi): \xi \in D_A\}.$$

We apply Theorem 2.6, with $X = S = \Omega(\alpha^+)$, and with G defined as above (with α^+ replacing α). The condition $r(G \cap S_\xi) \leq \alpha$ is satisfied for $\xi < \alpha^+$. Thus G is C^*-embedded in $\Omega(\alpha^+)$. Let $f \in C^*(\Omega(\alpha^+))$ be such that

$$f(x) = 1 \quad \text{if } x \in \bigcup_{\eta < \alpha} \{N_\eta : \eta \in \alpha \backslash A\},$$

$$= 0 \quad \text{if } x \in \bigcup \{N(c_\xi) : \xi \in D_A\}.$$

Let $\bar{f} \in C(\beta(\Omega(\alpha^+))$ be the continuous extension of f to $\beta(\Omega(\alpha^+))$. Let p be any element of $\beta(\Omega(\alpha^+))\backslash\Omega(\alpha^+)$ in the closure of $\{c_\xi : \xi \in D_A\}$. Then, since $C \subset B \subset W$ and $W' = F$, we have $p \in F$; further, $\bar{f}(p) = 0$. Also, if $q \in N'_\eta = V_\eta$ for some $\eta \in \alpha \backslash A$, we have that $\bar{f}(q) = 1$. We set $N = \{p \in \beta(\Omega(\alpha^+))\backslash\Omega(\alpha^+) : \bar{f}(p) \neq 1\}$. Then $p \in N$ and $\{\eta < \alpha : V_\eta \cap N \neq \varnothing\} \subset A$, hence $|\{\eta < \alpha : V_\eta \cap N \neq \varnothing\}| < \alpha$. This completes the proof of the theorem.

For $\alpha = \omega$, Theorem 3.2 is easily seen to be equivalent to the statement: Every nonempty zero-set of $\beta(\Omega(\omega^+))\backslash\Omega(\omega^+)$ has nonempty interior. The reader will be able to give a simple direct proof of this statement, along the lines of the proof of Theorem 3.2, using only the following known facts: every nonempty zero-set of $\beta\omega\backslash\omega$ has nonempty interior; $\Omega(\omega^+)$ is a pseudocompact, locally compact, zero-dimensional F_{ω^+}-space; and Theorem 4.1 of [3]. Note that it follows that $\beta(\Omega(\omega^+))\backslash\Omega(\omega^+)$ is not basically disconnected.

3.3. Lemma. *If* α, β *are regular cardinals and* $\omega \leq \beta < \alpha$, *then* $U(\alpha)$ *does not have property* Φ_β.

Proof. Let $d = (D_\eta)_{\eta < \beta}$ be a family of subsets of α, such that $|D_\eta| = \alpha$ for $\eta < \beta$, $D_\eta \cup D_{\eta'} = \varnothing$ for $\eta < \eta' < \beta$, and $\bigcup_{\eta < \beta} D_\eta = \alpha$. Let $F = \bigcap_{\zeta < \beta} (\alpha \backslash \bigcap_{\eta \leq \zeta} D_\eta)'$ and $V_\eta = D'_\eta$ for $\eta < \beta$. (We have set for $A \subset \alpha$, $A' = (\text{cl}_{\beta(\alpha)} A) \cap U(\alpha)$.) Then F is a closed, nonempty subset of $U(\alpha)$ of type at most β, and $\{V_\eta, \eta < \beta\}$ is a family of β pairwise disjoint nonempty open-and-closed subsets of $U(\alpha)$. But it is clear that if $p \in F$ and N is any open set of $U(\alpha)$ containing p, then $|\{\eta < \alpha : V_\eta \cap N \neq \varnothing\}| = \beta$.

3.4. Corollary. *If* $\alpha = \alpha^\omega$, *then* $\beta(\Omega(\alpha^+))\backslash\Omega(\alpha^+)$ *is not homeomorphic to* $U(\alpha^+)$. *In particular* $\Omega(\alpha^+)$ *is not* C^*-*embedded in* $\beta(\alpha^+)$.

Proof. By Theorem 3.2, $\beta(\Omega(\alpha^+))\backslash\Omega(\alpha^+)$ has property Φ_α. By Lemma 3.3, (with α^+, α replacing α, β, respectively) $U(\alpha^+)$ does not have property Φ_α.

The following argument is the natural extension of the proof given for Corollary 2 in [9].

3.5. Corollary. *Let* $\alpha = \alpha^\omega$.

(a) *If* $\alpha^+ \cup \Omega(\alpha^+) \subset X \subset \beta(\alpha^+)\backslash U(\alpha^+)$, *then* X *is not a normal space; and*

(b) $\Omega(\alpha^+)$ *is not a normal space.*

Proof. (a) follows from the fact that $\Omega(\alpha^+)$ is a closed subset of X which is not C^*-embedded in X (by 3.4).

(b) It follows from (a) that $\alpha^+ \cup \Omega(\alpha^+)$ is not a normal space. To prove (b) it is sufficient to prove that there is a closed subset of $\Omega(\alpha^+)$ which is homeomorphic to $\alpha^+ \cup \Omega(\alpha^+)$. Let $\phi: \alpha^+ \to \alpha^+$ be the unique one-to-one, order-preserving function such that

$$\phi[\alpha^+] = \{\xi < \alpha^+: cf(\xi) = \alpha\}.$$

We define

$$S_\xi = (\mathrm{cl}_{\beta(\alpha^+)} \phi(\xi)) \cap \Omega(\alpha^+) \quad \text{for } \xi < \alpha^+$$

and we note that

$\Omega(\alpha^+) = \bigcup_{\xi < \alpha^+} S_\xi$,

S_ξ is homeomorphic to $U(\alpha)$ for $\xi < \alpha^+$,

S_ξ is open-and-closed in $\Omega(\alpha^+)$ for $\xi < \alpha^+$, and

$S_\zeta \subsetneqq S_\xi$ for $\zeta < \xi < \alpha^+$.

Let $\{H_\xi: \xi < \alpha^+\}$ be a family of nonempty open-and-closed subsets of $\Omega(\alpha^+)$ such that $H_\xi \subset S_{\xi+1} \setminus S_\xi$ for $\xi < \alpha^+$ and define $H = \bigcup_{\xi < \alpha^+} H_\xi$. It follows from Theorem 2.6 and Corollary 2.7 that H is an open subset of $\Omega(\alpha^+)$ which is C^*-embedded in $\Omega(\alpha^+)$. Let $a_\xi \in H_\xi$ for $\xi < \alpha^+$ and let $A = \{a_\xi: \xi < \alpha^+\}$. Then A is a C^*-embedded subset of $\Omega(\alpha^+)$ homeomorphic to α^+. It is clear that if $B \subset A$ and $|B| \leq \alpha$ then $\mathrm{cl}_{\Omega(\alpha^+)} B$ is compact and hence homeomorphic to $\alpha^+ \cup \Omega(\alpha^+)$. This completes the proof.

We remark that the argument given in part (b) of the above corollary proves that for any infinite regular cardinal α there is a closed subset of $\Omega(\alpha^+)$ homeomorphic to $\alpha^+ \cup \Omega(\alpha^+)$.

For the statement of the classical Aronszajn-Specker theorem, which follows from our results, we need the following definitions. A ramification system is a partially ordered set $\langle A, \preceq \rangle$ with a least element (if it is nonempty) and such that the set

$$P(a) = \{x \in A: x \preceq a \text{ and } x \neq a\}$$

is well ordered by $\preceq$ for $a \in A$. The order of an element $a \in A$ is the order type of $P(a)$, i.e., the unique ordinal isomorphic to the well-ordered set $P(a)$. The order of the ramification system $\langle A, \preceq \rangle$ is the least ordinal ξ, such that the order of $P(a) < \xi$ for all $a \in A$. A cardinal α is ramifiable if every ramification system $\langle A, \preceq \rangle$ of order α, such that

$$|\{a \in A: \text{order type of } P(a) = \xi\}| < \alpha \quad \text{for } \xi < \alpha,$$

has a subset of cardinality α well ordered by $\preceq$.

3.6. **Corollary (Aronszajn [4]; Specker [8]; Monk [2, p. 76]).** *If $\alpha = \alpha^\circledS$ then α^+ is not a ramifiable cardinal.*

Proof. Since $\Omega(\alpha^+)$ is zero-dimensional (Corollary 2.7), and $\Omega(\alpha^+)$ is not C^*-embedded in $\beta(\alpha^+)$ it follows that there is a $\{0, 1\}$-valued continuous function f

on $\Omega(\alpha^+)$ which cannot be extended to a $\{0, 1\}$-valued continuous function on $\Sigma(\alpha^+)$. For every $\xi < \alpha^+$, we choose a $\{0, 1\}$-valued function $f_\xi: \xi \to \{0, 1\}$, such that

$$\overline{f}_\xi | \Omega(\alpha^+) \cap \mathrm{cl}_{\beta(\alpha^+)} \overline{\xi} = f | \Omega(\alpha^+) \cap \mathrm{cl}_{\beta(\alpha^+)} \xi.$$

Since $\Omega(\alpha^+) \cap \mathrm{cl}_{\beta(\alpha^+)} \xi$ is a compact subset of $\alpha^+ \cup (\Omega(\alpha^+) \cap \mathrm{cl}_{\beta(\alpha^+)} \xi)$, such a a function f_ξ exists for $\xi < \alpha^+$. For $\xi < \alpha^+$ we set

$$\mathfrak{A}_\xi = \{h \in \{0, 1\}^\xi: h = f_\eta | \xi \text{ for some } \eta \text{ with } \xi \leq \eta < \alpha^+\},$$

and we set $\mathfrak{A} = \bigcup_{\xi < \alpha^+} \mathfrak{A}_\xi$. We make $\mathfrak{A}$ into a partially ordered set by setting $h \leq h'$ if $h \subset h'$ (for h and h' in $\mathfrak{A}$). It is easy to verify that $\mathfrak{A}$ is a ramification system, that for $\xi < \alpha^+$ the set of elements of $\mathfrak{A}$ of order ξ is the set $\mathfrak{A}_\xi$, and (since $f_\xi \in \mathfrak{A}_\xi$) that $\mathfrak{A}_\xi \neq \varnothing$ for $\xi < \alpha^+$. Furthermore, if $h \in \mathfrak{A}_\xi$, then $|\{\zeta < \xi: h(\zeta) \neq f_\xi(\zeta)\}| < \alpha$ (because $\overline{h} | \Omega(\alpha^+) \cap \mathrm{cl}_{\beta(\alpha^+)} \xi = \overline{f}_\xi | \Omega(\alpha^+) \cap \mathrm{cl}_{\beta(\alpha^+)} \xi$; it then follows that $|\mathfrak{A}_\xi| \leq \alpha^\mathfrak{S} = \alpha < \alpha^+$ for $\xi < \alpha^+$). And finally, $\mathfrak{A}$ does not contain any well-ordered subset of order-type α^+; indeed if $\{h_\xi: \xi < \alpha^+\}$ were such a chain, then $h = \bigcup_{\xi < \alpha^+} h_\xi$ would be a well-defined, $\{0, 1\}$-valued function on α^+ such that $f \subset \overline{h} | \Sigma(\alpha^+)$, contrary to the choice of f. This completes the proof of the theorem.

We finally remark that results similar to those leading to Theorem 3.2 hold for the growth of the space $S_\alpha \backslash \{p\}$, when $\alpha = \alpha^\mathfrak{A}$, S_α is the Stone space of the α-homogeneous, α-universal Boolean algebra of cardinality α, and p is a P_α-point of S_α (see [6] for the theory of these spaces). The details, which are similar and somewhat simpler to those of the present paper, will be given elsewhere.

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